ON THE EXISTENCE RESULT FOR AN ABSTRACT NONLINEAR DIFFERENTIAL EQUATION

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1. Introduction

Let $X$ be a real Banach space with norm $||\cdot||$ and $U \subset X$ be an open set.

The purpose of this paper is to study the local existence of the integral solution in the sense of Benilan [2], for the initial value problem

\[
(E) \quad \frac{du(t)}{dt} + Au(t) \ni F(u(t)), \quad 0 \leq t \leq T,
\]

\[u(0) = u_0,
\]

where $A \subset X \times X$ is a $m$-accretive set, $F$ is a mapping from $C(0, T; U)$ into $C(0, T; X)$ and $u_0 \in \overline{D(A)} \cap U$.

Under different assumptions than ours, this problem has been studied in [1,6,7,9].

Section 2 is a preliminary part. In section 3, we establish the existence of integral solution of problem $(E)$ in the case which $(I + \lambda A)^{-1}$ is compact for all $\lambda > 0$ and $F$ satisfies an appropriate bounded condition. Section 4 is devoted to the continuation of solution and an example.

2. Preliminaries

Throughout this paper, $X$ is a real Banach space with norm $||\cdot||$ and $X^*$ is its dual space with the corresponding norm $||\cdot||_*$. Let $J : X \to 2^{X^*}$ be the duality mapping, i.e.
for each $x \in X$. For each $(x, y) \in X \times X$, define

(2.2) $\langle y, x \rangle = \sup \{ \langle y, x^* \rangle : x^* \in J(x) \}$.

We recall that the mapping $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is upper semicontinuous. If $A \subset X \times X$ and $x \in X$, we denote by $Ax = \{ y \in X : (x, y) \in A \}$, $D(A) = \{ x \in X : Ax \neq \emptyset \}$, $R(A) = \bigcup \{ Ax : x \in D(A) \}$ and $|Ax| = \inf \{ ||y|| : y \in Ax \}$. For each set $D \subset X$, $\overline{D}$ represents the closure of $D$. For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the closed ball with center $x$ and radius $r$.

**Definition 2.1.** ([9]) A continuous function $u : [0, T] \to D(A) \cap U$ with $u(0) = u_0$ is called an integral solution of (E) if

(2.3) $||u(t) - x||^2 \leq ||u(s) - x||^2 + 2 \int_s^t \langle F(u(\theta)) - y, u(\theta) - x \rangle \, d\theta$

for all $(x, y) \in A$ and $0 \leq s \leq t \leq T$.

Let $A \subset X \times X$ be an $m$-accretive set and $f \in L^1(0, T ; X)$. We recall that a function $u : [0, T] \to X$ is called a strong solution of the initial value problem

(2.4) $\frac{du(t)}{dt} + Au(t) = f(t) \quad 0 \leq t \leq T,$

(2.5) $u(0) = u_0$

if $u$ is differentiable almost everywhere on $[0, T]$, absolutely continuous and satisfies $u(0) = u_0$ and $u'(t) + Au(t) = f(t)$ almost everywhere on $[0, T]$. It is well known that the initial value problem (2.4), (2.5) has a unique integral solution on $[0, T]$. Every strong solution of (2.4), (2.5) is also an integral solution of (2.4), (2.5) (see [1], [2]). Moreover, if $u$ and $v$ are two integral solutions of (2.4), (2.5) corresponding to $f \in L^1(0, T ; X)$ and $g \in L^1(0, T ; X)$ respectively, then

(2.6) $||u(t) - v(t)||^2 \leq ||u(s) - v(s)||^2$.
for all $0 \leq s \leq t \leq T$. From (2.6), we also have that
\[
(2.7) \quad ||u(t) - v(t)|| \leq ||u(s) - v(s)|| + \int_{s}^{t} ||f(\theta) - g(\theta)|| d\theta
\]
for all $0 \leq s \leq t \leq T$. For the proof, see [1], [2].

