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<**Technical Paper**>

The Method of *J* Integral Analysis and Estimate

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*J*적분 해석과 산정방법

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Key Words: *J* Integral(*J*적분), Three-Point Bend Specimen(3점 굽힘 시험편), Center-Cracked Tension Specimen(중앙균열 인장 시험편), Compact Tension Specimen(콤팩트 인장 시험편), Stress Intensity Factor(응력세기 계수)

초 록

3점 굽힘 시험편, 중앙균열 인장 시험편, 콤팩트 인장 시험편에 대한 *J*적분식을 하나의 일반화된 형태로 유도한다. 이 일반식은 Eftis와 Liebowitz에 의해 제안된 하중과 하중점 변위 사이의 관계와 Sumpter에 의한 탄성과 소성성분 *J*적분의 중첩개념을 이용함으로써 유도된다. 일반식에 포함된 η계수를 위 3가지 시험편에 대해서 결정한다. 위 3가지 시험편에 대한 *J*적분의 최종식은 하중과 하중점 변위곡선 아래의 면적을 측정하지 않아도 되는 형태로 나타난다. 본 연구의 결과식은 Landes 등에 의한 실험치와 비교하여 매우 잘 일치함을 보인다.

1. Introduction

The *J*-integral defined in the form of the changing rate of potential energy per unit thickness with respect to crack length was used by many experimental researchers. Begley and Landes[1,2] measured the values of *J*-integral experimentally for the first time by using Rice's energy rate definition[3,4] along with compliance method. Their method gives accurate *J* value but has the disadvantage that five or more specimens are required for each *J* value.

Since then, several researchers have suggested equations to obtain *J* value from single specimen. Such equations were introduced by Rice, et. al.[5] for three-point bend (TPB) and center-cracked tension (CCT) and by Merkle and Corten[6] for compact tension (CT) specimen. However, to use such equations, it is necessary to measure the area under load (P) vs. load-point displacement (δ) curve.

Bucci, et. al.[7] analyzed the relations between *J* and δ in linear elastic and rigid plastic regions, respectively. They showed that for linear elastic deformation, *J* is proportional to δ^2 while in the case of rigid plastic deformation, it is proportional to δ. They did not obtain a single

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relation which can be applied in the entire displacement range.

The purpose of this paper is (a) to show a method to determine J value by taking the coordinate on P-δ curve instead of measuring the area under the curve, (b) to obtain a generalized form for the J-integral applicable to any configuration of specimen, (c) to derive a single relation between J and δ for the entire load-point displacement range, and (d) to show the validity of the present analysis by comparing with the results obtained by Landes, et. al.[8]

2. Previous Results

In the case of linear elastic deformation, the J-integral is equal to the strain energy release rate G, which in turn is related to the stress intensity factor K

$$J=G=\frac{K^2}{E'} \tag{1}$$

where

$$E'=\begin{cases} E & \text{plane stress} \\ E/(1-\nu^2) & \text{plane strain} \end{cases}$$

and E and ν are Young's modulus and Poisson's ratio, respectively. For the three types of specimens considered here, K is given in the form[9,10]

$$K=\frac{P}{BW^{1/2}}F\left(\frac{a}{W}\right) \tag{2}$$

where B, W and a are specimen thickness, specimen width and crack length, respectively and $F\left(\frac{a}{W}\right)$ are known expressions, different for each specimen.

In the case of plastic deformation, the J-integral is different from eqn (1). For example, for TPB specimen, Rice, et. al.[5] gives

$$J=\frac{2U}{B(W-a)} \tag{3}$$

where U is strain energy. They also developed an expression for J for CCT specimen

$$J=G+\frac{1}{B(W-a)}\left[2\int_0^{\delta_p}Pd\delta_p-P\delta_p\right] \tag{4}$$

where δ_p is the plastic portion of δ. Merkle and Corten[6] obtained the following J equation for CT specimen by taking into account bending effects as well as axial force effects and using a plastic limit load analysis

$$J=G+\frac{\varphi_r}{B(W-a)}\int_0^{\delta_p}Pd\delta_p + \frac{\varphi_c}{B(W-a)}\int_0^{P}\delta_p dP \tag{5a}$$

where

$$\varphi_r=\frac{2(1+\alpha)}{1+\alpha^2} \tag{5b}$$

$$\varphi_c=\frac{2\alpha(1-2\alpha-\alpha^2)}{(1+\alpha^2)^2} \tag{5c}$$

$$\alpha=2\left[\left(\frac{a}{b}\right)^2+\frac{a}{b}+\frac{1}{2}\right]^{1/2}-2\left[\frac{a}{b}+\frac{1}{2}\right] \tag{5d}$$

and b is the ligament length. As shown in eqns (3), (4) and (5), it is necessary in all these cases to measure the area under P-δ curve in order to determine the value of J.

