ASYMPTOTIC BEHAVIOR OF GENERALIZED SOLUTIONS IN BANACH SPACES

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1. Introduction

Let $X$ be a real Banach space with norm $| \cdot |$ and let $I$ denote the identity operator. Then an operator $A \subset X \times X$ with domain $D(A)$ and range $R(A)$ is said to be accretive if $|x_1 - x_2| \leq |x_1 - x_2 + r(y_1 - y_2)|$ for all $y_i \in Ax_i$, $i=1, 2$, and $r>0$. An accretive operator $A \subset X \times X$ is $m$-accretive if $R(I+rA)=X$ for all $r>0$.

We consider the initial value problem

$$\frac{du}{dt}(t) + Au(t) + G(u)(t) \ni f(t), \quad 0 < t < \infty$$
$$u(0) = x, \tag{1.1}$$

where $A$ is an $m$-accretive operator in a real Banach space $X$, $f \in L^1(0, T; X)$, $x \in D(A)$, and $G$ is given mapping

$$G : C([0, \infty); D(A)) \longrightarrow L^1(0, T; X), \text{ for } 0 < T < \infty. \tag{1.2}$$

By a recent result of Crandall and Nohel [3, Theorem 1], the problem (1.1) has a unique generalized solution $u \in C([0, \infty); X)$, provided that $G$ satisfies a Lipschitz type condition.

In this case $X$ is a Hilbert space, the asymptotic behavior of such a solution was studied by Aizicovici [1], Morosanu [9], Pazy [10]. The present paper is concerned with asymptotic behavior, as $t \longrightarrow \infty$, of generalized solution of (1.1) in a Banach space.

First, we prove that if $u(t)$ is a generalized solution of (1.1), then the closed convex set $\bigcap_{s \geq 0} \text{co} \{u(t) : t \geq s\} \cap A^{-1}0$ consists of at most one point, where $\text{co} \{u(t) : t \geq s\}$ is the closed convex hull of $\{u(t) : t \geq s\}$.

This result is applied to study the problem of weak convergence of the net $\{u(t) : t \geq 0\}$. 

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2. Preliminaries

Let $X$ be a real Banach space and let $X^*$ its dual. The value of $x^* \in X^*$ at $x \in X$ will be denote by $(x, x^*)$. With each $x \in X$, we associated the set

$$J(x) = \{x^* \in X^* : (x, x^*) = |x|^2 = |x^*|^2\}.$$

Using the Hahn–Banach theorem, it is immediately clear that $J(x) \neq \emptyset$ for any $x \in X$. Then multi-valued operator $J : X \longrightarrow X^*$ is called the duality mapping of $X$.

Let $U = \{x \in X : |x| = 1\}$ stand for the unit sphere of $X$. Then a Banach space $X$ is said to be smooth if

$$\lim_{t \to 0} \frac{|x + ty| - |x|}{t}$$

exists for each $x, y$ in $U$. When this is the case, the norm of $X$ is said to be Gâteaux differentiable. It is said to be Fréchet differentiable if for each $x$ in $U$, this limit is attained uniformly for $y$ in $U$. The space $X$ is said to be uniformly Gâteaux differentiable norm if for each $y$ in $U$, the limit is attained uniformly for $x$ in $U$. It is also known that if $X$ has a Fréchet differentiable norm, then $J$ is norm to norm continuous; [2] or [5] for more details.

Let $D$ be a subset of $X$. Then we denote by $\overline{D}$ the closure of $D$ and by $coD$ the closed convex hull of $D$. When $\{x_a\}$ is a net in $X$, then $x_a \longrightarrow x$ (resp. $x_a \rightharpoonup x$) will denote norm (resp. weak) convergence of the net $\{x_a\}$ to $x$.

