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A Distribution-Free Rank Test for Ordered Alternatives in a Randomized Block Design

Dong Hee Kim*, Moon Sup Song**, Woo-Chul Kim**

ABSTRACT

In this paper we propose a distribution-free rank test for ordered alternatives in a randomized block design and investigate the properties of the proposed test. The proposed test is an extension of the Page test to allow replications in each cell. Some asymptotic properties including ARE's are investigated. A small sample Monte Carlo study was performed to compare the powers of the tests considered in this paper for small samples. The results show that our proposed test is robust and efficient in the case of equally-spaced treatment effects.

1. Introduction

There has been considerable interests in nonparametric tests for ordered alternatives in a randomized block design. If any usual ANOVA test such as the Friedman test is applied, the informations of the hypothesized ordering are lost. We thus want some nonparametric tests which can easily detect an increasing (or decreasing) treatment effect.

The usual model for a randomized block design without interactions is of the form

$$X_{ijk}=\mu+\beta_i+\theta_j+\varepsilon_{ijk},\quad i=1,\dots,a\,;\ j=1,\dots,b\,;\ k=1,\dots,n_{ij} \tag{1.1}$$

* Department of Computer Science and Statistics, Pusan National University, Pusan 607, Korea.

** Department of Computer Science and Statistics, Seoul National University, Seoul 151, Korea.

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where μ represents the overall mean, the β_i's are random block effects, the θ_j's represent treatment effects satisfying $\sum_{j=1}^{b} \theta_j = 0$, and the ε_{ijk}'s are independent and identically distributed random variables from a continuous population with mean zero and variance σ^2.

We are interested in the testing problem of the null hypothesis

$$H_0 : \theta_1 = \ldots = \theta_b \tag{1.2}$$

against the ordered alternative

$$H_1 : \theta_1 \leq \ldots \leq \theta_b \tag{1.3}$$

where at least one inequality is strict. We consider $\beta_1, \ldots, \beta_a$ as nuisance parameters.

To test H_0 against H_1 we may use within-blocks information and also between-blocks information. Distribution-free tests using within-blocks information have been considered by Jonckheere (1954), Page (1963), Pirie and Hollander (1972), Hettmansperger (1975), and Skillings and Wolfe (1978), among others. Rank tests using between-blocks information have been proposed by Hollander (1967) and Puri and Sen (1968).

Pirie (1974) has shown that in many cases procedures based on within-blocks information are preferable to those employing between-blocks information. We thus consider a distribution-free test which uses within-blocks rankings and weights depending on the block sizes.

In this paper we propose a distribution-free rank test for ordered alternatives in a randomized block design, and investigate the properties of the proposed test. The proposed test is an extension of the Page test to allow replications in each cell. Hettmansperger (1975) has extended the Page test to the case of more than one observation per cell, by using average ranks on each cell. We propose a test statistic which is a weighted sum of the sum of ranks assigned to each cell with weights proportional to the inverse of block sizes. We also consider a generalization of the proposed test by defining the test statistic as a weighted sum of block statistics.

In constructing the test statistic we use the within-blocks ranking and therefore the test statistic is distribution-free. The exact distribution of the proposed test statistic is not prepared in this paper. But, using normal approximation a test can be easily performed.

Section 2 deals with the proposed test statistics. Asymptotic normality of the test statistics under H_0 is obtained under some mild regularity conditions. To investigate the asymptotic properties of the test statistics, asymptotic normality of the test statistics under the translation alternatives is also proved. Using these results the asymptotic properties including asymptotic relative efficiencies (ARE) are obtained.

In Section 3 we perform a small sample Monte Carlo study to compare the powers of the tests considred in this paper for small samples. Empirical powers of the test statistics are compared for various underlying distributions.

2. Proposed Test Statistics and Their Properties

2.1 Proposed Test Statistics

We now consider the model (1.1) for the randomized block design without interactions. For each block i, let R_{ijk} be the rank of X_{ijk}, $j=1, \ldots, b$, $k=1, \ldots, n_{ij}$, and let $R_{ij.}$ denote the sum of ranks of the observations in the (i, j) cell, that is,

$$R_{ij.}=\sum_{k=1}^{n_{ij}} R_{ijk}\,.$$

For each treatment j, let R_j be the modified sum of ranks given by

$$R_j=\sum_{i=1}^{a} R_{ij.}/n_{i.}, \quad j=1, \ldots, b \tag{2.1}$$

where $n_{i.}$ is the block size defined by $n_{i.}=\sum_{j=1}^{b} n_{ij}$. The proposed test statistic is then given by

$$W=\sum_{j=1}^{b} jR_j\,. \tag{2.2}$$

Note that the statistic W can also be written as

$$W=\sum_{i=1}^{a} W_i\,.$$

where

$$W_i=\sum_{j=1}^{b} jR_{ij.}/n_{i.}\,. \tag{2.3}$$

Since we are using the within-blocks ranking rather than the between-blocks ranking, the test statistic W in (2.2) is distribution-free. Using the asymptotic normality of the statistic W which will be shown in the next section, we reject the null hypothesis H_0 : $\theta_1=\ldots=\theta_b$ in favor of the ordered alternative H_1 : $\theta_1\leq\ldots\leq\theta_b$ for large values of W.