Sumpter[11] expressed the J-integral as the sum of elastic and plastic components in the form

$$J=J_e+J_p=\frac{\eta_e U_e}{B(W-a)}+\frac{\eta_p U_p}{B(W-a)} =G+\frac{\eta_p U_p}{B(W-a)} \tag{6}$$

where subscripts e and p denote elastic and plastic portions, respectively. η_e and η_p are constants[12,13]

Eftis and Liebowitz[14] suggested the following three-parameter relationship as a suitable approximation of the P-δ curve obtained from tensile test of the cracked specimen

$$\delta=\delta_e+\delta_p \tag{7a}$$

$$\delta_e=\frac{P}{M} \tag{7b}$$

$$\delta_p=k\left(\frac{P}{M}\right)^n \tag{7c}$$

where M is stiffness. δ_e and δ_p are shown in Fig. 1. In eqn (7c), k and n are determined

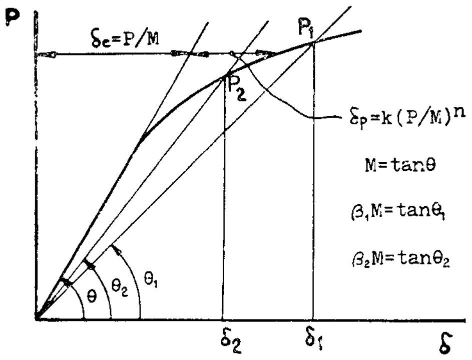

Fig. 1 Typical nonlinear load-displacement curve showing three-parameter representation and reduced modulus lines

from the two straight lines intersecting the P-δ curve by eqns (8a) and (8b), respectively

$$n=1+\frac{\log[(\beta_2/\beta_1)(1-\beta_1)/(1-\beta_2)]}{\log(P_1/P_2)} \tag{8 a}$$

$$k=\frac{1-\beta_1}{\beta_1}\left(\frac{P_1}{M}\right)^{1-n}$$
$$=\frac{1-\beta_2}{\beta_2}\left(\frac{P_2}{M}\right)^{1-n} \tag{8 b}$$

where β_1, β_2 as well as P_1, P_2 are defined in Fig. 1. Combination of the above results leads to an alternate method for evaluating J.

3. Modified J Equations

U_p in eqn (6) may be written in the form

$$U_p=\int_0^{\delta_p} P d\delta_p \tag{9}$$

From eqns (7c) and (9), it follows that

$$U_p=\frac{n}{n+1}P\delta_p \tag{10}$$

Substituting from eqn (10) into eqn (6) leads to the result

$$J=G+\frac{\eta_p}{B(W-a)}\ \frac{n}{n+1}P\delta_p \tag{11}$$

This a generalized form applicable to any specimen configurations. The term η_p in eqn (11) is determined below for the different types of test specimens.

(i) TPB specimen

Eqn (3) can be written in the form

$$J=G+\frac{2U_p}{B(W-a)} \tag{12}$$

Substituting from eqn (10) into eqn (12) shows that

$$J=G+\frac{2}{B(W-a)}\ \frac{n}{n+1}P\delta_p \tag{13}$$

and comparing eqn (11) and eqn (13) it follows that

$$\eta_p=2 \tag{14}$$

(ii) CCT specimen

Using eqn (10) permits eqn (4) to be written in the form

$$J=G+\frac{1}{B(W-a)}\ \frac{n-1}{n+1}P\delta_p \tag{15}$$

From eqn (11) and eqn (15) it follows that

$$\eta_p=\frac{n-1}{n} \tag{16}$$

(iii) CT specimen

Upon introducing eqn (10), eqn (5a) has the following form

$$J=G+\frac{1}{B(W-a)}\left(\varphi_r+\frac{\varphi_c}{n}\right)$$
$$\frac{n}{n+1}P\delta_p \tag{17}$$

Comparision between eqns (11) and (17) gives

$$\eta_p=\varphi_r+\frac{\varphi_c}{n} \tag{18}$$

As shown above, η_p depends on the specimen geometry and size as well as work hardening exponent. In order to express J in terms of δ alone, by substituting from eqns (1) and (2) into eqn (11) and enforcing eqn (7b) it follows that

$$J=\frac{M^2}{E'B^2W}F^2\left(\frac{a}{W}\right)\delta_e^2+\frac{\eta_p M}{B(W-a)}$$
$$\frac{n}{n+1}\delta_e\delta_p \tag{19}$$

Eqn (19) is in agreement with Bucci, et. al's results[7] mentioned in the Introduction.