For $x$ and $y$ in $X$, let $\langle y, x \rangle_s = \max \{(y, j) : j \in J(x)\}$. An operator $A \subset X \times X$ is accretive if and only if $\langle y_1 - y_2, x_1 - x_2 \rangle_s \geq 0$ for all $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. Let $A \subset X \times X$ be $m$-accretive. Then we put

$$A^{-1}0 = \{x : x \in D(A), \ Ax \equiv 0\}$$

For a function $u : [0, \infty) \longrightarrow X$, we denote by

$$\omega(u) = \{y \in X : u(t_n) \longrightarrow y \text{ for some } \{t_n\} \text{ with } t_n \longrightarrow \infty\}.$$

Consider now the initial value problem (1.1), where $G$ satisfies (1.2), $x$ in $\overline{D(A)}$, and $f \in L^1([0, \infty); X)$.

Recall that if $\mathcal{I}$ is an interval, then $u \in W^{1, 1}(\mathcal{I}; X)$ means that there is a function $v : \mathcal{I} \longrightarrow X$ which is strongly integrable on $\mathcal{I}$. (i.e., $v \in L^1(\mathcal{I}; X)$ such that $u(t) - u(s) = \int_s^t v(\tau)\, d\tau$ $(t, s \in \mathcal{I})$ then $u'(t) = v(t)$
Asymptotic behavior of generalized solutions in Banach spaces

DEFINITION 2.1. A strong solution of (1.1) on $[0, \infty)$ is a function $u \in W^{1,1}_{\text{loc}}([0, \infty); X) \cap C([0, \infty); \overline{D(A)})$, satisfying $u(0) = x$ and $u'(t) + Au(t) + G(u)(t) \ni f(t)$, a.e. on $(0, \infty)$.

DEFINITION 2.2. A function $u \in C([0, \infty); \overline{D(A)})$ is said to be a generalized solution of equation of (1,1) if there are sequence $x_n \in \overline{D(A)}$, $f_n \in L^1_{\text{loc}}([0, \infty); X)$ and $u_n \in C([0, \infty); X)$ such that $u_n$ is a strong solution of

$$
\frac{du_n}{dt} + Au_n + G(u_n) \ni f_n,
$$

$$
u_n(0) = x_n,
$$

$x_n \to x$, $f_n \to f$ on $L^1(0, T; X)$ and $u_n \to u$ in $C([0, T]; X)$, for each $0 < T < \infty$.

The following result is direct consequence of [3, Theorem 1 and 2]

PROPOSITION 2.1. Let $G$ satisfy (1,2) and

$$
|G(u) - G(v)|_{L^1(0, T; X)} \leq \int_0^T \gamma(s) |u(s) - v(s)|_{L^\infty(0, t; X)} ds,
$$

$0 \leq s \leq t < \infty$ (2.1)

for some $\gamma \in L^1_{\text{loc}}([0, \infty); \mathbb{R})$ and every $u, v \in C([0, \infty); \overline{D(A)})$.

For each $T \in (0, \infty)$, there is $\alpha_T : [0, \infty) \to [0, \infty)$ such that

$$
\text{Var}(G(u) : [0, t]) \leq \alpha_T(R)(1 + \text{Var}(u : [0, t])),
$$

$0 \leq t \leq T$ (2.2)

and

$$
|G(u)(0^+) - G(u)(0^-)| \leq \alpha_T(R),
$$

whenever $u \in C([0, T]; \overline{D(A)})$ is of bounded variation and $|u|_{L^\infty(0, T; X)} \leq R$. Then, for each $x \in \overline{D(A)}$ and $f \in BV_{\text{loc}}([0, \infty); X)$, problem (1.1) has a unique strong solution. If $x \in \overline{D(A)}$ and $f \in L^1_{\text{loc}}([0, \infty); X)$, then (1.1) has a unique generalized solution.

It is assume throughout

$$
\int_s^t <G(v)(\tau) - G(w)(\tau), v(\tau) - w(\tau)> d\tau \geq 0,
$$

$0 \leq s \leq t < \infty$, $v, w \in C([0, \infty); \overline{D(A)})$ (2.3)

$$
G(v)(t) \in L^1(0, \infty; X), \text{ for each constant function } v(t) \equiv \nu \in D(A)
$$

(2.4)
\[ f \in L^1(0, \infty; X) \quad (2.5) \]
\[ x \in D(A). \quad (2.6) \]

The following lemma is a consequence of [8, Remark 3.2].