The statistic R_j in (2.1) was originally considered by Mack and Skillings (1980) to form a Friedman-type rank test for main effects in a two-way analysis of variance. Note that the statistic R_j is a weighted sum of the $R_{ij.}$'s with weights equal to the inverse of the block size $n_{i..}$ Mack and Skillings used these weights by considering the fact that the potential size of the ranks used to form $R_{ij.}$ is a function of $n_{i..}$

For the purpose of theoretical interest we consider a class of test statistics defined by

$$W^* = \sum_{i=1}^{a} c_i W_i \tag{2.4}$$

where W_i is the statistic defined by (2.3) and c_i's are nonnegative weighting constants. The statistic W^* is a weighted sum of block statistics, which is a generalization of the unweighted sum W.

The Hettmansperger test statistic T is given by

$$T = \sum_{i=1}^{a} T_i \tag{2.5}$$

where T_i is the rank correlation between the hypothesized ordering and the observed average ordering defined by

$$T_i = \sum_{j=1}^{b} jR_{ij.}/n_{ij}\,. \tag{2.6}$$

We note that the proposed test statistic W in (2.2) is linearly related to the Hettmansperger test statistic T in (2.5) when the cell sizes n_{ij} are all equal. Note also that if we let the cell sizes be equal to multiples of treatment numbers, i.e. $n_{ij}=c\cdot j$, then the Hettmansperger statistic T is the sum of the ranks of all observations, which is a constant. Thus, in this case, the Hettmansperger statistic T has variance zero, and the test based on T cannot be used. But the proposed test statistic W does not suffer from this kind of difficulties.

2.2 Asymptotic Distributions

In this section we consider the null mean and variance and the asymptotic null distribution of the proposed test statistic. Using the results in Hettmansperger (1975, p. 58), we have under H_0,

$$E_0(R_{ij.}) = n_{ij}(n_{i.}+1)/2,$$
$$Var_0(R_{ij.}) = n_{ij}(n_{i.}-n_{ij})(n_{i.}+1)/12,$$
$$Cov_0(R_{ij.}, R_{ik.}) = -n_{ij}n_{ik}(n_{i.}+1)/12.$$

Thus, under H_0, the mean and variance of $W^* = \sum_{i=1}^{a} c_i W_i$, where $W_i = \sum_{j=1}^{a} jR_{ij\cdot}/n_{i\cdot}$, are given by

$$E_0(W^*) = \sum_{i=1}^{a} c_i \sum_{j=1}^{b} j \; n_{ij}(n_{i\cdot}+1)/2n_{i\cdot} \qquad (2.7)$$

and

$$Var_0(W^*) = \sum_{i=1}^{a} c_i^2 (n_{i\cdot}+1) \left(\sum_{j=1}^{b} j^2 n_{ij}(n_{i\cdot} - n_{ij}) - 2 \sum_{j<k}\sum j\, k n_{ij} n_{ik} \right) / 12 n_{i\cdot}^2 . \qquad (2.8)$$

The null mean and variance of W can also be obtained by setting $c_i = 1$ in (2.7) and (2.8), respectively.

The following proof of the asymptotic normality of W^* as $a \to \infty$ is similar to that of Theorem 1 and Theorem 2 in Skillings and Wolfe (1978). The detailed proofs of the theorems in this section can be found in Kim(1986). To discuss the asymptotic distributions we make the following assumptions:

A_1 : The nonnegative weighting constants $c_1, \ldots, c_a$ remain bounded as $a \to \infty$.

A_2 : There exists some constant $M < \infty$ such that $n_{i\cdot} \leq M$ for all i.

A_3 : $\lim_{a\to\infty} \left(\sum_{i=1}^{a} c_i^3 \right) \Big/ \left(\sum_{i=1}^{a} c_i^2 \right)^{3/2} = 0.$

Theorem 2.2.1. Suppose that the assumptions A_1, A_2 and A_3 are satisfied. Then, under H_0, as $a \to \infty$,

$$(W^* - E_0(W^*)) / [Var_0(W^*)]^{\frac{1}{2}}$$

has a limiting standard normal distribution, where $E_0(W^*)$ and $Var_0(W^*)$ are given by (2.7) and (2.8), respectively.

Proof : By applying the Liapounov central limit theorem to W^*, the theorem can be easily proved. ▨

From Theorem 2.2.1 and moments of W^* in (2.7) and (2.8), we have the following corollary.

