SOME PROPERTIES OF BITOPOLOGICAL SPACES

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1. Introduction

A bitopological space \((X, P, Q)\) is a set \(X\) together with two topologies \(P\) and \(Q\) on \(X\). In this paper we study some properties of bitopological spaces.

In J. C. Kelly[2], Theorem 2.7 is the generalization of Urysohn's Lemma. In this paper we prove the sufficiency condition also holds.

In section 3, when we define a bitopological space \((X, P, Q)\) to be pairwise paracompact if \((X, P, Q)\) is pairwise Hausdorff and each \(P\)-open covering of \(X\) has a \(Q\)-open nbd-finite refinement, and each \(Q\)-open covering of \(X\) has a \(P\)-open nbd-finite refinement, we prove that every pairwise paracompact space is pairwise normal.

2. Pairwise Hausdorff, pairwise regular, pairwise normal bitopological spaces

The following definition extends to a bitopological space \((X, P, Q)\) the notions of separation properties of a topological space \((X, F)\).

DEFINITION 2.1. [2]. In a space \((X, P, Q)\), \(P\) is said to be regular with respect to \(Q\) if, for each point \(x\) in \(X\), there is a \(P\)-neighbourhood base of \(Q\)-closed sets.

THEOREM 2.2. In a space \((X, P, Q)\), \(P\) is regular with respect to \(Q\) if and only if, for each point \(x\) in \(X\) and
each P-closed set P such that \( x \in P \), there are a P-open set U and a Q-open set V such that \( x \in U, P \subseteq V \), and \( U \cap V = \emptyset \).

PROOF. Let \( x \) be an arbitrary point in \( X \), and \( P \) is a P-closed set such that \( x \in P \). Then \( x \in P^c \) and \( P^c \) is a P-open set. By hypothesis, there exists a \( P \)-neighbourhood of \( Q \)-closed set \( U \) of \( x \) such that \( U \subseteq P^c \). Then, \( x \in U, P \subseteq U^c \), \( U \) is a P-open set, \( U^c \) is a Q-open set, and \( U \cap U^c = \emptyset \), i.e. there exist a P-open set \( U \) and a Q-open set \( U^c \) such that \( x \in U, P \subseteq U^c \), and \( U \cap U^c = \emptyset \). Conversely, let \( x \) be an arbitrary point in \( X \) and \( N(x) \) be a P-open neighbourhood base of \( x \). Then for each element \( W \) of \( N(x) \), \( W^c \) is P-closed and \( x \in W^c \).

By hypothesis, there exist a P-open set \( U \) and a Q-open set \( V \) such that \( x \in U, W^c \subseteq V \) and \( U \cap V = \emptyset \). Then there are a P-open set \( U \) and Q-closed set \( V^c \) such that \( x \in U \subseteq V^c \subseteq W \).

Put \( W(x) = \{ V^c \} \), then \( W(x) \) is a \( P \)-neighbourhood base of \( Q \)-closed sets.

\((X, P, Q)\) is or \( P \) and \( Q \) are, pairwise regular if \( P \) is regular with respect to \( Q \) and vice versa.

DEFINITION 2.3. [2]. A space \((X, P, Q)\) is said to be pairwise Hausdorff if for each two distinct points \( x \) and \( y \), there are a \( P \)-neighbourhood \( U \) of \( x \) and a \( Q \)-neighbourhood \( V \) of \( y \) such \( U \cap V = \emptyset \).

THEOREM 2.4. [5]. If a bitopological space \((X, P, Q)\) is pairwise Hausdorff then sets which are compact with respect to one are closed with respect to the other.

DEFINITION 2.5. [2]. A space \((X, P, Q)\) is said to be pairwise normal if, given a \( P \)-closed set \( A \) and a \( Q \)-closed set \( B \)
with $A \cap B = \phi$, there exist a $Q$-open set $U$ and a $P$-open set $V$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \phi$.

**Theorem 2.6.** A space $(X, P, Q)$ is pairwise normal if and only if, given a $Q$-closed set $C$ and a $P$-open set $D$ such that $C \subseteq D$, there are a $P$-open set $G$ and a $Q$-closed set $F$ such that $C \subseteq G \subseteq F \subseteq D$.