**Lemma 2.1.** Let \( x, y \in D(A), \ f, g \in L^1([0, \infty); X), \) and let \( u, v \) be the corresponding generalized solution of (1.1). If (1.2), (2.1), (2.2), and (2.3) are satisfied, then we have

\[ |u(t) - v(t)| \leq |u(s) - v(s)| + \int_s^t |f(\tau) - g(\tau)| d\tau \quad (2.7) \]

whenever \( 0 \leq s \leq t < \infty. \)

**Lemma 2.2** [1, Lemma 3.1]. Suppose that (1.2), (2.1), (2.2), (2.3), (2.4), and (2.6) hold. Let \( u \) be the generalized solution of (1.1) corresponding to \( f \in L^1([0, \infty); X). \) Then

\[ \frac{1}{2} |u(t) - y|^2 - \frac{1}{2} |u(s) - y|^2 \leq \int_s^t (f(\tau) - G(y)(\tau) - z, u(\tau) - y) d\tau \quad (2.8) \]

\[ |u(t) - y| \leq |u(s) - y| + \int_s^t |f(\tau) - G(y)(\tau) - z| d\tau \quad (2.9) \]

whenever \( 0 \leq s \leq t < \infty, \) and \((y, z) \in A.\)

If \( A^{-1}0 \neq \phi, \) we can take \( y \in A^{-1}0 \) in (2.9) to obtain

\[ |u(t) - y| - |u(s) - y| \leq \int_s^t |f(\tau) - G(y)(\tau)| d\tau, \quad t \geq s \geq 0. \quad (2.10) \]

Consequently, \( u(t) \) is bounded on \([0, \infty)\) and the function \( t \mapsto |u(t) - y| - \int_0^t |f(\tau) - G(y)(\tau)| d\tau \) is nonincreasing on \([0, \infty).\) Then, since \( G(v)(t) \in L^1(0, \infty; X), \) for each constant function \( v(t) \equiv v \in D(A) \) and \( f \in L^1(0, \infty; X), \) we deduce that

\[ \lim_{t \to \phi} |u(t) - y| = \rho(x) \]

exist for each \( y \in A^{-1}0. \)

Finally, let \( \{S(t) : t \geq 0\} \) be a nonexpansive semigroup generated by \(-A\) and let \( u(t) \) be the generalized solution of (1.1). Then we obtain that

\[ |S(t)u(s) - u(t+s)| \leq \int_s^{s+t} |f(\tau) - G(y)(\tau)| d\tau \quad (2.11) \]

whenever \( 0 \leq s, t < \infty. \)
From (2.11), we have
\[ \lim_{t \to 0} \sup_{t > \delta} |S(t)u(s) - u(t+s)| = 0. \] (2.12)

3. Asymptotic behavior

Unless other specified, \( \{ S(t) : t \geq 0 \} \) denote a nonexpansive semigroup generated by \(-A\).

**Lemma 3.1** Let \( X \) be a uniformly convex Banach space and \( A \) be an \( m \)-accretive operator in \( X \times X \). Let (1.2), (2.1) \( \sim \) (2.6) be satisfied. Let \( u : [0, \infty) \to X \) be a generalized solution of (1.1) and \( A^{-1}0 \neq \phi \). Let \( y \in A^{-1}0, 0 < \alpha \leq \beta < 1 \) and \( r = \lim_{t \to \infty} |u(t) - y| \). Then, for any \( \varepsilon > 0 \), there is \( t_0 \geq 0 \) such that
\[ |S(t) (\lambda u(s) + (1 - \lambda) y) - (\lambda S(t) u(s) + (1 - \lambda) y) | < \varepsilon \]
for all \( s \geq t_0, t \geq 0 \) and \( \lambda \in \mathbb{R} \) with \( \alpha \leq \lambda \leq \beta \).