Corollary 2.2.1. Suppose that the assumption A_2 holds. Then the limiting (as $a \to \infty$) null distribution of

$$(W - E_0(W)) / [Var_0(W)]^{\frac{1}{2}}$$

is standard normal with

$$E_0(W) = \sum_{i=1}^{a} \sum_{j=1}^{b} j \; n_{ij}(n_{i\cdot}+1)/2n_{i\cdot}.$$

and

$$Var_0(W)=\sum_{i=1}^{a}(n_{i.}+1)\left(\sum_{j=1}^{b}j^2\ n_{ij}(n_{i.}-n_{ij})-2\sum\sum_{j<k}jkn_{ij}n_{ik}\right)/12n_{i.}^2. \qquad (2.9)$$

2.3 Asymptotic Relative Efficiencies

We consider the ARE's of the tests based on W and W^* relative to the other parametric and nonparametric tests. As nonparametric tests we consider the Hettmansperger test and the Skillings and Wolfe test. The Skillings and Wolfe test K^* which is an extension of the Jonckheere (1954) test is defined by

$$K^*=\sum_{i=1}^{a}b_i\sum_{s=1}^{b-1}\sum_{t=s+1}^{b}U_{ist} \qquad (2.10)$$

where $b_1,\dots, b_a$ are nonnegative weighting constants and U_{ist} is the Mann-Whitney statistic applied to the observations in cells (i, s) and (i, t). As a parametric test we include the following statistic A^* which was considered by Skillings and Wolfe(1978) as a generalization of a statistic discussed by Puri(1965) for testing ordered alternatives in a one-way analysis of variance. Let

$$A^*=\sum_{i=1}^{a}\sum_{s=1}^{b-1}\sum_{t=s+1}^{b}d_in_{is}n_{it}(\overline{X}_{it.}-\overline{X}_{is.}) \qquad (2.11)$$

where $d_1,\dots,d_a$ are nonnegative constants, and $\overline{X}_{ij.}$ is the average of the observations in (i,j) cell, *i.e.* $\overline{X}_{ij.}=\sum_{k=1}^{n_{ij}}X_{ijk}/n_{ij}$.

We shall follow the discussions for ARE in Skillings and Wolfe (1977, 1978). To evaluate ARE we consider the limiting distributions of the test statistics considered in this paper under translation alternatives. The translation alternatives we will consider are as follows: let $X_{ijk}, k=1,\dots,n_{ij}$, have continuous unknown distribution function

$$F_{ij}(x)=F(x-\mu-\beta_i-(j-1)\theta/\sqrt{a}) \qquad (2.12)$$

for $i=1,\dots, a$, $j=1,\dots,b$. The alternatives (2.12) were considered by Hollander (1967) and were called S alternatives. From Theorem 4 of Skillings and Wolfe (1977), we have the following theorem about the asymptotic normality of A^* in (2.11)

Theorem 2.3.1. Assume that the assumptions A_1, A_2 and A_3 in Section 2.2 are satisfied with weights d_i instead of c_i. If $\sigma^2=Var(X_{ijk})<\infty$, then, under the S alternatives (2.12),

$$(A^*-E(A^*))/\sigma_{A^*}$$

has a limiting (as $a \to \infty$) standard normal distribution, where

$$E(A^*)=\sum_{i=1}^{a}\sum_{s=1}^{b-1}\sum_{t=s+1}^{b} d_i n_{is} n_{it}(t-s)\theta/\sqrt{a}$$

and

$$\sigma_{A^*}{}^2=\sigma^2\sum_{i=1}^{a} d_1{}^2\left(\left(\sum_{j=1}^{b} n_{ij}\right)^3-\sum_{j=1}^{b} n_{ij}{}^3\right)/3.$$

As a special case of A^*, we consider the test statistic A with $d_i=1, i=1, \ldots, a$, in the definition of A^* in (2.11).

Corollary 2.3.1. Assume that the assumption A_2 holds. If $\sigma^2=Var(X_{ijk})<\infty$, then under the S alternatives (2.12)

$$(A-E(A))/\sigma_A$$

has a limiting (as $a \to \infty$) standard normal distribuiton, where

$$E(A)=\sum_{i=1}^{a}\sum_{s=1}^{b-1}\sum_{t=s+1}^{b} n_{is} n_{it}(t-s)\theta/\sqrt{a}$$

and

$$\sigma_A{}^2=\sigma^2 \sum_{i=2}^{a}\left(\left(\sum_{j=1}^{b} n_{ij}\right)^3-\sum_{j=1}^{a} n_{ij}\right)^3/3. \qquad (2.13)$$

For the purpose of theoretical comparison we consider a generalized Hettmansperger statistic defined by

$$T^*=\sum_{i=1}^{a} e_i \sum_{j=1}^{b} j\ R_{ij\cdot}/n_{ij}, \qquad (2.14)$$

where $e_1, \ldots, e_a$ are nonnegative weighting constants. That is, T^* is a weighted sum of the rank correlation T_i in (2.6).