**Proof.** Let $C$ be a $Q$-closed set and $D$ be a $P$-open set such that $C \subseteq D$. Then $D^c$ is a $P$-closed set and $C \cap D^c = \phi$. By hypothesis there exists a $Q$-open set $U$ and a $P$-open set $V$ such that $D^c \subseteq U$, $C \subseteq V$, and $U \cap V = \phi$.

Then $U^c \subseteq D$, $C \subseteq V$, $V \subseteq U^c$, and $U^c$ is a $Q$-closed set. Put $F = U^c$, and $G = V$, then there exist a $P$-open set $G$ and a $Q$-closed set $F$ such that $C \subseteq G \subseteq F \subseteq D$.

Conversely, let $A$ be a $P$-closed set and $B$ be a $Q$-closed set such that $A \cap B = \phi$. Then $A^c$ is a $P$-open set and $B$ is a $Q$-closed set such that $B \subseteq A^c$.

By hypothesis there exist a $P$ open set $G$ and $Q$-closed set $F$ such that $B \subseteq G \subseteq F \subseteq A^c$. Then $B \subseteq G$, $A \subseteq F^c$, $F^c$ is a $Q$-open set and $G \cap F^c = \phi$. Thus there exist a $P$-open set $G$ and a $Q$-open set $F^c$ such that $B \subseteq G$, $A \subseteq F^c$, and $G \cap F^c = \phi$.

In J.C. Kelly [2], Theorem 2.7. is the generalization of Urysohn’s Lemma. In this paper we prove that the sufficient condition also holds.

**Theorem 2.7.** A space $(X, P, Q)$ is pairwise normal, if and only if, given a $Q$-closed set $F$ and a $P$-closed set $H$ with $F \cap H = \phi$, there exists a real-valued function $g$ on $X$ such that,

1. $g(x) = 0$ ($x \in F$), $g(x) = 1$ ($x \in H$), and $0 \leq g \leq 1$.
2. $g$ is $P$-upper semi-continuous and $Q$-lower semi-con-
Proof. Necessity. [2, theorem 2.7]

Sufficiency. Let $A$ and $B$ be subsets of $X$ such that $A$ is a $Q$-closed set, $B$ is a $P$-closed set, and $A \cap B = \emptyset$. By hypothesis there exists a real valued function $g$ on $X$ such that,

1. $g(x) = 0$ ($x \in A$), $g(x) = 1$ ($x \in H$), and
2. $g$ is $P$-upper semi-continuous and $Q$-lower semi-continuous.

Put $U = \{x | g(x) < \frac{1}{2}\}$, $V = \{x | g(x) > \frac{1}{2}\}$. Then $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$, and $U$ is a $P$-open set $V$ is a $Q$-open set. Hence $(X, P, Q)$ is pairwise normal.

Another necessary and sufficient conditions that $(X, P, Q)$ is pairwise normal is in E. P. Lane [4].

Theorem 2.8. In order for $(X, P, Q)$ to be pairwise normal, it is necessary and sufficient that for every pair of functions $f$ and $g$ defined on $X$ such that $f$ is $P$-lower semi-continuous and $g$ is $Q$-upper semi-continuous, and $g \leq f$, there exists a $P$-lower semi-continuous and $Q$-upper semi-continuous function $h$ on $X$ such that $g \leq h \leq f$.

Proof. Necessity. [4, theorem 2.5]

