A NOTE ON THE OPERATOR EQUATION $\alpha + \alpha^{-1} = \beta + \beta^{-1}$

A. B. Thaieem

1. Introduction

Let $M$ be a von Neumann algebra and $\alpha, \beta$ be $*$-automorphisms of $M$ satisfying the operator equation

$$\alpha + \alpha^{-1} = \beta + \beta^{-1}$$

This operator equation has been extensively studied and many important decomposition theorems have been obtained by several authors (for instance see [4], [5], [2], [1]). Originally, this operator equation arose in the paper of Van Daele on the new approach of the Tomita–Takesaki theory in the case of modular operators ([7]). In the case of one-parameter automorphism groups, this equation has produced a bounded and completely positive map which can play a role similar to the infinitesimal generator (for details see [6] and [1]). A recent and one of the most important applications of this equation has been in developing an analogue of the Tomita–Takesaki theory for Jordan algebras by Haagerup [3]. One general result of this theory is the following:

**Theorem.** Let $M$ be a Von Neumann algebra and $\alpha, \beta$ be commuting $*$-automorphisms of $M$ satisfying the operator equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Then there exists a central projection $p$ in $M$ such that $\alpha(px) = \beta(px)$ and $\alpha((1-p)x) = \beta^{-1}((1-p)x)$ for all $x$ in $M$ (see [4], [5] and [2]). The proof of this theorem depends, in one or the other form, on the deep algebraic and topological properties of von Neumann algebras.

Important applications of the above theorem are reflected in the case when the von Neumann algebra is $B(H)$ (the algebra of all bounded operators on a Hilbert space $H$) or a factor in which case this operator equation has a complete solution in the sense that either $\alpha = \beta$ or $\alpha = \beta^{-1}$ which can be obtained from the above theorem by putting $p=1$ (see, for instance, [4] and [5]). In view of the importance of this

Received April 25, 1986.
conclusion for $B(H)$ or factors, it is worthwhile to look for an independent and simpler proof of this result in this case. We give here a direct and simpler proof of this result without using the technical properties of von Neumann algebras and thus is easily comprehensible to a reader with elementary background in operator theory.

2. Results

We prove the following result.

**Theorem 2.1.** Let $\alpha, \beta$ be $*$-automorphisms on $B(H)$ such that

$$\alpha + \alpha^{-1} = \beta + \beta^{-1}$$

then either $\alpha = \beta$ or $\alpha = \beta^{-1}$.

**Proof.** It is well-known that $\alpha, \beta$ are inner, so there exist unitaries $u$ and $v$ such that

$$\alpha(x) = u x u^*$$
$$\beta(x) = v x v^*$$

for all $x$ in $B(H)$.

Thus, we have

$$u x u^* + u^* x u = v x v^* + v^* x v.$$  \hspace{1cm} (1)

We can rewrite equation (1) in terms of Hilbert–Schmidt operators on $H \otimes \overline{H}$ as:

$$u \otimes \overline{u} + u^* \otimes \overline{u}^* = v \otimes \overline{v} + v^* \otimes \overline{v}^*$$  \hspace{1cm} (2)

Assume that $u^* \neq \lambda u$ for any complex number $\lambda$ (and similarly for $v$). Let $W$ be in $B(H)_*$ such that $W(v) = 1$ and $W(v^*) = 0$. Applying $(1 \otimes \overline{W})$, we get

$$\overline{W}(\overline{u}) u + \overline{W}(\overline{u}^*) u^* = v.$$ 

So there exist numbers $k_1$ and $k_2$ such that

$$v = k_1 u + k_2 u^*$$
$$v^* = k_1 u^* + k_2 u$$

and hence from equation (2), we have

$$u \otimes \overline{u} + u^* \otimes \overline{u}^* = \{ |k_1|^2 + |k_2|^2 \} u \otimes \overline{u} + (k_1 \overline{k}_2 + k_2 \overline{k}_1) u \otimes \overline{u}^*$$
$$+ (k_1 k_2 + k_2 k_1) u^* \otimes \overline{u} + \{ |k_1|^2 + |k_2|^2 \} u^* \otimes \overline{u}^*.$$ 