It is well known that each $m$-accretive set $A \subset X \times X$ generates a nonlinear semigroup of contractions $S(t) : \overline{D(A)} \rightarrow \overline{D(A)}$, strongly continuous at the origin (see [1], [5]).

Using (2.7), we verify easily the following lemma (see also [10]).

**Lemma 2.1.** Let $u : [0, T] \rightarrow \overline{D(A)}$ be an integral solution of (2.4), (2.5), $t \in [0, T)$, $s \in (0, T]$, $h \in \mathbb{R}$ with $t + h \in [0, T]$ and $s - h \in [0, T]$. Then, the following inequalities hold:
\[
(2.8) \quad ||u(t) - u(t + h)|| \leq ||S(h)u_0 - u_0|| + \int_{0}^{s} ||f(\theta)|| d\theta
\]
\[
+ \int_{s}^{t} ||f(\theta + h) - f(\theta)|| d\theta
\]
and
\[
(2.9) \quad ||u(s) - u(s - h)|| \leq ||S(h)u_0 - u_0|| + \int_{s}^{t} ||f(\theta)|| d\theta
\]
\[
+ \int_{t}^{s} ||f(\theta - h) - f(\theta)|| d\theta.
\]

The following lemma has been obtained by Brezis [4].

**Lemma 2.2.** Let $A \subset X \times X$ be a $m$-accretive set and $J_\lambda = (I + \lambda A)^{-1}$ for $\lambda > 0$. Then, for every $x \in \overline{D(A)}$, $t > 0$ and $\lambda > 0$, we have the following inequality
\[
(2.10) \quad ||x - J_\lambda x|| \leq (1 + \frac{\lambda}{t}) \frac{1}{2} \int_{0}^{t} ||x - S(\theta)x|| d\theta,
\]
and in particular
\[
(2.11) \quad ||x - J_\lambda x|| \leq \frac{4}{t} \int_{0}^{t} ||x - S(\theta)x|| d\theta.
\]

**Definition 2.2.** A mapping $F : C([0, a]; U) \rightarrow C([0, a]; X)$ is
called $L^\infty$-bounded if for each $T \in (0, a]$, $v \in C(0, T; U)$ and $r > 0$, there exists $g \in L^\infty(0, T; \mathbb{R})$ such that
\begin{equation}
||F(u)(t)|| \leq g(t) \quad a.e. \text{ on } [0, T],
\end{equation}
for all $u \in C(0, T; U)$ with $||u(t) - v(t)|| \leq r$, for $0 \leq t \leq T$.

We conclude this Section with following lemma.

**Lemma 2.3.** Let $F$ be $L^\infty$-bounded and $T \in (0, a]$. Then, for $v \in C(0, T; U)$ and $r > 0$, the following relations:
\begin{align}
(2.13) \quad & \lim_{h \to 0} \int_0^t ||F(u)(s) - F(u)(s + h)|| \, ds = 0, \quad 0 \leq t < T, \\
(2.14) \quad & \lim_{h \to 0} \int_0^t ||F(u)(s) - F(u)(s - h)|| \, ds = 0, \quad 0 < t \leq T,
\end{align}
hold, uniformly with respect to all $u \in C(0, T; U)$ with $||u(t) - v(t)|| \leq r$ for $0 \leq t \leq T$.

**Proof.** This follows directly from Lebesgue convergence theorem.

3. Existence

**Theorem 3.1.** Assume that
\begin{enumerate}
\item[(C_1)] $X$ is a real Banach space and $U \subset X$ is a given open set,
\item[(C_2)] $A \subset X \times X$ is a $m$-accretive set and $(I + \lambda A)^{-1}$ is compact for all $\lambda > 0$
\item[(C_3)] $F : C(0, a; U) \rightarrow C(0, a; X)$ is continuous and $L^\infty$-bounded.
\end{enumerate}
Then, for each $u_0 \in \overline{D(A)} \cap U$, there exists $T \in (0, a]$ such that problem (E) has at least one integral solution on $[0, T]$.