Theorem 2.3.2. Suppose that the assumptions A_1, A_2 and A_3 in Section 2.2 are satisfied with weights e_i rather than c_i. Then, under the S alternatives (2.12),

$$(T^*-E(T^*))/\sigma_{T^*}$$

has a limiting (as $a \to \infty$) standard normal distribution, where

$$E(T^*)=\sum_{i=1}^{a} e_i \sum_{s=1}^{b} s\ \frac{1}{n_{is}}\left\{\sum_{t=1}^{b} n_{is} n_{it}\int[1-F(y-(s-t)\theta/\sqrt{a})]f(y)dy+n_{is}(n_{is}+1)/2\right\}$$

and

$$\sigma_{T^*}{}^2=\sum_{i=1}^{a} e_i{}^2\left\{\sum_{j=1}^{b} n_{i\cdot}(n_{i\cdot}+1)j^2/12n_{ij}-b^2(b+1)^2(n_{i\cdot}+1)/48\right\}.$$

Proof : Since we have

$$E(R_{is\cdot})=\sum_{t=1}^{b} n_{is} n_{it}\int\left[1-F(y-(s-t)\theta/\sqrt{a})\right]f(y)dy+n_{is}(n_{is}+1)/2,$$

$E(T^*)$ can be easily found. The asymptotic normality can also be obtained by using the same argument as in Theorem 2.2.1. ▨

Corollary 2.3.2. Assume that the assumption A_2 holds. Then, under the S alternatives (2.12),

$$(T-E(T))/\sigma_T$$

has a limiting (as $a\to\infty$) standard normal distribution, where

$$E(T)=\sum_{i=1}^{a}\sum_{s=1}^{b} s\frac{1}{n_{is}}\left\{\sum_{t=1}^{b} n_{is}n_{it}\int[1-F(y-(s-t)\theta/\sqrt{a})]f(y)dy+n_{is}(n_{is}+1)/2\right\}$$

and

$$\sigma_T^2=\sum_{i=1}^{a}\sum_{j=1}^{b} n_{i\cdot}(n_{i\cdot}+1)j^2/12n_{ij}-b^2(b+1)^2(n_{i\cdot}+1)/48. \qquad (2.15)$$

The asymptotic normality of the Skillings and Wolfe statistic K^* can be obtained by using the results in Skillings and Wolfe (1977, 1978).

Theorem 2.3.3. Assume that the assumptions A_1, A_2 and A_3 in Section 2.2 are satisfied with weights b_i rather than c_i. Then, under the S alternatives (2.12),

$$(K^*-E(K^*))/\sigma_{K^*}$$

has a limiting (as $a\to\infty$) standard normal distribution, where

$$E(K^*)=\sum_{i=1}^{a} b_i\sum_{s=1}^{b-1}\sum_{t=s+1}^{b} n_{is}n_{it}(t-s)\int[1-F(y-(t-s)\theta/\sqrt{a})]f(y)dy$$

and

$$\sigma_{K^*}^2=\sum_{i=1}^{a} b_i^2(2n_{i\cdot}^3+3n_{i\cdot}^2-\sum_{j=1}^{b} n_{ij}^2(2n_{ij}+3))/72.$$

Proof : Since the statistic K^* can be written by

$$K^*=\sum_{i=1}^{a} b_i\sum_{s=1}^{b-1}\sum_{t=s+1}^{b}\sum_{u=1}^{n}\sum_{v=1}^{n_{it}}\Psi(X_{itv}-X_{isu})$$

where $\Psi(z)=1$ or 0 as $z>0$ or $z\leq 0$, we can easily find $E(K^*)$. Since the asymptotic variance of K^* under S alternatives is the same as that of K^* under H_0, $\sigma_{K^*}^2$ is obtained. ▨

Corollary 2.3.3. Assume that the assumption A_2 holds. Then, under the S alternatives (2.12),

$$(K-E(K))/\sigma_K$$

has a limiting (as $a\to\infty$) standard normal distribution, where

$$E(K)=\sum_{i=1}^{a}\sum_{s=1}^{b-1}\sum_{t=s+1}^{b} n_{is}n_{it}\int[1-F(y-(t-s)\theta/\sqrt{a})]f(y)dy$$

and

$$\sigma_K^2=\sum_{i=1}^{a}(2n_{i\cdot}^3+3n_{i\cdot}^2-\sum_{j=1}^{b}n_{ij}^2(2n_{ij}+3))/72. \qquad (2.16)$$

The asymptotic normality of W^* under S alternatives can also be obtained as follows.

Theorem 2.3.4. Assume that the assumptions A_1, A_2 and A_3 are satisfied. Then, under the S alternatives (2.12),

$$(W^*-E(W^*))/\sigma_{W^*}$$

has a limiting (as $a\to\infty$) standard normal distribution, where

$$E(W^*)=\sum_{i=1}^{a}c_i\frac{1}{n_{i\cdot}}\sum_{s=1}^{b}s\left\{\sum_{t=1}^{b}n_{is}n_{it}\int[1-F(y-(s-t)\theta/\sqrt{a})]f(y)dy+n_{is}(n_{is}+1)/2\right\}$$

and $\sigma_{W^*}^2$ is the same as $\mathrm{Var}_0(W^*)$ given by (2.8).