4. Discussion

The experimental P-δ curves of Landes, et.

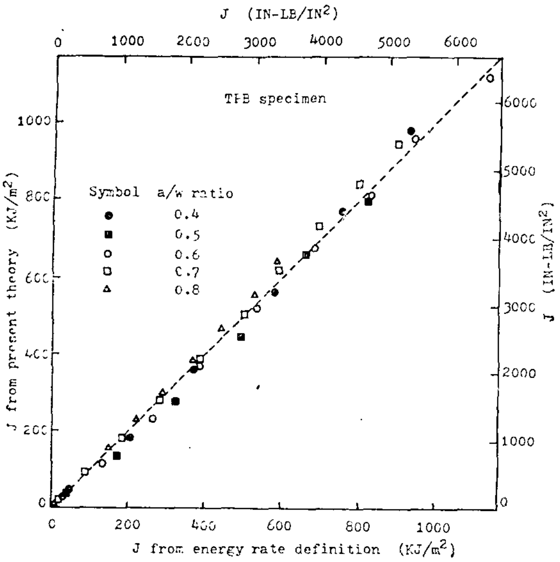

Fig. 2 Comparison of J from present theory with J from energy rate definition[8] for three point bend specimens

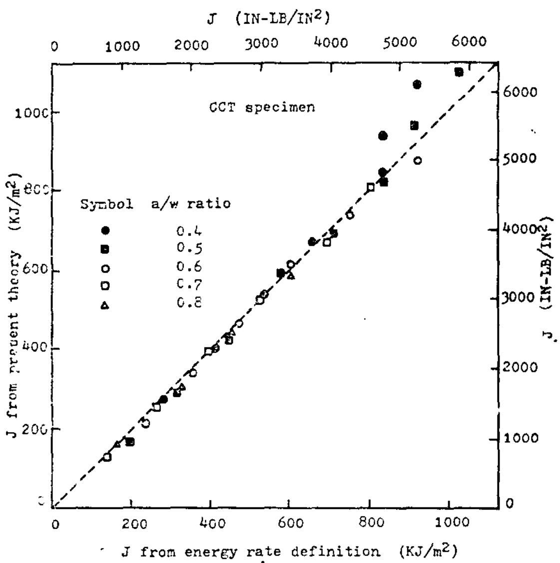

Fig. 3 Comparison of J from present theory with J from energy rate definition[8] for center cracked tension specimens

al.[8] are used. The specimens were TPB, CCT and CT types with a thickness of 22.86 mm and a width of 50.8 mm. The material was HY-30 steel with elastic constants $E=206,850$ MPa and $\nu=0.3$. Figs. 2, 3 and 4 show the

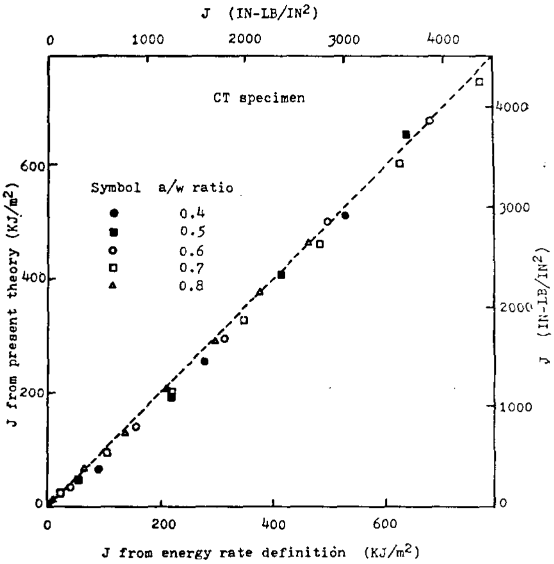

Fig. 4 Comparison of J from present theory with J from energy rate definition[8] for compact tension specimens

comparisions between J values obtained by Landes, et. al.[8] from energy rate definition and values calculated by eqns (13), (15) and (17). The excellent agreement between two methods is observed. It is noted that n was determined by using eqn (8 a) and drawing two straight lines on the P-δ curve, one from the origin to the load at which each J is calculated and the other from the origin to the load located at the half distance of the P-δ curve between the elastic limit and the load at which each J is determined. Alternatively, one might select a single value of n for all J values, however, this leads to less accurate results.

5. Conclusions

In summary, the following results have been developed with respect to the J-integral.

(1) The alternate method of the determination of J-integral without measuring the area under load and load-point displacement curve has been developed.

(2) A generalized form of J-integral applicable to any configuration of specimen has been obtained.

(3) A single relation between J and load-point displacement has been derived.

(4) J values obtained by the present method are in excellent agreement with ones by Landes, et. al's compliance method.

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