**Remark 3.1.** We note that in the case in which $X$ is a real reflexive Banach space and $F(u)$ is of bounded variation, where $u \in C(0, a; U)$ is of bounded variation, the integral solution $u$ provided by Theorem 3.1 is a strong solution (see [1] [7]), as well as, in the case in which $X$ is a real refle-
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xive Banach space, \( u \) is a weak solution (see [1]).

Proof. Let \( u_0 \in \overline{D(A)} \cap U \) and choose \( T > 0, \; r > 0 \) and \( g \in L^{\infty}(0, T; R) \) such that

\[
(3.1) \quad B(u_0, r) \subset \overline{D(A)} \cap U,
\]
\[
(3.2) \quad || F(u)(t) || \leq g(t) \quad a.e. [0,T]
\]

for all \( u \in C(0, T; U) \) with \( || u(t) - u_0 || \leq r \) for \( 0 \leq t \leq T \) and in addition,

\[
(3.3) \quad \int_0^T g(s) ds + || S(t) u_0 - u_0 || \leq r
\]

for all \( 0 \leq t \leq T \). From condition \((C_3)\) and continuity of \( S(t) \) at the origin, we can choose such constants \( T, r \) and \( g \in L^{\infty}(0, T; R) \). Now we set

\[
(3.4) \quad K = \{ u \in C(0, T; U) ; u(t) \in B(u_0, r) \} \text{ for all } 0 \leq t \leq T
\]

and we deduce easily that \( K \) is nonempty, convex and closed in \( C(0, T; X) \). Let \( v \in K \). Then, by Benilan's existence and uniqueness theorem, the initial value problem

\[
(3.5) \quad \frac{du(t)}{dt} + Au(t) \ni F(v)(t), \quad 0 \leq t \leq T,
\]
\[
(3.6) \quad u(0) = u_0
\]

has a unique integral solution \( u \in C(0, T; X) \). Therefore we define the operator \( Q : K \to C(0, T; X) \) by \( Qv = u \), where \( u \) and \( v \) satisfy together \((3.5),(3.6)\). Now let us observe that the problem \((E)\) has at least one integral solution if and only if the operator \( Q \) has at least one fixed point. Thus it suffices to show the operator \( Q \) maps \( K \) into \( K \) and is completely continuous. To this end, we apply \((2.7)\) to \( Qv \) and to the solution \( w \) of the problem

\[
(3.7) \quad \frac{dw(t)}{dt} + Aw(t) \ni 0, \quad 0 \leq t \leq T,
\]
\[
(3.8) \quad w(0) = u_0
\]

Since the integral solution of \((3.7),(3.8)\) is expressed as
(3.9) \( w(t) = S(t)u_0 \),

we obtain from (2.7)

(3.10) \[ \| (Qv)(t) - S(t)u_0 \| \leq \int_0^t \| F(v(s)) \| \, ds. \]

Combining (3.10) with

(3.11) \[ \| (Qv)(t) - u_0 \| \leq \| (Qv)(t) - S(t)u_0 \| + \| S(t)u_0 - u_0 \|, \]

we get from (3.2)

(3.12) \[ \| (Qv)(t) - u_0 \| \leq \int_0^t g(s) \, ds + \| S(t)u_0 - u_0 \|, \]

which in view of (3.3) implies

(3.13) \[ \| (Qv)(t) - u_0 \| \leq r \text{ for all } 0 \leq t \leq T. \]

But (3.13) show that \( QK \subset K \). It is easy to show that \( Q \)
is continuous on \( K \) in the uniform convergence topology (see [9]). To conclude that \( Q \) is completely continuous, we shall use-Ascoli’s theorem. First we show that the family \( \{Qv: v \in K\} \) is equicontinuous on \([0, T]\). Using Lemma 2.1, we get

(3.14) \[ \| (Qv)(t) - (Qv)(t+h) \| \]

\[ \leq \| S(h)u_0 - u_0 \| + \int_0^h \| F(v(s)) \| \, ds \]

\[ + \int_0^t \| F(v(s)) - F(v(s+h)) \| \, ds \]

for each \( 0 \leq t < T \) and \( h > 0 \). From (3.2), we deduce

(3.15) \[ \| (Qv)(t) - (Qv)(t+h) \| \]

\[ \leq \| S(h)u_0 - u_0 \| + \int_0^h g(s) \, ds \]

\[ + \int_0^t \| F(v(s)) - F(v(s+h)) \| \, ds. \]

