A NOTE ON HOLOMORPHIC VECTOR BUNDLES OVER COMPLEX TORI

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1. Let \( L \) be a lattice in \( \mathbb{C}^n \). A holomorphic automorphic factor of rank \( r \) for the lattice \( L \) is a holomorphic mapping

\[
J : L \times \mathbb{C}^n \rightarrow GL(r; \mathbb{C})
\]

such that

(1) \( J(\alpha, z) \ (\alpha \in L, \ z \in \mathbb{C}^n) \) is holomorphic in \( z \),

(2) \( J(\alpha + \beta, z) = J(\alpha, z + \beta)J(\beta, z) \) for all \( \alpha, \beta \in L \) and \( z \in \mathbb{C}^n \).

We have the following free action of \( L \) on \( \mathbb{C}^n \times \mathbb{C}^r \) defined by

\[
(z, \xi)\alpha = (z + \alpha, J(\alpha, z)\xi), \alpha \in L, z \in \mathbb{C}^n, \xi \in \mathbb{C}^r.
\]

The quotient of \( \mathbb{C}^n \times \mathbb{C}^r \) by this group action of \( L \) is a holomorphic vector bundle \( E_J \) over the complex torus \( M = \mathbb{C}^n/L \). Holomorphic vector bundles over the complex torus \( M = \mathbb{C}^n/L \) are always obtained in this way.

In this short paper, we characterize projectively flat vector bundles over a complex torus which is simple.

2. A holomorphic vector bundle over a complex manifold is said to be simple if its endomorphisms are all scalars. It is easy to show that the vector bundle \( E_J \) over the complex torus \( M = \mathbb{C}^n/L \) defined by an automorphic factor \( J \) is simple if and only if scalars are holomorphic maps \( B : \mathbb{C}^n \rightarrow GL(r; \mathbb{C}) \) such that

\[
B(z + \alpha) J(\alpha, z) = J(\alpha, z)B(z) \text{ for all } \alpha \in L.
\]

In his paper [1], Morikawa shows the following theorem.

**Theorem.** Let \( J \) be a holomorphic automorphic factor of rank \( r \) for the lattice \( L \) in \( \mathbb{C}^n \) such that

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(1) the associated vector bundle $E_J$ is simple,
(2) $J(\alpha, z + \beta)J(\alpha, z)^{-1}$ $(\alpha, \beta \in L)$ are constants.

Then there exists an isogeny $\psi : N \to \mathbb{C}^n / L$ of degree $r$ and a line bundle $F$ over the complex torus $N$ such that $E_J$ is the direct image of $F$ under the isogeny $\psi$.

Remark. We can also show that $T_g^*F \cong F$ for all $g \in \ker \psi$, where $T_g$ is the translation of the complex torus $N$ by $g \in N$.

Let $J$ be an automorphic factor of rank $r$ for the lattice $L$ in $\mathbb{C}^n$ satisfying the hypotheses in the above theorem. Then, by Oda [2], the homomorphism $H^j(M, \mathcal{O}) \to \text{End}(E_J)$ induced by $\mathcal{O} \to \text{End}(E_J)$ is an isomorphism for each $j$, where $M = \mathbb{C}^n / L$. Therefore we obtain, for each $j$,

$$\dim \mathbb{C} H^j(M, \text{End}(E_J)) = \dim \mathbb{C} H^j(M, \mathcal{O}) = \binom{n}{j}.$$

3. Let $E$ a holomorphic vector bundle of rank $r$ over a complex torus $M = \mathbb{C}^n / L$ with the corresponding automorphic factor $J : L \times \mathbb{C}^n \to GL (r; \mathbb{C})$. Let

$$\mathbb{p} : GL(r; \mathbb{C}) \to PGL(r; \mathbb{C})$$

be the natural projection to the projective general linear group and define

$$\hat{J} = \mathbb{p} \circ J : L \times \mathbb{C}^n \to PGL(r; \mathbb{C}).$$

Then $\hat{J}$ is an automorphic factor for the projective bundle $P(E)$ over $M$.

We assume that $E$ is projectively flat. Then $P(E)$ is defined by a representation of $L$ into $PGL(r; \mathbb{C})$. Replacing $J$ by an equivalent automorphic factor, we may assume that $\hat{J}$ is a representation of $L$ into $PGL(r; \mathbb{C})$. Since $\hat{J}$ is independent of $z$, we can write

$$J(\alpha, z) = f(\alpha, z)J(\alpha, 0), \quad \alpha \in L, \quad z \in \mathbb{C}^n,$$

where $f : L \times \mathbb{C}^n \to \mathbb{C}^*$ is a scalar function. Since $\det J$ is an automorphic factor for the line bundle $\det E$, we may write

$$\det J(\alpha, z) = \chi(\alpha) \exp \left\{ H(z, \alpha) + \frac{1}{2} H(\alpha, \alpha) \right\}, \quad \alpha \in L, \quad z \in \mathbb{C}^n,$$

where $\chi : L \to \mathbb{C}^*$ is a semi-character of $L$, and $H$ is an Hermitian form on $\mathbb{C}^n$. Since $f(\alpha, 0) = f(0, z) = 1$,