Sufficiency. Let $A$ and $B$ be subsets of $X$ such that $A$ is a $P$-closed set, $B$ is a $Q$-closed set, $A \cap B = \emptyset$. Define real functions $\chi_{X-A}: X \rightarrow R$, $\chi_{X-B}: X \rightarrow R$ by $\chi_{X-A}(x) = \begin{cases} 1 & x \in X - A, \\ 0 & x \in A \end{cases}$, $\chi_{X-B}(x) = \begin{cases} 1 & x \in B, \\ 0 & x \in X - B. \end{cases}$ Then $\chi_{X-A}$ is $P$-lower semi-continuous, $\chi_B$ is $Q$-upper semi-continuous and $\chi_B \leq \chi_{X-A}$. By hypothesis there exists a $P$-lower semi-continuous and $Q$-upper semi-continuous function $h$ on $X$ such that $\chi_B \leq h \leq \chi_{X-A}$. Put $U = \{x : h(x) > \frac{1}{2}\}$, $V = \{x : h(x) < \frac{1}{2}\}$. Then $B \subseteq U$, $A \subseteq V$, $U \cap V$
Definition 2.9. Suppose that $P$ and $Q$ are topologies on a set $X$. We say that $P$ is completely regular with respect to $Q$ in case every $P$-closed subset $F$ of $X$ and each point $x$ in $X \setminus F$, there is a $P$-lower semicontinuous and $Q$-upper semicontinuous function $f$ on $X$ such that $f = 0$ on $F$, $f(x) = 1$, and $0 \leq f \leq 1$. The space $(X, P, Q)$ is pairwise completely regular if $P$ is $P$ is completely regular with respect to $Q$ and $Q$ is completely regular with respect to $P$.

Definition 2.10. A subset $A$ of $X$ is SC-embedded (resp. SC-embedded) in $X$ if every real-valued (resp. bounded real-valued) $P$-lower semicontinuous and $Q$-upper semicontinuous function on $A$ can be extended to a $P$-lower semicontinuous and $Q$-upper semicontinuous function on $X$.

Theorem 2.11. Every $P$-closed and $Q$-closed subset of a pairwise normal space $(X, P, Q)$ is SC*-embedded.

Definition 3.14. Let $(X, P, Q)$ be a bitopological space. If $f$ is a real-valued function on $X$ that is $P$-lower semicontinuous and $Q$-upper semicontinuous, then $\{x \in X \mid f(x) \leq 0\}$ is a $P$-zero-set with respect to $Q$, and $\{x \in X \mid 0 \leq f(x)\}$ is a $Q$-zero-set with respect to $P$.

The terminology will be abbreviated as follows; a $P$-zero-set with respect to $Q$ will be called a $P$-zero-set, and a $Q$-zero-set with respect to $P$ will be called a $Q$-zero-set.

Let $f$ be a $P$-upper semicontinuous and $Q$-lower semicontinuous function on $X$. Then, for every real number $r$, $\{x \in X \mid r \leq f(x)\}$ is a $P$-zero-set and $\{x \in X \mid f(x) \leq r\}$ is a $Q$-zero-set. Also, because $\{x \in X \mid g(x) \leq 0\} = \{x \in X \mid (gV_0)(x) = 0\}$, any $P$-zero-set is of the form $\{x \in X \mid h(x) = 0\}$, where $h$ is $P$-lower semicontinuous and $Q$-upper semicontinuous and $h \geq \phi$, and $U$ is a $P$-open set $V$ is a $Q$-open set.
0. Similarly, any $Q$-zero-set is of the from $\{x \in X | h(x) = 0\}$, where $h$ is $P$-upper semicontinuous and $Q$-lower semicontinuous and $h \geq 0$.

**Theorem 2.12.** The space $(X, P, Q)$ is pairwise completely regular if and only if the $P$-zero-sets from a base for the $P$-closed sets and the $Q$-zero-sets from a base for the $Q$-closed sets.

Before proving Theorem 2.12, we need the following Lemma.

**Lemma 2.13.** For every $P$-zero-set $F$ and each point $x$ in $X \setminus F$, there exists a $P$-lower semicontinuous and $Q$-upper semicontinuous function $f$ on $X$ such that $f(F) = 0$, $f(x) = 1$, and $0 \leq f(x) \leq 1$ ($x \in X$)

**Proof.** Let $F$ be a $P$-zero-set. Then $F = \{x \in X | h(x) = 0\}$ where $h$ is $P$-lower semicontinuous and $Q$-upper semicontinuous, and $h \geq 0$. Let $x_0 \in X \setminus F$, then by definition of $F$, $h(x) > 0$. Put $f(x) = \min \left\{ \frac{h(x)}{h(x_0)}, 1 \right\}$, then $f(F) = 0$, $f(x_0) = 1$, and $0 \leq f \leq 1$. Furthermore, $f$ is $P$-lower semicontinuous and $Q$-upper semicontinuous. Since $h$ is $P$-lower semicontinuous and $Q$-upper semicontinuous, and $h(x_0) > 0$, $\frac{h(x)}{h(x_0)}$ is $P$-lower semicontinuous $Q$-upper semicontinuous. So $f(x) = \min \left\{ \frac{h(x)}{h(x_0)}, 1 \right\}$ is $P$-lower semicontinuous and $Q$-upper semicontinuous.