In a similar manner we get, for each \( 0 < t \leq T \) and \( 0 < h \leq t \),

(3.16) \[ \| (Qv)(t) - (Qv)(t-h) \| \]

\[ \leq \| S(h)u_0 - u_0 \| + \int_0^h g(s) \, ds \]

\[ + \int_0^t \| F(v(s)) - F(v(s-h)) \| \, ds. \]

Considering (2.13) and (2.14) in Lemma 2.3 and the continuity of the semigroup \( S(t) \) at the origin, we obtain from
(3.15), (3.16) the equicontinuity of the family \( \{Qv; v \in K\} \). Next we show that for each \( 0 \leq t \leq T \), the set \( \{Qv(t); v \in K\} \) is precompact in \( X \). Obviously for \( t=0 \), the set above being a singleton \( \{u_0\} \) is precompact. Then, let \( 0 < t \leq T \) and \( 0 < \theta < t \), \( v \in K \) and consider the following initial value problem

\[
\frac{dv^\theta(s)}{ds} + Av^\theta(s) \geq 0, \quad t - \theta \leq s \leq t + \theta,
\]

\[
v^\theta(t - \theta) = (Qv)(t - \theta).
\]

Applying (2.7) to \( v^\theta \) and to \( Qv \) on \([t-\theta, t]\), we get

\[
||Qv(t) - v^\theta(t)|| \leq \int_{t-\theta}^{t} |F(v)(r)||dr.
\]

From (3.2) and (3.19), it follows that

\[
||Qv(t) - v^\theta(t)|| \leq \int_{t-\theta}^{t} g(r)dr.
\]

Now by the inequality (2.19) of Lemma 2.2, we get

\[
||Qv(t) - J_\lambda(Qv)(t)|| \leq (1 + \frac{1}{s}) \frac{2}{s} \int_{0}^{s} ||Qv(t) - S(\theta)(Qv)(t)||d\theta.
\]

Let us observe that

\[
||Qv(t) - S(\theta)(Qv)(t)|| \leq ||Qv(t) - v^\theta(t)|| + ||S(\theta)(Qv)(t - \theta)|| + ||S(\theta)(Qv)(t - \theta) - S(\theta)(Qv)(t)||
\]

for all \( 0 < \theta < t \). In view of the equality

\[
S(\theta)(Qv)(t - \theta) = v^\theta(t),
\]

we get from (3.20) and (3.23)

\[
||Qv(t) - S(\theta)(Qv)(t)|| \leq \int_{t-\theta}^{t} g(r)dr + ||Qv(t - \theta) - (Qv)(t)||.
\]

Using (3.24), we get from (3.21) for \( \lambda = s \in (0, t) \),

\[
||Qv(t) - J_\lambda(Qv)(t)|| \leq 4 \int_{t-\theta}^{t} g(r)dr + \frac{4}{\lambda} \int_{0}^{\lambda} ||Qv(t - \theta) - (Qv)(t)||d\theta.
\]
As family \( \{ Qv : v \in K \} \) is equicontinuous at each \( 0 \leq t \leq T \), it follows that
\[
(3.26) \quad \| (Qv)(t) - (Qv)(s) - (Qv)(s) \| \leq \epsilon(t),
\]
where \( \epsilon(t) \) is nondecreasing and \( \epsilon(t) \to 0 \) as \( t \to 0 \). Thus from (3.25) and (3.26), we obtain
\[
(3.27) \quad \| (Qv)(t) - J\lambda(Qv)(t) \| \leq 4\left( \int_{t-\lambda}^{t} g(s) ds + \epsilon(\lambda) \right).
\]

Since \( J\lambda = (I + \lambda A)^{-1} \) is compact by (C3) and \( \{(Qv)(t) : v \in K \} \) is bounded for all \( 0 \leq t \leq T \), we conclude from (3.27) that the set \( \{(Qv)(t) : v \in K \} \) is precompact in \( X \) for all \( 0 \leq t \leq T \). Therefore by Ascoli's theorem \( QK \) is relatively compact in \( C(0, T; X) \), which complete the completely continuity of \( Q \).