$$f(\alpha, z) = \exp \left\{ \frac{1}{r} H(z, \alpha) \right\}, \quad \alpha \in L, \quad z \in \mathbb{C}^n.$$
Now we assume that \( E \) admits a projectively flat Hermitian structure \( h \). Then we may assume that \( \hat{J} \) is a representation of \( L \) into \( PU(r) \). Let \( \hat{h} \) be the induced Hermitian structure in \( \hat{E} = \pi^* E = C^n \times C^r \), where \( \pi : C^n \to C^n/L \) is the natural projection. Then
\[
(A) \quad \hat{h}(z) = \hat{J}(\alpha, z) \hat{h}(z+\alpha) J(\alpha, z), \quad \alpha \in L, \ z \in C^n.
\]

The curvature form \( \Omega \) of \( E \) is of the form
\[
\Omega = \hat{\delta} I_r, \quad \hat{\delta} \text{ is a 2-form}.
\]

Since its trace is the curvature of \( \det E \), we have
\[
\hat{\delta} = \frac{1}{r} \sum_{j,k} H_{j\bar{k}} \ dz^j \wedge d\bar{z}^k,
\]
with constant coefficient \( H_{j\bar{k}} \). Thus
\[
\hat{\omega}(z) = \hat{h}(z)^{-1} \partial \hat{h}(z)
\]
\[
= -\frac{1}{r} H(dz, z) I_r + \mathcal{E}(z),
\]
where \( H(dz, z) = \sum H_{j\bar{k}} \bar{z}^k dz^j \) and \( \mathcal{E} \) is a holomorphic 1-form with values in the Lie algebra of \( CU(r) = \{ cU; c \in C^*, \ U \subset U(r) \} \). Since \( \mathcal{E} + i\overline{\mathcal{E}} = \phi I_r \) (\( \phi \) is a 1-form) and \( \mathcal{E} \) is holomorphic, it follows that
\[
\mathcal{E} = \theta I_r, \quad \theta \text{ is a holomorphic 1-form}.
\]

By a simple calculation, we obtain
\[
\mathcal{E}(z+\alpha) = \mathcal{E}(z), \quad \alpha \in L.
\]
That is, \( \theta \) is a holomorphic 1-form on \( M = C^n/L \). Hence
\[
\theta = \sum_{j=1}^n C_j dz_j, \quad C_j \text{'s are constants}.
\]

If we solve the differential equation
\[
\hat{h}^{-1} \partial \hat{h} = \hat{\omega}(z) = -\frac{1}{r} \sum H_{j\bar{k}} \bar{z}^k dz^j + \sum_{j=1}^n C_j dz_j,
\]
we obtain
\[
\hat{h}(z) = \hat{h}(0) \ \exp \left\{ -\frac{1}{r} H(z, z) + C(z) + \overline{C(z)} \right\},
\]
where \( C(z) = \sum_{j=1}^n C_j z^j \).

Using the isomorphism of the bundle \( \hat{E} \) defined by
\[
(z, \xi) \in \hat{E} \longrightarrow (z, \ \exp \{ C(z) \} \xi) \in \hat{E},
\]
we may assume that \( C(z) = 0 \). By a linear change of coordinates in \( C^r \), we may assume that \( \hat{h}(0) = I_r \). Therefore
\[ h(z) = \exp \left\{ -\frac{1}{r} H(z, z) \right\} I_r, \quad z \in \mathbb{C}^n. \]

By the formula (A)

\[ J(\alpha, z) = U(\alpha) \exp \left\{ \frac{1}{r} H(z, \alpha) + \frac{1}{2r} H(\alpha, \alpha) \right\}, \]

where \( U(\alpha) = J(\alpha, 0) \exp \left\{ -\frac{1}{2r} H(\alpha, \alpha) \right\} \) is a unitary matrix.

In summary, if \( E \) admits a projectively flat Hermitian structure \( h \), its associated automorphic factor \( J \) can be written as follows:

(B) \[ J(\alpha, z) = U(\alpha) \exp \left\{ \frac{1}{r} H(z, \alpha) + \frac{1}{2r} H(\alpha, \alpha) \right\}, \quad \alpha \in L, \quad z \in \mathbb{C}^n, \]

where

(i) \( H \) is an Hermitian form on \( \mathbb{C}^n \) and its imaginary part \( A \) satisfies

\[ \frac{1}{\pi} A(\alpha, \beta) \in \mathbb{Z} \quad \text{for} \quad \alpha, \beta \in L, \]

(ii) \( U : L \to U(r) \) is a semi-representation in the sense that it satisfies

\[ U(\alpha + \beta) = U(\alpha) U(\beta) \exp \left\{ \frac{i}{r} A(\beta, \alpha) \right\}, \quad \alpha, \beta \in L. \]