Proof : The results can be easily proved by the same argument as in the proof of Theorem 2.3.2. ▨

Corollary 2.3.4. Assume that the assumption A_2 holds. Then, under the S alternatives,

$$(W-E(W))/\sigma_W$$

has a limiting (as $a\to\infty$) standard normal distribution, where

$$E(W)=\sum_{i=1}^{a}\frac{1}{n_{i\cdot}}\sum_{s=1}^{b}s\left\{\sum_{t=1}^{b}n_{is}n_{it}\int[1-F(y-(s-t)\theta/\sqrt{a})]f(y)dy+n_{is}(n_{is}+1)/2\right\}$$

and σ_W^2 is the same as $\mathrm{Var}_0(W)$ given in (2.9).

Using the above results, we obtain the following theorem.

Theorem 2.3.5. The ARE's of W relative to A, T and K are, respectively, given by

$$\mathrm{ARE}(W,A)=\lim_{a\to\infty}\frac{\sigma_A^2\left\{\sum_{i=1}^{a}\frac{1}{n_{i\cdot}}\sum_{s=1}^{b}s\sum_{t=1}^{b}n_{is}n_{it}(s-t)\int f^2(y)dy\right\}^2}{\sigma_W^2\left\{\sum_{i=1}^{a}\sum_{s=1}^{b-1}\sum_{t=s+1}^{b}n_{is}n_{it}(t-s)\right\}^2}$$

$$\mathrm{ARE}(W,T)=\lim_{a\to\infty}\frac{\sigma_T^2\left\{\sum_{i=1}^{a}\frac{1}{n_{i\cdot}}\sum_{s=1}^{b}s\sum_{t=1}^{b}n_{is}n_{it}(s-t)\right\}}{\sigma_W^2\left\{\sum_{i=1}^{a}\sum_{s=1}^{b}s\sum_{t=1}^{b}n_{it}(s-t)\right\}^2}$$

$$\mathrm{ARE}(W,K)=\lim_{a\to\infty}\frac{\sigma_K^2\left\{\sum_{i=1}^{a}\frac{1}{n_{i\cdot}}\sum_{s=1}^{b}s\sum_{t=1}^{b}n_{is}n_{it}(s-t)\right\}^2}{\sigma_W^2\left\{\sum_{i=1}^{a}\sum_{s=1}^{b-1}\sum_{t=s+1}^{b}n_{is}n_{it}(t-s)\right\}^2}$$

where σ_W^2, σ_A^2, σ_T^2 and σ_k^2 are given by (2.9), (2.13), (2.15), (2.16), respectively.

Note that in the case of $n_{ij}=n_{i\cdot}/b$, i.e., when the cell sizes are equal in each block, the W statistic is reduced to T/b. Thus in this case the two tests based on W and T are equivalent and the ARE(W, T) is reduced to 1.

When the cell sizes are all equal, i.e., $n_{ij}=n$ for each i and j, the ARE of W relative to K is reduced to

$$\text{ARE}(W,K)=\frac{2nb^2+2nb+3b}{2nb^2+2nb+2b+2}$$

Thus, in this case, ARE(W,K) is always ≥ 1. The equality holds only when $b=2$, and in this case the two tests based on W and K are actually equivalent.

In the case of $n_{ij}=n$ for every i and j, the ARE of W relative to the parametric test A is reduced to

$$\text{ARE}(W,A)=\frac{nb}{nb+1}\ 12\sigma^2\ \left(\int f^2(y)dy\right)^2.$$

Note that when $n=1$, the ARE(W, A) is reduced to the ARE of the Friedman test relative to the parametric F test in a randomized block design problem.

We now consider the ARE's of the weighted statistics W^* and T^*. The efficacy of W^* can be obtained as follows.

$$\text{eff}(W^*)=\lim_{a\to\infty}\frac{\sum_{i=1}^{a} c_i\ \frac{1}{n_{i\cdot}}\sum_{s=1}^{b} s\sum_{t=1}^{b} n_{is}n_{it}(s-t)\int f^2(y)dy}{a\sigma_{W^*}}$$

where

$$\sigma_{W^*}^2=\sum_{i=1}^{a} c_i^2\ \text{Var}(W_i)$$

with

$$\text{Var}(W_i)=(n_{i\cdot}+1)\left\{\sum_{j=1}^{b} j^2n_{ij}(n_{i\cdot}-n_{ij})-2\sum_{j<k}\sum jkn_{ij}\,n_{ik}\right\}\Big/12n_{i\cdot}^2. \qquad (2.17)$$