**Proof of Theorem 2.12.** Let $F$ be a $P$-closed set and $x$ be an arbitrary point in $X \setminus F$. By hypothesis, $F = \bigcap F_L$ where $F_L$ is $P$-zero-set for each $L$. Since $F = \bigcap F_L$ and $x \in F$, there exists a $F$-zero-set $F_{t_0}$ such that $F \subseteq F_{t_0}$ and $x \in F_{t_0}$. Then by Lemma 3.16, there exists a $P$-lower semicontinuous and
Q-upper semicontinuous function $f$ on $X$ such that $f(F_0) = 0$, $f(x) = 1$, and $0 \leq f \leq 1$. Then since $F \subseteq F_0$, $f(F) = 0$. Thus $F$ is $P$-completely regular with respect to $Q$. Similarly $Q$ is completely regular with respect to $P$.

Conversely, let $F$ be a $P$-closed set. Then since $(X, P, Q)$ is pairwise completely regular, for each point $x_0$ in $X-F_0$ there is a $P$-lower semicontinuous and $Q$-upper semicontinuous function $f_{x_0}$ on $X$ such that $f_{x_0}(F)=0$, $f_{x_0}(x)=1$, and $0 \leq f_{x_0} \leq 1$. Put $F_0 = \{ x \in X | f_{x_0}(x) \leq 0 \}$, then $F_0$ is a Q-zero set and $F = \bigcap F_{x_0}$. Thus $P$-zero sets form a base for the $P$-closed sets. Similarly $Q$-zero sets form a base for the $Q$-closed sets.

Theorem 2.14. [4]. If $X$ is pairwise normal and if a subset $A$ of $X$ is a $P$-zero-set and a $Q$-zero-set, then $A$ is $SC$-embedded in $X$.

3. Pairwise paracompact spaces

Definition 3.1. Let $\{A_\alpha | \alpha \in \Lambda \}$ and $\{B_\beta | \beta \in B\}$ be two coverings of a space $X$. $(A_\alpha)$ is said to refine (or be a refinement of) $(B_\beta)$ if for each $A_\alpha$ there is some $B_\beta$ with $A_\alpha \subseteq B_\beta$.

Definition 3.2. A refinement $\{A_\alpha | \alpha \in \Lambda \}$ of $\{B_\beta | \beta \in B\}$ is called precise if $A = B$ and $A_\alpha \subseteq B_\beta$ for each $\alpha$.

Lemma 3.3. If the space $(X, P, Q)$ is pairwise Hausdorff. Then for a fixed point $p$ in $X$, and for each point $q \neq p$ in $X$, there is a $P$-open neighbourhood $U_{(p)}$ such that $q \subseteq Q\text{-}cl U_{(p)}$. Similarly there is a $Q$-open neighbourhood $V_{(p)}$ such that $q \subseteq P\text{-}cl V_{(p)}$.

Proof. Let $q$ be an arbitrary point in $X$ such that $p \neq q$. Since $(X, P, Q)$ is pairwise Hausdorff there exist a $P$-open...
neighbourhood $U_{(q)}$ and a $Q$-open neighbourhood $V(q)$ such that $U_{(q)} \cap V(q) = \emptyset$. Then $U_{(q)} \subseteq V(q)$, $q \in V(q)$, and $V(q)$ is $Q$-closed. Thus $Q-cl U_{(q)} \subseteq V(q)$. So $q \in Q-cl U_{(q)}$.

**Lemma 3.4.** If the $P$-covering $\{A_\alpha | \alpha \in A\}$ of $X$ has a $Q$-neighbourhood-finite refinement $\{B_\beta | \beta \in B\}$, then it also has a precise $Q$-neighbourhood-finite refinement $\{C_\alpha | \alpha \in A\}$. Furthermore, if each $B_\beta$ is a $Q$-open set, then each $C_\alpha$ can be chosen to be an $Q$-open set also.