4. Let \( E \) be a simple holomorphic vector bundle of rank \( r \) over the complex torus \( M = \mathbb{C}^n/L \) which admits a projectively flat Hermitian structure. Then its automorphic factor \( J \) is given by the formula (B). Then, for all \( \alpha, \beta \in L \),

\[ J(\alpha, z + \beta) J(\alpha, z)^{-1} \]

\[ = U(\alpha) \exp \left\{ \frac{1}{r} H(z + \beta, \alpha) + \frac{1}{2r} H(\alpha, \alpha) \right\} \]

\[ \cdot U(\alpha)^{-1} \exp \left\{ -\frac{1}{r} H(z, \alpha) - \frac{1}{2r} H(\alpha, \alpha) \right\} \]

\[ = \exp \left\{ \frac{1}{r} H(\beta, \alpha) \right\}. \]

By Theorem \( \text{(Morikawa)} \), there exists a sublattice \( \tilde{L} \) of \( L \) and a line bundle \( F \) over \( N = \mathbb{C}^n/\tilde{L} \) such that \([L : \tilde{L}] = r \) and \( \phi_* F \cong E \), where \( \phi : \mathbb{C}^n/\tilde{L} \to \mathbb{C}^n/L \) is the natural isogeny. And we have

\[ H^j(M, \mathcal{O}) \cong H^j(M, \text{End}(E)) \quad \text{for all} \quad j \]

and

\[ \dim_{\mathbb{C}} H^j(M, \text{End}(E)) = \binom{n}{j}. \]
We set
\[ \omega_a(z) = J(\alpha, z)^{-1} \, dJ(\alpha, z), \quad \alpha \in L. \]
Then we obtain a system of integrable connections satisfying

1. \( d\omega_a(z) + \omega_a(z) \wedge \omega_a(z) = 0 \)
2. \( \omega_a + \beta(z) = \omega_a(z) + J(\alpha, z)^{-1} \, \omega_\beta(z + \alpha)J(\alpha, z), \quad \alpha, \beta \in L. \)

If we write
\[ \omega_a(z) = \sum_{l=1}^{n} A_{al}(z) \, dz_l \quad (\alpha \in L, \ 1 \leq l \leq n), \]
then all \( A_{al}(z) \) are constants. Indeed,
\[
\begin{align*}
\omega_a(z + \beta) - \omega_a(z) &= J(\alpha, z + \beta)^{-1} \, dJ(\alpha, z + \beta) - J(\alpha, z)^{-1} \, dJ(\alpha, z) \\
&= J(\alpha, z + \beta)^{-1} \, d(J(\alpha, z + \beta) \, J(\alpha, z)^{-1}) \, J(\alpha, z) \\
&= 0.
\end{align*}
\]
Since \( M = \mathbb{C}^n/L \) is compact and \( A_{al}(z + \beta) = A_{al}(z) \quad (\beta \in L) \), all \( A_{al} \) are constants.

And we have \( \omega_a(z) \wedge \omega_a(z) = 0 \) and so \([A_{al}, A_{am}] = 0 \) for all \( \alpha \in L, \ 1 \leq l, m \leq n \). If we define
\[ \tilde{J}(\alpha, z) = J(\alpha, 0) \, \exp\left( \sum_{l=1}^{n} A_{al} \, z_l \right), \]
then we have
\[ \tilde{J}(\alpha, z)^{-1} \, d\tilde{J}(\alpha, z) = J(\alpha, z)^{-1} \, dJ(\alpha, z). \]

Since \( \tilde{J}(\alpha, 0) = J(\alpha, 0) \), we have \( \tilde{J}(\alpha, z) = J(\alpha, z) \quad (\alpha \in L) \). It is easy
to show that the total Chern class of \( E \) is given by
\[ c(E) = \left( \frac{1 + c_1(E)}{r} \right)^r. \]
That is, the \( k \)-th Chern class of \( E \) is given by
\[ c_k(E) = \left( \frac{r}{k} \right)^{1/r} c_1(E)^k. \]
It is also easy to show the following identity:
\[ c_2(\text{End } E) = -(r-1) c_1^2(E) + 2rc_2(E) = 0. \]

In summary, we have

**Theorem.** Let \( E \) be a simple holomorphic vector bundle of rank \( r \) over
the complex torus \( M = \mathbb{C}^n/L \). Assume \( E \) admits a projectively flat Hermitian structure. Let \( J \) be its associated automorphic factor for \( L \) given
by the formula (B). Let \( \omega_\alpha(z) = J(\alpha, z)^{-1}dJ(\alpha, z) \) \( (\alpha \in L) \) be a system of integrable connections. Then

1. There exists an isogeny \( \phi : N \rightarrow M \) of degree \( r \) and a line bundle \( F \) such that \( \phi^*F \cong E \).

2. \( H^j(M, \mathcal{O}) \cong H^j(M, \text{End}(E)) \) for all \( j \).

3. All \( A_{\alpha l}(\alpha \in L, \ 1 \leq l \leq n) \) are constants and \( J(\alpha, z) = J(\alpha, 0) \exp (\sum A_{\alpha l}z_l), \ \alpha \in L, \ z \in \mathbb{C}^n. \)

4. \( c(E) = \left( 1 + \frac{c_1(E)}{r} \right)^r \)

and

\[ c_2(\text{End } E) = 0. \]

References


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