**Proof.** Let \( r > 0 \). Then we can choose \( d > 0 \) so small that
\[ (r+d) (1-c\delta (\frac{\varepsilon}{r+d})) = r_0 < r, \]
where \( \delta \) is the modulus of convexity of norm and \( c = \min \{ 2\lambda (1-\lambda) : \alpha \leq \lambda \leq \beta \} \). Let \( \varepsilon_1 > 0 \) with \( r_0 + 2\varepsilon_1 < r \). Then we can choose \( t_0 \geq 0 \) such that \( |u(s) - y| \geq r - \varepsilon_1 \) for all \( s \geq t_0 \) and \( |S(t) u(s) - u(t+s) | < \varepsilon_1 \) for all \( t \geq 0 \) and \( s \geq t_0 \). Suppose that
\[ |S(t) (\lambda u(s) + (1 - \lambda) y) - (\lambda S(t) u(s) + (1 - \lambda) y) | \geq \varepsilon \]
for some \( s \geq t_0, t \geq 0 \) and \( \lambda \in \mathbb{R} \) with \( \alpha \leq \lambda \leq \beta \).

Put \( u = (1-\lambda) (S(t)z - y) \) and \( v = \lambda (S(t) u(s) - S(t)z) \), where \( z = \lambda u(s) + (1 - \lambda) y \). Then \( |u| \leq \lambda (1-\lambda) |u(s) - y| \) and \( |v| \leq \lambda |u(s) - z| = \lambda (1 - \lambda) |y - u(s)| \). We also have that \( |u - v| = |S(t) z - (\lambda S(t) u(s) + (1 - \lambda) y) | \geq \varepsilon \) and \( \lambda u + (1 - \lambda) v = \lambda (1-\lambda) (S(t) u(s) - y) \). So by using the lemma in [6],
\[ |\lambda (1-\lambda) (S(t) u(s) - y) | = |\lambda u(s) + (1-\lambda) v(s) | \]
\[ \leq \lambda (1-\lambda) |u(s) - y| (1 - 2\lambda (1-\lambda) \delta (\frac{\varepsilon}{|u(s) - y|})) \]
\[ \leq \lambda (1-\lambda) (r + d) (1-c\delta (\frac{\varepsilon}{r+d})) \]
\[ = \lambda (1-\lambda) r_0 \]
and hence \( |S(t) u(s) - y| \leq r_0 \). This implies
\[ |u(t+s) - y| \leq |S(t)u(s) - y| + \varepsilon \leq r_0 + \varepsilon \leq r - \varepsilon. \]

On the other hand, since \(|u(t+s) - y| \geq r - \varepsilon_1\), this is contradiction. In the case when \(r = 0\), for any \(t, s \geq 0\), \(y \in A^{-1}0\) and \(\lambda \in \mathbb{R}\) with \(0 \leq \lambda \leq 1\),
\[
\begin{align*}
|S(t)(\lambda u(s) + (1-\lambda)y - (\lambda S(t)u(s) + (1-\lambda)y) | \\
\leq \lambda |S(t)(\lambda u(s) + (1-\lambda)y) - S(t)u(s)| \\
+ (1-\lambda) |S(t)(\lambda u(s) + (1-\lambda)y - y)| \\
\leq \lambda |\lambda u(s) + (1-\lambda)y - u(s)| + (1-\lambda) |\lambda u(s) + (1-\lambda)y - y| \\
\leq 2\lambda(1-\lambda)|y - u(s)|
\end{align*}
\]

So, we obtain the desired result.

Let \(x\) and \(y\) be element of a Banach space \(X\). Then we denote by \([x, y]\) the set \(\{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}\).