**Proof.** Define a map $\varphi : B \to A$ by assigning to each $\beta \in B$ some $\alpha \in A$ such that $B_\beta \subseteq A_\alpha$. For each $\alpha \in A$ let $C_\alpha = \bigcup \{B_\beta | \varphi(\beta) = \alpha\}$, some $C_\alpha$ may be empty. Clearly, $C_\alpha \subseteq A_\alpha$ for each $\alpha$, and also $\{C_\alpha | \alpha \in A\}$ is a covering because each $B_\beta$ appears somewhere; $C_\alpha$ is evidently $Q$-open whenever each $B_\beta$ is $Q$-open. If $\{B_\beta\}$ is $Q$-neighbourhood-finite, then each point $x$ in $X$ has a $Q$-neighbourhood $V$ such that $V \cap B_\beta \neq \emptyset$ for at most finitely many indices $\beta$. And the number of $\{C_\alpha\}$ such that $V \cap C_\alpha \neq \emptyset$ is less than $\{B_\beta\}$. Thus $V \cap C_\alpha \neq \emptyset$ for at most finitely many indices $\alpha$, $\{C_\alpha\}$ is therefore $Q$-neighbourhood-finite refinement.

**Lemma 3.5.** Let $\{A_\alpha | \alpha \in A\}$ be a neighbourhood-finite family in $X$. Then:

1. $\{cl A_\alpha | \alpha \in A\}$ is also neighbourhood-finite.
2. For each $B \subseteq A$, $\cup \{A_\beta | \beta \in B\}$ is closed in $X$.

**Definition 3.6.** A pairwise Hausdorff space $(X, P, Q)$ is pairwise paracompact if each $P$-open covering of $X$ has a $Q$-open neighbourhood-finite refinement and each $Q$-open covering of $X$ has a $P$-open neighbourhood-finite refinement.

**Theorem 3.7.** Every pairwise paracompact space is pairwise normal.
Proof. We first show that the pairwise paracompact space is pairwise regular. Let \( A \) be a \( P \)-closed set in \( X \) and \( y \) be a given point in \( X \) such that \( y \notin A \). Since \((X, P, Q)\) is pairwise Hausdorff, we find by Lemma 3.3, that each \( a \in A \) has a \( P \)-open neighbourhood \( U_a \) with \( y \notin Q-cl(U_a) \). Since \( \{ U_a \mid a \in A \} \cup \{ A^c \} \) is a \( P \)-open covering of \( X \), we use pairwise paracompactness and Lemma 3.4, to get a precise \( Q \)-neighbourhood-finite open refinement \( \{ V_a \mid a \in A \} \cup \{ G \} \). Then \( M = \bigcup \{ V_a \mid a \in A \} \) is \( Q \)-open and contains \( A \). Furthermore, because \( \{ V_a \} \) is \( Q \)-neighbourhood-finite, Lemma 3.5, shows that \( Q-clM = \bigcup \{ Q-clV_a \mid a \in A \} \) and since \( y \notin Q-cl(U_a) \subseteq Q-clV_a \) for each \( a \in A \), we find \( y \notin Q-clM \). Since \((X, P, Q)\) is pairwise Hausdorff, and \( y \notin Q-clM \), we find by Lemma 3.3, that each \( b \in Q-clM \) has a \( Q \)-neighbourhood \( W_b \) with \( y \notin P-clW_b \). Since \( \{ W_b \mid b \in Q-clM \} \cup \{ Q-clM^c \} \) is a \( Q \)-open covering of \( X \), we use pairwise paracompactness and Lemma 3.4, to get a precise \( P \)-neighbourhood-finite open refinement \( \{ T_b \mid b \in Q-clM \} \cup \{ H \} \). Then \( N = \bigcup \{ T_b \mid b \in Q-clM \} \) is \( P \)-open contains \( Q-clM \). Furthermore, because \( \{ T_b \} \) is \( P \)-neighbourhood finite Lemma 3.5, shows that \( P-clN = \bigcup \{ P-clT_b \mid b \in Q-clM \} \), and since \( y \notin P-clW_b \) for each \( b \in Q-clM \), we find \( y \notin P-clN \). Then \( y \notin [P-clN]^c \), \( [P-clN]^c \cap M = \phi \), i.e. there exist a \( P \)-open set \([P-clN]^c \), \( Q \)-open set \( M \), \( y \notin [P-clN]^c \), \( A \subseteq M \) such that \([P-clN]^c \cap M = \phi \). So \( P \) is regular with respect to \( Q \), similarly \( Q \) is regular with respect to \( P \). Thus \((X, P, Q)\) is pairwise regular. We now prove that \( X \) is pairwise normal. Let \( A \) be a \( P \)-closed set, \( B \) be a \( Q \)-closed set with \( A \cap B = \phi \). Then for each \( b \in B \), \( b \notin A \). Since \((X, P, Q)\) is pairwise Hausdorff, we proved above that for each \( b \in B \) there is a \( Q \)-open set \( M_b \) such that \( A \subseteq M_b \) and \( b \notin Q-clM_b \).
Then $A \subseteq \bigcap_{b \in B} Mb$, \ \[ \bigcap_{b \in B} Q \cdot cl Mb \cap B = \phi \]. Put $M = \bigcap_{b \in B} Mb$, then