Finally, to prove Theorem 3.1 we have only to mention that the operator \( Q \) satisfies the hypotheses of Schauder's fixed point theorem, and thus \( Q \) has at least one fixed point \( u \in K \), which is an integral solution of the problem \((E)\).

**Corollary 3.1.** Assume that \( X \) is a real finite dimensional Hilbert space, \( A \subset X \times X \) is a \( m \)-accretive set and (C3) is satisfied. Then, for each \( u_0 \in D(A) \cap U \) there exists \( T > 0 \) such that the problem \((E)\) has at least one strong solution on \([0, T]\).

**Proof.** As \( X \) is finite dimensional and \( A \subset X \times X \) is a \( m \)-accretive set, it follows that the operator \( J\lambda = (I + \lambda A)^{-1} \) is compact for all \( \lambda > 0 \) and therefore we are in the hypotheses of Theorem 3.1. Now we have to remark that in the case in which \( X \) is finite dimensional Hilbert space, each integral solution is a strong solution (see [31]).

**4. Continuation of solution, example**

In this section, we shall give a result about the continuation of solution and an example which is an application of
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our results.

**Theorem 4.1.** Assume that \( (C_1), (C_2) \) and \( (C_3) \) are satisfied. Then, for each \( u_0 \in \mathcal{D}(A) \cap U \), there exists an integral solution \( u \) of \( (E) \) defined on a maximal interval of existence \([0, T_{\text{max}}] \), where either \( T_{\text{max}} = a \) or if \( T_{\text{max}} < a \), then \( ||u(t)|| \to \infty \) as \( t \uparrow T_{\text{max}} \).

**Proof.** Let \( u(t) \) be an integral solution of problem \((E)\) on \([0, t_1] \). We extend \( u(t) \) to the interval \([0, t_1 + \delta] \) with \( \delta > 0 \) by defining

\[
u(t + t_1) = w(t),
\]

where \( w(t) \) is an integral solution of the problem

\[
\frac{dw(t)}{dt} + Aw(t) = F(w)(t),
\]

\[
w(0) = u(t_1).
\]

From Theorem 3.1, there exists an integral solution \( w(t) \) on an interval of positive length \( \delta > 0 \). Let \([0, T_{\text{max}}] \) be the maximal interval to which the integral solution \( u(t) \) of problem \((E)\) can be extended. Now we shall prove that if \( T_{\text{max}} < a \), then \( ||u(t)|| \to \infty \) as \( t \uparrow T_{\text{max}} \). First we show that if \( T_{\text{max}} < a \), then \( \lim \sup_{t \uparrow T_{\text{max}}} ||u(t)|| = \infty \). To this end, assume that if \( T_{\text{max}} < a \), \( \lim \sup_{t \uparrow T_{\text{max}}} ||u(t)|| < \infty \). Then there exists \( K > 0 \) such that for \( 0 \leq t < T_{\text{max}} \),

\[
||u(t)|| \leq K.
\]

and by our assumption on function \( F \), there exists

\[
g \in L^\infty(0, T_{\text{max}}; R_+)
\]

such that

\[
||F(u)(t)|| \leq g(t) \quad \text{a.e. on } [0, T_{\text{max}}].
\]