Since $u$ and $u^*$ are linearly independent, it follows that $k_1 \overline{k}_2 = 0$ and
A note on the operator equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$

$|k_1|^2 + |k_2|^2 = 1$. Now $k_1 k_2 = 0$ implies either $k_1 = 0$ or $k_2 = 0$. So, if $k_1 = 0$ then $|k_2| = 1$ and if $k_2 = 0$ then $|k_1| = 1$. In the first case $v = k_2 u^*$ and hence

$$\beta^{-1}(x) = v^* xv = k_2 u^* k_2 u^* = |k_2|^2 |u^* u^*| = \alpha(x)$$

for all $x$ in $B(H)$.

In the second case, $v = k_1 u$ and by similar calculations, we have that

$$\alpha(x) = \beta(x)$$

for all $x$ in $B(H)$. Thus we have either $\alpha(x) = \beta(x)$ for all $x$ in $B(H)$ or $\alpha(x) = \beta^{-1}(x)$ for all $x$ in $B(H)$.

Similarly, when $u^* = \lambda u$ for a complex number $\lambda$, with $|\lambda| = 1$, we get

$$2(u \otimes \bar{u}) = v \otimes \bar{v} + v^* \otimes \bar{v}^*.$$ Again choosing $W$ in $B(H)_*$ with $W(v) = 1$ and $W(v^*) = 0$, we have

$$2 \overline{W(u)} u = v.$$ This is possible only when $|2 \overline{W(u)}| = 1$ and this implies that $\alpha = \beta$.

So in any case, either $\alpha = \beta$ or $\alpha = \beta^{-1}$. Q.E.D.

The following corollary is, in fact, an improvement of Theorem 2.1.

**Corollary 2.2.** Let $\alpha, \beta$ be inner $*$-automorphisms of a factor $M$ acting on a Hilbert space $H$ such that

$$\alpha + \alpha^{-1} = \beta + \beta^{-1}$$

then, either $\alpha = \beta$ or $\alpha = \beta^{-1}$.

**Proof.** Let $u$ and $v$ be unitaries such that

$$\alpha(x) = u x u^*$$

and

$$\beta(x) = v x v^*$$

for all $x \in M$.

We define $\tilde{\alpha}$ and $\tilde{\beta}$ on $B(H)$ by the same formulas.

Because $M$ is a factor and $(M \cup M')' = M \cap M' = C_1$, we get that $(M \cup M')'' = B(H)$. Therefore, the algebra generated by $M$ and $M'$ is $B(H)$ and hence for any $x \in B(H)$, we can write

$$x = \sum_{i=1}^{n} a_i a_i'$$

where $a_i \in M$ and $a_i' \in M'$.

Remark that $\tilde{\alpha}(a_i') = a_i$ and $\tilde{\alpha}^{-1}(a_i') = a_i'$ (because $u \in M$).

Applying $\tilde{\alpha} + \tilde{\alpha}^{-1}$ on $B(H)$, we get
(\tilde{\alpha} + \tilde{\alpha}^{-1}) (x) = \sum_{i=1}^{n} \alpha (a_i) a_i' + \sum_{i=1}^{n} \alpha^{-1} (a_i) a_i' \\
= \sum_{i=1}^{n} (\alpha + \alpha^{-1}) (a_i) a_i'.

Since \alpha + \alpha^{-1} = \beta + \beta^{-1} then \tilde{\alpha} + \tilde{\alpha}^{-1} = \tilde{\beta} + \tilde{\beta}^{-1} on B(H). By the theorem above, it follows that \tilde{\alpha} = \tilde{\beta} or \tilde{\alpha} = \tilde{\beta}^{-1} and hence \alpha = \beta or \alpha = \beta^{-1} on M.

We conclude the note with the following

PROBLEM. Can we improve the above corollary by dropping the innerness of automorphisms?

References


Department of Mathematics
Quaid–i–Azam University
Islamabad, Pakistan