**Lemma 3.2.** Let \(C\) be a closed convex subset of a Banach space \(X\) with a Fréchet differentiable norm and \(\{u_t\}\) a bounded net in \(C\). Let \(z \in \overline{\text{co}}\{u_t : t \geq s\}, y \in C\) and \(\{y_t\}\) a net of element in \(C\) with \(y_t \in [y, u_t]\) and \(|y_t - z| = \min \{|u - z| : u \in [y, u_t]\}\). If \(y_t \longrightarrow y\), then \(y = z\).

**Proof.** Since \(J\) is single-valued, it follows from Theorem 2.5. in [3] that \((u - y_t, J(y_t - z)) \geq 0\) for all \(u \in [y, u_t]\). Putting \(u = u_t\), we have
\[
(u_t - y_t, J(y_t - z)) \geq 0. \tag{3.1}
\]
Since \(y_t \longrightarrow y\) and \(\{u_t\}\) is bounded, there exist \(K > 0\) and \(t_0\) such that
\[
|u_{t_0} - y| \leq K \quad \text{and} \quad |y_t - z| \leq K \quad \text{for all} \quad t \geq t_0.
\]
Let \(\varepsilon > 0\) and choose \(\delta > 0\) so small that \(2\delta K < \varepsilon\). Since the norm of \(X\) is Fréchet differentiable, we can choose \(t_1 \geq t_0\) such that \(|y_t - y| \leq \delta\) and \(|J(y_t - z) - J(y - z)| \leq \delta\) for all \(t \geq t_1\). Since for \(t \geq t_1\)
\[
|\left( u_t - y_t, J(y_t - z) \right) - \left( u_t - y, J(y - z) \right) |
= |(u_t - y_t, J(y_t - z)) - (u_t - y, J(y_t - z))|
+ \left| J((y_t - z) - J(y - z)) \right| |
\leq |y_t - z| |y_t - y| + |u_t - y| |J(y_t - z) - J(y - z)|
\leq 2\delta K < \varepsilon
\]
by using (3.1), we have
\[
(u_t - y, J(y - z)) \geq (u_t - y_t, J(y_t - z)) - \varepsilon
\geq 0 - \varepsilon
= -\varepsilon.
\]
Since \(z \in \overline{\text{co}}\{u_t : t \geq s\}\), we have \((z - y, J(y - z)) \geq -\varepsilon\). This implies \(-|z - y|^2 \geq 0\) and hence \(z = y\).
Lemma 3.3. Let $X$ be a uniformly convex Banach space with a Fréchet differentiable norm, let $u(t)$ be the unique generalized solution of (1.1), and $A^{-1}0 \neq \phi$. Let $z \in \bigcap_{t \geq 0} \{ u(t) : t \geq s \} \cap A^{-1}0$ and $y \in A^{-1}0$. Then, for any positive number $\varepsilon$, there is $t_0 \geq 0$ such that

$$
(u(t) - y, J(y - z)) \leq \varepsilon |y - z| \text{ for every } t \geq t_0.
$$

Proof. Let $z \in \bigcap_{t \geq 0} \{ u(t) : t \geq s \} \cap A^{-1}0$, $y \in A^{-1}0$ and $\varepsilon > 0$. If $y = z$, lemma 3.3 is obvious. So, let $y \neq z$. For any $t \geq 0$, define a unique element $y_t$ such that $y_t \in [y, u(t)]$ and $|y_t - z| = \min \{|u - z| : u \in [y, u(t)]\}$. Then, since $y \neq z$, by lemma 3.2, we have $y_t \rightarrow y$. So, we obtain $c > 0$ and $\{ t_n \}$ with $t_n \rightarrow \infty$ and $|y_{t_n} - y| \geq c$. Setting

$$
y_t = a_{t_n}u(t_n) + (1 - a_{t_n}) y, \quad 0 \leq a_{t_n} \leq 1,
$$

we also obtain $c_0 > 0$ so small that $a_{t_n} \geq c_0$ for every $n$. In fact since