Hence the ARE of W^* relative to W is given by

$$\text{ARE}(W^*,\ W)=\lim_{a\to\infty}\frac{\left\{\sum_{i=1}^{a}\text{Var}(W_i)\right\}\left\{\sum_{i=1}^{a} c_i\ \frac{1}{n_{i\cdot}}\sum_{s=1}^{b} s\sum_{t=1}^{b} n_{is}\,n_{it}(s-t)\right\}^2}{\left\{\sum_{i=1}^{a} c_i^2\ \text{Var}(W_i)\right\}\left\{\sum_{i=1}^{a}\frac{1}{n_{i\cdot}}\sum_{s=1}^{b} s\sum_{t=1}^{b} n_{is}\,n_{it}(s-t)\right\}^2} \qquad (2.18)$$

To find optimal weights c_i to maximize ARE(W^*, W) in (2.18), we need the following lemma.

Lemma 1. (Lemma 1 of Skillings and Wolfe (1978)) Let $V=\left(\sum_{i=1}^{a} c_i G_i\right)^2\Big/\left(\sum_{i=1}^{a} c_i^2 B_i\right)$, where all $B_i>0$. Then V is maximized with respect to the c_i's by any solution of the system of equations

$$c_j=\left(G_j \sum_{i=1}^{a} c_i^2 B_i\right)\Big/\left(B_j \sum_{i=1}^{a} c_i G_i\right),\quad j=1,\cdots,a. \tag{2.19}$$

If in (2.18) we let

$$G_i=\frac{1}{n_{i\cdot}} \sum_{s=1}^{b} s \sum_{t=1}^{b} n_{is} n_{it}(s-t)$$

and $B_i=\mathrm{Var}(W_i)$, given by (2.17), then we find that ARE(W^*, W) in (2.18) is maximized with respect to the c_i's by any solution of the system of equation (2.19). Thus, by the above lemma, if $n_{ij}=n_{i\cdot}/b\geq 1$, ARE($W^*$, W) in (2.18) is maximal with respect to the c_i's when $c_i=n_{i\cdot}/b(n_{i\cdot}+1)$. Then the ARE of $W^{*\prime}$ (employing the optimal weights c_i) relative to W is of the form

$$\mathrm{ARE}(W^{*\prime}, W)=\lim_{a\to\infty}\frac{\left\{\sum_{i=1}^{a} n_{i\cdot}^2(n_{i\cdot}+1)^{-1}\right\}\left\{\sum_{i=1}^{a}(n_{i\cdot}+1)\right\}}{\left\{\sum_{i=1}^{a} n_{i\cdot}\right\}^2}$$

The efficacy of the weighted Hettmansperger statistic T^* can be obtained as follows.

$$\mathrm{eff}(T^*)=\lim_{a\to\infty}\frac{\sum_{i=1}^{e} e_{ib}\sum_{s=1}^{b} s\sum_{t=1}^{b} n_{it}(s-t)\int f^2(y)\,dy}{a\sigma_{T^*}}$$

where

$$\sigma_{T^*}^2=\sum_{i=1}^{a} e_i^2\,\mathrm{Var}(T_i)$$

with

$$\mathrm{Var}(T_i)=\sum_{j=1}^{b} n_{i\cdot}(n_{i\cdot}+1)j^2/12n_{ij}-b^2(b+1)^2(n_{i\cdot}+1)/48. \tag{2.20}$$

Hence the ARE of T^* relative to T is given by

$$\mathrm{ARE}(T^*, T)=\lim_{a\to\infty}\frac{\left\{\sum_{i=1}^{a}\mathrm{Var}(T_i)\right\}\left\{\sum_{i=1}^{a} e_i\sum_{s=1}^{b} s\sum_{t=1}^{b} n_{it}(s-t)\right\}^2}{\left\{\sum_{i=1}^{a} e_i^2\,\mathrm{Var}(T_i)\right\}\left\{\sum_{i=1}^{a}\sum_{s=1}^{b} s\sum_{t=1}^{b} n_{it}(s-t)\right\}^2} \tag{2.21}$$

Using the above lemma, we can easily find the optimal weights to maximize ARE (T^*, T) in (2.21), The optimal weights are given by

$$e_j=\left(H_j \sum_{i=1}^{a} e_i^2 J_i\right)\Big/\left(J_j \sum_{i=1}^{a} e_i H_i\right),\quad j=1,\cdots,a, \tag{2.22}$$

where

$$H_i=\sum_{s=1}^{b} s\sum_{t=1}^{b} n_{is}(s-t)$$

and $J_i=\mathrm{Var}(T_i)$ is given by (2.20). Thus ARE(T^*, T) is maximized with respect to the e_i's by any solution of the system of equation (2.22). In particular, when $n_{ij}=n_{i\cdot}/b$, ARE($T^{*\prime}$, T) is equivalent to ARE($W^{*\prime}$, W).

