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An Enhancement of Dynamic Range for the Active Realization of Elliptic Filters

(能動 楕圓필터의 最大 動的範圍 實現化)

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要 約

타원함수는 $j\omega$ 축상에 일연의 영점(zero)들을 갖고있다. 따라서 타원함수를 능동회로망으로 실현할때에는 주어진 함수를 biquad의 적(product)으로 분해한다.

본 논문에서는 먼저 동적범위의 확장을 위한 간단한 pole-zero pairing방법을 제시한다. 그리고, 각 biquad의순서를 최적화 시키는 방안을 제시하며, 일반적으로 사용되는 순서최적화 방법이 타원함수의 경우에는 적용되지 않음을 보인다. 본 논문에서 제안된 것은 일종의 Leuder 방법에 대한 approximation에 해당된다.

Abstract

The elliptic functions have a set of zeros on the jw axis. In active realization we decompose a function into the product of biquads. In this paper, a simple method of pole-zero pairing is proposed for the enhancement of overall dynamic range. Secondly, the optimum sequencing of individual biquads is developed and it is demonstrated that the commonly accepted sequencing technique does not hold in the case of elliptic functions. This work is an approximation for Leuder's (1970) method.

I. Introduction

In the cascade realization of active networks, the given function of order n must be decomposed into a number of biquads (for n even) or into a first-order function in addition to biquads (for n odd). In the decomposition, there are the problems of pairing the poles and zeros, sequencing of individual biquads and the gain distribution. We may therefore optimize the decomposition process in order to enhance an important performance measure such as the dynamic range or sensitivity.

Lueder decomposed the function and maximized the dynamic range of a whole network while minimizing inband losses (1970)[1], and also proposed the use of a dynamic programming method to simultaneously implement gain distribution (1975)[2]. Halfin (1970)[3,4] solved a similar proplem taking a slightly different approach. Moschytz (1970) utilized the degrees of freedom of the given problem to minimize the transmission sensitivity and suggested the second-order pole-zero pairing method.

In this paper we deal with the pole-zero pairing, gain distribution and sensitivity minimization of an elliptic filter making use of specific properties pertaining to elliptic func-

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tions in general. Elliptic functions have transmission zeros at specific points on the jw axis (Figs. 1 (*a*) and (*b*)). A lowpass filter realized from an elliptic function produces the magnitude characteristic which is an equal ripple in both the passband and the stopband as shown in figs. 1(*c*) and (*d*). As a consequence, the elliptic filter is optimal, in the sense that for a given order n and given ripple specifications in the passband and stopband, the transition band is minimal.

We shall introduce a systematic procedure to generate an *F matrix* from which the pairing of poles and zeros are assigned so as to yield each biquad having the flattest possible magnitude response. Subsequently, the particular cascading sequence that results in the transfer functions, from the first input to the biquad outputs, having the flattest magnitude will be developed for the case of elliptic functions with reference to the Q of individual biquads.

Owing to the nature of the elliptic function, every pair of poles must take a pair of finite zeros on the jw axis to form a lowpass notch function. A relatively simple method pole-zero pairing will be developed to improve the overall dynamic range of the realized filter.

II. Elliptic functions and pole-zero pairing

Elliptic functions, first introduced by Cauer, are rational functions of order n in which the degree of the numerator is equal to that of the denominator for even n, and one less than that of the denominator for odd n. Since poles p_i and zeros z_i are tabulated (Huelsman and Allen 1980)[8] we may write the function in factored form for active realization as follows

$$T(s) = K \prod_{i,j}^{n/2} t_{ij}(s) = K \prod_{i,j}^{n/2} \frac{(s-z_j)(s-z_j^*)}{(s-p_i)(s-p_i^*)} \quad \text{for even } n \quad (1a)$$

$$T(s) = K t_0(s) \prod_{i,j}^{(n-1)/2} t_{ij}(s) = K \frac{1}{s+\sigma} \prod_{i,j}^{(n-1)/2} \frac{(s-z_j)(s-z_j^*)}{(s-p_i)(s-p_i^*)} \quad \text{for odd } n \quad (1b)$$