$A \subseteq Q \cdot cl M$ and since $Q \cdot cl M = Q \cdot cl \left( \bigcap_{b \in B} Mb \right) \subseteq \bigcap_{b \in B} Q \cdot cl Mb$,

$Q \cdot cl M \cap B = \phi$. Since $(X, P, Q)$ is pairwise regular, for each $a \in Q \cdot cl M$, there is a $Q$-open set $U_a$, and $P$-open set $V_a$ such that $a \in U_a$, $B \subset V_a$, and $U_a \cap V_a = \phi$. Therefore for each $a \in Q \cdot cl M$, there is a $Q$-open set $U_a$ such that $a \in U_a$ and $P \cdot cl U_a \cap B = \phi$. Since \{ \{U_a\} | a \in Q \cdot cl M\} \cup \{ (Q \cdot cl M) \} is a $Q$-open covering of $X$, we use pairwise paracompactness and Lemma 3.4 to get a precise $P$-neighbourhood-finite open refinement \{ \{V_a\} | a \in Q \cdot cl M\} \cup G. Put $N = \bigcup \{V_a\} | a \in Q \cdot cl M\}$, then $N$ is $P$-open and $Q \cdot cl M \subseteq N$. Furthermore, because $\{V_a\}$ is $P$-neighbourhood-finite, Lemma 3.5 shows that $P \cdot cl N = \bigcup \{P \cdot cl V_a\} | a \in Q \cdot cl M\}$. Since $P \cdot cl U_a \supseteq P \cdot cl V_a$ for each $a \in Q \cdot cl M$, and $B \cap P \cdot cl U_a = \phi$, we find $B \cap P \cdot cl N = \phi$. So we find a $Q$-closed set $Q \cdot cl M$, $P$-closed set $N$, such that $A \subseteq Q \cdot cl M$, $B \subset N$, and $Q \cdot cl M \cap N = \phi$. Since $Q \cdot cl M$ is $Q$-closed, $N$ is $P$-closed, and $Q \cdot cl M \cap N = \phi$, as we proved above there is a set $H$ such that $N \subseteq H \subseteq Q \cdot cl H$ and $Q \cdot cl H \cap Q \cdot cl M = \phi$. Then $A \subseteq Q \cdot cl M \subseteq [Q \cdot cl H]^c$, $B \subseteq (P \cdot cl N)^c$ since $[P \cdot cl N]^c \subseteq N \subseteq H \subseteq Q \cdot cl H$, $[Q \cdot cl H]^c \cap [P \cdot cl N]^c = \phi$. In other word, there exist $P$-open set $[P \cdot cl N]^c$ and $Q$-open set $[Q \cdot cl H]^c$ such that $A \subseteq [Q \cdot cl H]^c$, $B \subseteq [P \cdot cl N]^c$ and $[Q \cdot cl H]^c \cap [P \cdot cl N]^c = \phi$.

**Example 3.8.** In a real line $R$, let $P$ be the usual topology and $Q$ be the topology generated by the open-closed interval $(a, b)$ $(a, b \in R, a < b)$. Then $(R, P, Q)$ is a pairwise paracom pact space.
References


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