3. Small Sample Monte Carlo Study

3.1 Design of the Simulation

This section treats the results of a small sample Monte Carlo study to compare the empirical powers of our proposed statistic W with the Hettmansperger statistic T, and the Puri's parametric statistic A. In this Monte Carlo study we compare the empirical significance levels and empirical powers of the test statistics for various distributions including the uniform, normal, contaminated normal, double exponential and Cauchy distributions. Here the cdf of an ε-contaminated normal distribution is given by

$$F(x)=(1-\varepsilon)\Phi(x)+\varepsilon\Phi(x/\sigma)$$

where $\Phi(x)$ is the cdf of a standard normal distribution.

To generate random variates in our simulation study, we used the package IMSL (International Mathematical and Statistical Libraries) on VAX 780. The uniform random variates are generated by the subroutine GGUBT. The normal random variates with and without contamination are also generated by using the subroutine GGNML. The inverse integral transformation is applied to generate the double exponential and Cauchy random variates. All computations are also carried out on VAX 780.

In our simulation study the number of blocks and treatments we have chosen are four and three, respectively. The cell sizes we have considered are the following three cases.

CASE A :	$n_{11}=3$	$n_{12}=5$	$n_{13}=7$
	$n_{21}=3$	$n_{22}=5$	$n_{23}=7$
	$n_{31}=3$	$n_{32}=5$	$n_{33}=7$
	$n_{41}=3$	$n_{42}=5$	$n_{43}=7$
CASE B :	$n_{11}=3$	$n_{12}=5$	$n_{13}=7$
	$n_{21}=4$	$n_{22}=6$	$n_{23}=8$
	$n_{31}=5$	$n_{32}=7$	$n_{33}=9$
	$n_{41}=6$	$n_{42}=8$	$n_{43}=10$
CASE C :	$n_{11}=18$	$n_{12}=12$	$n_{13}=6$
	$n_{21}=15$	$n_{22}=10$	$n_{23}=5$
	$n_{31}=12$	$n_{32}=8$	$n_{33}=4$
	$n_{41}=9$	$n_{42}=6$	$n_{43}=3$

We have considered the following equally-spaced treatment effects:

$$(\theta_1,\ \theta_2,\ \theta_3)=(-\delta\sigma,\ 0,\ \delta\sigma).$$

Here δ ranges from 0.0 to 0.5 with an increment of 0.1 and σ is a standard deviation of each population. In the case of Cauchy distribution σ is chosen to satisfy

$$\int_{-\sigma}^{\sigma}\frac{1}{\pi(1+x^2)}\,dx=0.6827=\Phi(1)-\Phi(-1).$$

Thus, the value of σ in this case is 1.8326. For the contaminated normal distribution, $\varepsilon=0.1$ and $\sigma=3.0$ are used.

For each sample generated according to the randomized block design model (1.1), the values of the test statistics A, T, K and W are calculated and compared with their respective critical values at significance level of $\alpha=0.05$. In our simulation, 1000 replications were performed for each value of design constants.

3.2 Monte Carlo Results

To compute the empirical powers of the test statistics discussed in this paper, we count the number of times that the null hypothesis H_0 is rejected for each combination of design constants. Then, the number of times of rejecting H_0 divided by 1000 is the empirical power. The empirical power when $\delta=0.0$ is the empirical significance level. The results of the empirical powers are tabulated in Table 1 (CASE A), Table 2 (CASE B), and Table 3 (CASE C).

It is observed from the tables that the behavior of the empirical powers of the tests depends on the underlying distributions. The parametric procedure A has the greatest empiricl power when the underlying distributions are uniform and normal. The Skillings and Wolfe test K and our proposed test W have almost the same powers for these short or medium tailed distributions in Case A. In CASE B and C, the test based on W is somewhat better than the test based on K. For the distributions with heavy tails, such as contaminated normal, double exponential and Cauchy distributions, the two rank tests based on K and W are better than the other two tests. Among K and W, our proposed test W has in most cases slightly better powers than the Skillings and Wolfe test K. The power of the parametric test A drops off dramatically for the Cauchy distribution as expected. For example, the powers of A remain around 0.2~0.3 when the powers of the nonparametric tests are around 0.9 for all CASE A, B and C. Our proposed test

W, which is an extension of the Hettmansperger test T, is always significanly better than the test based on T. Generally speaking, our proposed test W and the Skillings and Wolfe test K are quite close for all cases. These two tests are more robust than the other two tests. As a conclusion, the proposed test is slighty more efficient than the other tests and significantly more robust than the parametric test in the case of equally-spaced treatment effects.