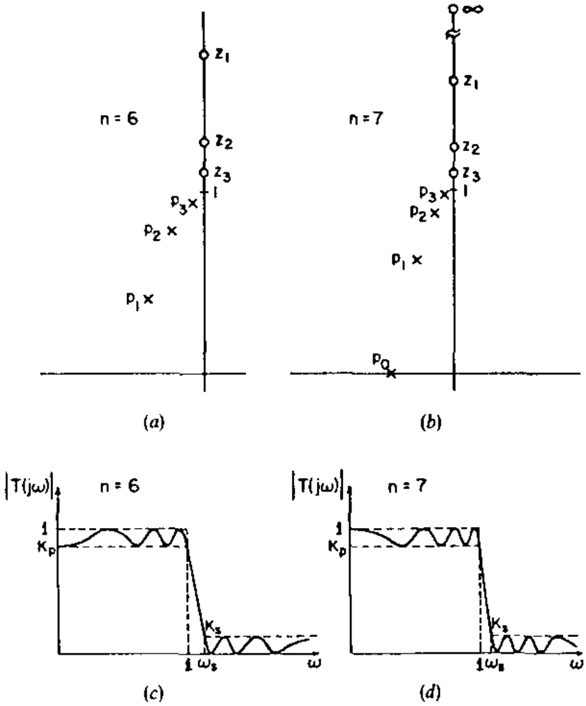


Fig. 1. Pole-zero plot of typical elliptic function of (*a*) n=6 and (*b*) n=7. Magnitude characteristic for (*c*) n=6 and (*d*) n=7.

where $|p_i| < \omega_s < |z_j|$, and ω_s is the frequency at which the stopband begins. To be general, the subscripts i and j may taken on any combination of 1, 2, 3, ... as shown in Figs. 1 (*a*) and (*b*).

For specified passband ripple K_p, stopband ripple K_s and stopband frequency ω_s as illustrated in Figs. 1(*c*) and 1(*d*), we can refer to a table (Huelsman and Allen) and find the order n, poles p_i and zeros z_j. Subsequently we write the desired function in the form of (1) for active cascade realization of the individual lowpass notch functions of the type

$$t_{ij}(s) = \frac{(s-z_j)(s-z_j^*)}{(s-p_i)(s-p_i^*)} = \frac{s^2+\omega_{zj}^2}{s^2+\dfrac{\omega_{pi}}{Q_i}s+\omega_{pi}^2} \quad (2)$$

where $\omega_{zj} = |z_j|$ and $\omega_{pi} = |p_i|$, and Q_i is the Q of pole p_i. The above implies various ways of associating the pair of zeros to a given pair of poles. There are $(n/2)!$ (for even n) combinations that suggest a possible optimum pairing for maximum dynamic range. This problem will be solved by pairing the poles and zeros such that the magnitude of each biquad $t_{ij}(s)$ is as flat as possible over the filter passband. The magnitude and the frequency which make the magnitude maximum are obtained as follows

$$|t_{ij}| = \frac{\omega_{zj}^2 - \omega^2}{\left[(\omega_{pi}^2 - \omega^2)^2 + \left(\frac{\omega_{pi}}{Q_i}\right)^2 \omega^2\right]^{1/2}} \tag{3a}$$

$$\omega_{mij} = \left[\frac{\left(\frac{\omega_{zj}}{\omega_{pi}}\right)^2\left(1 - \frac{1}{2Q_i^2}\right) - 1}{\left(\frac{\omega_{zj}}{\omega_{pi}}\right)^2 + \frac{1}{2Q_i^2} - 1}\right]^{1/2} \omega_{pi} \tag{3b}$$

$$|t_{ij}|_{\max} = |t_{ij}|_{\omega = \omega_{mij}} \tag{3c}$$

Owing to the nature of elliptic functions, the minimum value $|t_{ij}|_{\min}$ occurs at the edge of the passband, namely at ω=0 or ω=1 depending on the value of the pole Q. In order to find the condition for Q_i such that $|t_{ij}|_{\min}$ occurs at ω=0, we impose $|t_{ij}(0)| \leqslant |t_{ij}(1)|$ from which we obtain the condition

$$Q_i \geqslant \frac{\frac{\omega_{pi}}{(\omega_{zj}^2 - 1)}}{\left[\left(\frac{\omega_{pi}}{\omega_{zj}}\right)^4 - \left(\frac{\omega_{pi}^2 - 1}{\omega_{zj}^2 - 1}\right)^2\right]^{1/2}} \equiv \alpha_{ij} \tag{4}$$