If \( 0 < t < t' < T_{\text{max}} \), then we have from (2.6) that

\[
||u(t') - u(t)||^2 \leq 2 \int\limits_{t'}^{t''} \langle F(u)(\theta) - y, u(\theta) - u(t) \rangle \, d\theta
\]

\[
\leq 2 \int\limits_{t'}^{\min(t', t'')} ||F(u)(\theta) - y|| ||u(\theta) - u(t)|| \, d\theta
\]
\[
\begin{align*}
\leq 4K \int_{t}^{t'} (||F(u(\theta))|| + |Au(\theta)|) d\theta \\
\leq 4K [||g||_{L^\infty(0,T_{\text{max}};R_+)} (t' - t) \\
+ \int_{t}^{t'} |Au(\theta)| d\theta],
\end{align*}
\]

where \((u(t), y) \in A\). From (4.5) we get \(|u(t') - u(t)| \to 0\) as \(t', t \uparrow T_{\text{max}}\). Thus \(\lim_{t \to T_{\text{max}}} u(t) = u(T_{\text{max}})\) exists and by the first part of the proof, the solution \(u\) can be extended beyond \(T_{\text{max}}\), contradicting the maximality of \(T_{\text{max}}\). Therefore the assumption \(T_{\text{max}} < a\) implies \(\lim_{t \to T_{\text{max}}} \sup_{t} |u(t)| = \infty\). To complete the proof, we show that \(|u(t)| \to \infty\) as \(t \uparrow T_{\text{max}}\). If it is false, then there exist \(K > 0\) and a sequence \(\{t_n\}\) such that \(t_n \to T_{\text{max}}\) and for all \(n\) and \(x \in D(A)\),

\[ (4.6) \quad |u(t_n) - x|^2 \leq K. \]

We also deduce that

\[ (4.7) \quad ||F(u)(t)|| + |Ax| \leq g(t) + M \quad \text{a.e. on } [t, T_{\text{max}}] \]

for some \(g \in L^\infty(0,T_{\text{max}};R_+)\) and some \(M > 0\). Since \(t \to |u(t) - x|\) is continuous and \(\lim_{t \to T_{\text{max}}} \sup_{t} |u(t)| = \infty\), there exists a sequence \(\{h_n\}\) with the following properties: \(h_n \to 0\) as \(n \to \infty\),

\[ (4.8) \quad |u(t) - x|^2 \leq K + 1 \]

for \(t_n \leq t \leq t_n + h_n\) and

\[ (4.9) \quad |u(t_n + h_n) - x|^2 = K + 1. \]

Then we have from (2.6)

\[ (4.10) \quad K + 1 = |u(t_n + h_n) - x|^2 \]

\[ \leq |u(t_n) - x|^2 + 2 \int_{t_n}^{t_n + h_n} \langle F(u(\theta)) - y, u(\theta) - x \rangle \, d\theta \]

\[ \leq K + 2 \int_{t_n}^{t_n + h_n} ||F(u(\theta)) - y|| \cdot ||u(\theta) - x|| \, d\theta \]

\[ \leq K + 2 \sqrt{K + 1} (||g||_{L^\infty(0,T_{\text{max}};R_+)} + M) h_n, \]
where $(x, y) \in A$. This gives a contraction as $n \to \infty$. Thus we conclude that $|u(t)| \to \infty$ as $t \uparrow T_{m}$, which completes the proof of Theorem 4.1.

We note that each continuous function $f : [0, a] \times U \to X$ generates a unique continuous function $F : C(0, a; U) \to C(0, a; X)$. Thus we obtain the following.

**Corollary 4.1.** Assume that $(C_{1})$ and $(C_{2})$ are satisfied and in addition that $f : [0, \infty) \times U \to X$ is a continuous mapping. Assume further that

$$
(4.11) \quad ||f(t, x)|| \leq K_{1}(t)||x|| + K_{2}
$$

for each $(t, x) \in [0, \infty) \times U$, where $K_{1} \in L^{\infty}_{loc}(0, \infty; R)$ and $K_{2} \in R$. Then, for each $u_{0} \in \overline{D(A)} \cap \mathbb{C}$, there exists an integral solution of $(E)$ defined on the whole positive half axis.

**Proof.** From (4.11), it is easy to show that a function $F$ generated by $f$ is $L^{\infty}$-bounded, and hence, by Theorem 4.1, the Corollary follows.