Table 1. Empirical Powers of the Statistics T, K, W and A Based on 1000 Replications (CASE A)

distribution	δ	T	K	W	A
	0.0	0.041	0.041	0.042	0.053
	0.1	0.138	0.146	0.158	0.179
uniform	0.2	0.277	0.293	0.294	0.347
	0.3	0.465	0.492	0.512	0.597
	0.4	0.668	0.688	0.699	0.774
	0.5	0.835	0.854	0.860	0.935
	0.0	0.037	0.038	0.039	0.050
	0.1	0.132	0.132	0.135	0.157
normal	0.2	0.292	0.294	0.317	0.352
	0.3	0.501	0.521	0.536	0.598
	0.4	0.705	0.725	0.744	0.791
	0.5	0.861	0.877	0.886	0.917
	0.0	0.044	0.045	0.045	0.056
	0.1	0.150	0.161	0.164	0.169
contaminated normal	0.2	0.401	0.400	0.419	0.419
	0.3	0.629	0.641	0.639	0.598
	0.4	0.817	0.824	0.839	0.784
	0.5	0.939	0.946	0.950	0.901
	0.0	0.046	0.039	0.045	0.068
	0.1	0.169	0.164	0.180	0.175
double exponential	0.2	0.368	0.379	0.383	0.350
	0.3	0.618	0.640	0.651	0.593
	0.4	0.821	0.852	0.853	0.807
	0.5	0.931	0.951	0.951	0.920
	0.0	0.045	0.039	0.040	0.053
	0.1	0.142	0.140	0.155	0.080
Cauchy	0.2	0.299	0.304	0.307	0.096
	0.3	0.475	0.491	0.504	0.144
	0.4	0.663	0.678	0.697	0.206
	0.5	0.806	0.835	0.831	0.264

Table 2. Empirical Powers of the Statistics T, K, W and A Based on 1000 Replications (CASE B)

distribution	δ	T	K	W	A
	0.0	0.057	0.049	0.050	0.077
	0.1	0.162	0.171	0.165	0.203
uniform	0.2	0.380	0.375	0.389	0.459
	0.3	0.567	0.591	0.615	0.707
	0.4	0.792	0.825	0.833	0.906
	0.5	0.929	0.935	0.938	0.979
	0.0	0.060	0.063	0.057	0.074
	0.1	0.137	0.142	0.142	0.187
normal	0.2	0.343	0.345	0.358	0.442
	0.3	0.588	0.590	0.623	0.696
	0.4	0.815	0.834	0.850	0.891
	0.5	0.926	0.939	0.952	0.965
	0.0	0.048	0.051	0.053	0.060
	0.1	0.176	0.174	0.194	0.202
contaminated normal	0.2	0.422	0.430	0.451	0.446
	0.3	0.719	0.739	0.741	0.718
	0.4	0.901	0.915	0.932	0.888
	0.5	0.984	0.986	0.991	0.969
	0.0	0.058	0.056	0.055	0.079
	0.1	0.189	0.192	0.203	0.207
double exponential	0.2	0.497	0.507	0.509	0.466
	0.3	0.744	0.765	0.775	0.718
	0.4	0.905	0.921	0.917	0.885
	0.5	0.978	0.982	0.983	0.966
	0.0	0.057	0.059	0.056	0.082
	0.1	0.148	0.165	0.168	0.093
Cauchy	0.2	0.346	0.361	0.381	0.125
	0.3	0.614	0.624	0.633	0.162
	0.4	0.782	0.814	0.817	0.205
	0.5	0.892	0.915	0.917	0.250

Table 3. Empirical Powers of the Statistics T, K, W and A Based on 1000 Replications (CASE C)

distribution	δ	T	K	W	A
	0.0	0.051	0.046	0.043	0.069
	0.1	0.204	0.202	0.208	0.236
uniform	0.2	0.345	0.383	0.391	0.454
	0.3	0.675	0.692	0.711	0.768
	0.4	0.858	0.890	0.889	0.935
	0.5	0.962	0.963	0.972	0.998
	0.0	0.048	0.046	0.047	0.063
	0.1	0.170	0.180	0.199	0.225
normal	0.2	0.407	0.396	0.415	0.450
	0.3	0.683	0.688	0.723	0.758
	0.4	0.882	0.889	0.911	0.929
	0.5	0.967	0.969	0.979	0.982
	0.0	0.050	0.052	0.058	0.063
	0.1	0.235	0.226	0.233	0.240
contaminated normal	0.2	0.518	0.533	0.537	0.507
	0.3	0.829	0.819	0.849	0.784
	0.4	0.953	0.963	0.963	0.925
	0.5	0.996	0.999	0.997	0.985
	0.0	0.033	0.044	0.040	0.050
	0.1	0.234	0.237	0.242	0.223
double exponential	0.2	0.546	0.541	0.572	0.495
	0.3	0.810	0.830	0.846	0.758
	0.4	0.946	0.962	0.972	0.922
	0.5	0.991	0.993	0.997	0.986
	0.0	0.049	0.052	0.050	0.050
	0.1	0.190	0.190	0.202	0.090
Cauchy	0.2	0.419	0.432	0.461	0.107
	0.3	0.642	0.685	0.691	0.140
	0.4	0.850	0.858	0.878	0.182
	0.5	0.934	0.959	0.961	0.214

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