$$
c \leq |y_{t_n} - y| = a_{t_n} |u(t_n) - y| \leq a_{t_n} (|x - y| + \int_{0}^{t_n} |f(\tau)| \, d\tau)
$$

$$
\leq a_{t_n} (|x - y| + \int_{0}^{\infty} |f(\tau)| \, d\tau),
$$

we may put $c_0 = c/\left( |x - y| + \int_{0}^{\infty} |f(\tau)| \, d\tau \right)$. Since the limit of $|u(t) - y|$ exists, putting $k = \lim_{t \rightarrow \infty} |u(t) - y|$. We have $k > 0$. If not, we have $u(t) \rightarrow y$ and hence $y_t \rightarrow y$, which contradicts $y_t \rightarrow y$. Let $r$ be a positive number such that $\varepsilon > r$ and $k > 2r$. And choose $a > 0$ so small that

$$
(R + a) \left( 1 - \delta \left( \frac{c_0 r}{R + a} \right) \right) < R,
$$

where $\delta$ is the modulus of convexity of the norm and $R = |z - y|$. Then, by lemma 3.1., there exists $s_1 \geq 0$ such that

$$
|S(s)(c_0 u(t) + (1 - c_0) y) - (c_0 S(s) u(t) + (1 - c_0) y)| < a \quad (3.2)
$$

for all $s \geq 0$ and $t \geq s_1$. We also choose $s_2 \geq 0$ such that $|u(t) - y| \geq 2r$ for every $t \geq s_2$ and $|u(t + s) - S(t) u(s)| < r$ for every $t \geq 0$ and $s \geq s_2$. Fix $t_n$ with $t_n \geq \max(s_1, s_2)$. Then since $a_{t_n} \geq c_0$, we have

$$
c_0 u(t_n) + (1 - c_0) y \in [y, a_{t_n} u(t_n) + (1 - a_{t_n}) y] = [y, y_{t_n}].
$$

Hence

$$
|c_0 u(t_n) + (1 - c_0) y - z| \leq \max \{|z - y|, |z - y_{t_n}|\} = |z - y| = R.
$$

By using (3.2), we obtain
\[ |c_0 S(s)u(t_n) + (1-c_0)y-z| \leq |S(s)(c_0u(t_n) + (1-c_0)y) - z| + a \]
\[ \leq |c_0u(t_n) + (1-c_0)y-z| + a \]
\[ \leq R + a \text{ for every } s \geq 0. \]

On the other hand, since \( y-z = R < R+a \) and
\[ |c_0 S(s)u(t_n) + (1-c_0)y-y| = c_0 |S(s)u(t_n) - y| \]
\[ \geq c_0 (|u(s+t_n) - y| - r) \]
\[ \geq c_0 r \]
for all \( s \geq 0 \), we have, by uniform convexity,
\[ \frac{1}{2} ((c_0 S(s)u(t_n) + (1-c_0)y-z) + (y-z)) \leq (R+a) (1-\delta \left( \frac{c_0 r}{R+a} \right)) \]
\[ < R \]
and hence
\[ |\frac{c_0}{2} S(s)u(t_n) + \left(1-\frac{c_0}{2}\right)y-z| < R \text{ for all } s \geq 0. \]

This implies that if \( u_t = \frac{c_0}{2} S(s)u(t_n) + \left(1-\frac{c_0}{2}\right)y \), then
\[ |u_t + \alpha (y-u_t) - z| > |y-z| \text{ for all } \alpha \geq 1. \]

By Theorem 2.5., in [4], we have \( (u_t + \alpha (y-u_t) - y, J(y-z)) \geq 0 \) and hence \( (u_t - y, J(y-z)) \leq 0 \). Then \( (S(s)u(t_n) - y, J(y-z)) \leq 0 \). Therefore
\[ (u(s+t_n) - y, J(y-z)) \leq |u(s+t_n) - S(s)u(t_n)| + |y-z| \]
\[ + (S(s)u(t_n) - y, J(y-z)) \]
\[ < \varepsilon |y-z| \text{ for all } s \geq 0. \]

This completes the proof.