**Example 4.1.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^{n}$ with sufficiently smooth boundary $\Gamma$. Let $W^{1,1}(\Omega), W^{1, r_{0}}, \ldots, W^{1, r_{n}}(\Omega), H^{s}(\Omega)$ and $H^{s}_{0}(\Omega)$ stand for Sobolev spaces on $\Omega$.

We consider a nonlinear differential operator of the form

$$
(4.12) \quad Au = \sum_{i=1}^{m} (-1)^{s} D^{s} A_{s}(x, u, Du, \ldots, D^{s} u),
$$

where $A_{s}(x, z)$ are real functions defined on $\Omega \times \mathbb{R}^{n}$ and satisfy the following conditions:

(I) $A_{s}$ are measurable $x$ and continuous in $z$ for all $\alpha$.

There exist $p > 1, g \in L^{p}(\Omega) \left(\frac{1}{p} + \frac{1}{q} = 1\right)$ and a positive constant $C$ such that

$$
(4.13) \quad |A_{s}(x, z)| \leq C|z|^{{p-1}} + g(x) \quad a.e. \quad x \in \Omega.
$$

(II) For any $(y, z) \in \mathbb{R}^{n} \times \mathbb{R}^{n}$ and for almost every $x \in \Omega$,
the following inequality holds:
\[ \sum_{|d| \leq m} (A_d(x, z) - A_d(x, y))(z - y) \geq \omega( \sum_{|d| \leq m} |z - y|^3), \]
where \( \omega > 0. \)

We recall that the operator \( A: W_0^{m, \beta}(\Omega) \rightarrow W^{-m, \beta}(\Omega) \)
defined by
\[ \langle Au, v \rangle = \sum_{|d| \leq m} \int_{\Omega} A_d(x, u, Du, ..., D^\alpha u) D^\alpha v dx \]
for \( u, v \in W_0^{m, \beta}(\Omega), \) is monotone and demicontinuous and
\[ ||Au||_{-m, \beta} \leq C(1 + ||u||_{m, \beta}^p) \]
(see [11]).

Now we consider the nonlinear boundary value problem of the parabolic type
\[ \frac{\partial u}{\partial t} + \sum_{|d| \leq m} (-1)^d D^\alpha A_d(x, u, Du, ..., D^\alpha u) = f \]
with Dirichlet boundary conditions
\[ D^\alpha u = 0 \text{ on } [0, T] \times \Gamma \text{ for } |\alpha| \leq m - 1 \]
and initial condition
\[ u(0, x) = u_0(x) \text{ on } \Omega. \]

**Theorem 4.2.** Let \( H = L^2(\Omega), V = H_0^m(\Omega) \) and \( A: H_0^m(\Omega) \rightarrow H^{-m}(\Omega) \) be the nonlinear operator defined above. Let \( f: [0, \infty) \times H \rightarrow H \) be a continuous function. Then, for \( u_0 \in H, \) there exists \( T > 0 \) such that (4.17), (4.18), (4.19) has at least one integral solution on \([0, T] \).

**Proof.** Let \( A_H \) be a operator defined by
\[ A_H u = Au \]
for \( u \in D(A_H) = \{ u \in V : Au \in H \}. \) Then \( A_H \) is a \( m \)-accretive operator on \( H \) (see [1], [9]). Now let us remark that (4.17), (4.18), (4.19) can be rewritten in the form
\[ \frac{du(t)}{dt} + A_H(t) = F(u(t)), \]
THE EXISTENCE RESULT

\[ u(0) = u_0, \]
\[ F(u)(t) = f(t, u(t)). \]

Since \( A_H \) is coercive by (II) and the inclusion mapping from \( V \) into \( H \) is completely continuous, we can show that \( (I + \lambda A_H)^{-1} \) is compact for \( \lambda > 0 \). It is easy to see that for each \( T > 0 \), \( F : C(0, T; H) \rightarrow C(0, T; H) \) is continuous. Since \( f(t, u(t)) \in H \) for any \( u \in C(0, T; H) \), it is obvious that \( F \) is \( L^\infty \)-bounded. Therefore by applying Theorem 3.1, we obtain the conclusion of Theorem 4.2.

References


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