**Theorem 3.1.** Let \( X \) be a uniformly convex Banach space with a Fréchet differentiable norm, let \( u(t) \) be the unique generalized solution of (1.1), and \( A^{-1}0 \neq \phi \). Then the set \( \bigcap_{s \geq 0} \overline{co} \{u(t) : t \geq s\} \cap A^{-1}0 \) consists of at most one point.

**Proof.** Let \( y, z \in \bigcap_{s \geq 0} \overline{co} \{u(t) : t \geq s\} \cap A^{-1}0 \). Then, since \( \frac{y+z}{2} \in A^{-1}0 \), it follows from lemma 3.3, that for any \( \varepsilon > 0 \), there is \( t_0 \geq 0 \) such that
\[ (u(t+t_0) - \frac{y+z}{2}, J\left(\frac{y+z}{2} - z\right)) \leq \varepsilon |\frac{y+z}{2} - z| \text{ for every } t \geq 0. \]

Since \( y \in \overline{co} \{u(t+t_0) : t \geq 0\} \), we have \( \left(y - \frac{y+z}{2}, J\left(\frac{y+z}{2} - z\right)\right) \leq \varepsilon |\frac{y+z}{2} - z| \)
and hence \( (y-z, J(y-z)) \leq 2\varepsilon |y-z| \). Then we have \( |y-z| \leq 2\varepsilon. \)
Since $\varepsilon$ is arbitrary, we have $y = \varepsilon$.

**Theorem 3.2.** Let $X$ be a uniformly convex Banach space with a Fréchet differentiable norm, let $u(t)$ be the unique generalized solution of (1.1), and $A^{-1}0 \neq \phi$. If $\omega(u) \subset A^{-1}0$, then the set $\{u(t) : t \geq 0\}$ converges weakly to some $z \in A^{-1}0$.

**Proof.** Since $A^{-1}0 \neq \phi$, $\{u(t) : t \geq 0\}$ is bounded. So, any sequence $\{u(t_n)\}$ with $t_n \to \infty$ must contain a subsequence $\{u(t_{n_i})\}$ which converges weakly to some $z \in \overline{D(A)}$. Since $\omega(u) \subset A^{-1}0$ and $z \in \bigcap_{t \geq 0} \text{co} \{u(t) : t \geq s\}$, we obtain $z \in \bigcap_{t \geq 0} \text{co} \{u(t) : t \geq s\} \cap A^{-1}0$. Therefore, it follows from Theorem 3.1., that $\{u(t) : t \geq 0\}$ converges weakly to $z \in A^{-1}0$.

**Theorem 3.3.** Let $X$ be a uniformly convex Banach space with a Fréchet differentiable norm, let $u(t)$ be the unique generalized solution of (1.1), and $A^{-1}0 \neq \phi$. If $\lim_{t \to \infty} |u(t+h) - u(t)| = 0$ for all $h \geq 0$, then the net $\{u(t) : t \geq 0\}$ converges weakly to some $y \in A^{-1}0$.

**Proof.** By Theorem 3.2., it suffices to show that $\omega(u) \subset A^{-1}0$. Let $\{u(t_n)\}$ be a sequence converges weakly to some $y \in \overline{D(A)}$, as $t_n \to \infty$. Let $\varepsilon > 0$. Then $|u(t+t_n) - S(t)u(t_n)| < \varepsilon/2$ and $|u(t+t_n) - u(t_n)| < \varepsilon/2$ for all large enough $n$ and $t \geq 0$. Since $|S(t)u(t_n) - u(t_n)| \leq |S(t)u(t_n) - u(t+t_n)| + |u(t+t_n) - u(t_n)|,$ we obtain $|S(t)u(t_n) - u(t_n)| < \varepsilon$ for all large enough $n$ and $t \geq 0$. Since $(I-S(t))$ is demiclosed [5], we have $S(t)y = y$. Thus $y \in A^{-1}0$.

**References**


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