Thus, we have from (3*a*) and (4) that

$$|t_{ij}|_{\min} = \left(\frac{\omega_{zj}}{\omega_{pi}}\right)^2 \quad \text{if } Q_i \geqslant \alpha_{ij} \tag{5a}$$

and

$$|t_{ij}|_{\min} = \frac{\omega_{zj}^2 - 1}{\left[(\omega_{pi}^2 - 1)^2 + \left(\frac{\omega_{pi}}{Q_i}\right)^2\right]^{1/2}} \quad \text{otherwise}$$

otherwise (5b)

To find optimum pole-zero pairing let us define the flatness matrix *F and* its elements as follows (we can take the case of even n without lack of generality)

	z_1	z_2	$\cdots$	$z_{n/2}$
p_1	f_{11}	f_{12}	$\cdots$	$f_{1(n/2)}$
p_2	f_{21}	f_{22}	$\cdots$	$f_{2(n/2)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$p_{n/2}$	$f_{(n/2)1}$	$f_{(n/2)2}$	$\cdots$	$f_{(n/2)(n/2)}$

$$F = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1(n/2)} \\ f_{21} & f_{22} & \cdots & f_{2(n/2)} \\ \vdots & \vdots & & \vdots \\ f_{(n/2)1} & f_{(n/2)2} & \cdots & f_{(n/2)(n/2)} \end{bmatrix}$$

where

$$f_{ij} = \frac{|t_{ij}|_{\min}}{|t_{ij}|_{\max}}$$

As shown in Fig.2, the normalized magnitude $|t_{ij}|_n = |t_{ij}|/|t_{ij}|_{\max}$ is plotted for various Q_i for the typical cases of n=6 and n=7 for illustrative purpose with j=1 as a special case.

Because the maximum value of t_{ij} is unity for each biquad and f_{ij} has its minimum in the passband such that $0 < f_{ij} < 1$, the elements in F are the direct indices of the flatness of all possible pole-zero pairings. The diterminant of the F matrix is taken and expanded into $(n/2)!$ terms, each term consisting of $n/2$ elements. If the element of minimum value in the kth term is greater than the minimum elements of the $[(n/2)!-1]$ remaining terms, then the pole-zero pairings corresponding to the subscripts of the K_{th} term yield maximum dynamic range.

Theorem 1

Let the poles p_i be numbered beginning with the lowest Q pole, and let the zeros be numbered beginning with the farthest on the $j\omega$ axis as shown in Figs. 1(*a*) and (*b*). The optimum pole-zero pairing for the enhancement of the dynam-

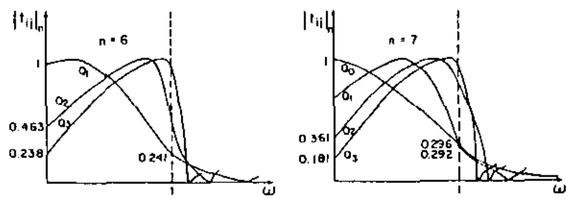


Fig. 2. Plots of the normalized magnitude of lospass notch sections.

ic range coincides with the term consisting of the diagonal elements of the flatness matrix F.

Thus the elliptic functions may be written as

$$T(s) = K \prod_{i=1}^{n/2} t_i(s) = K \prod_{i=1}^{n/2} \frac{(s-z_i)(s-z_i^*)}{(s-p_i)(s-p_i^*)} \quad \text{for even } n \qquad (7a)$$

$$T(s) = Kt_0 \prod_{i=1}^{(n-1/2)} t_i(s) = K \frac{1}{s+\sigma} \prod_{i=1}^{(n-1/2)} \frac{(s-z_i)(s-z_i^*)}{(s-p_i)(s-p_i^*)} \quad \text{for odd } n \qquad (7b)$$

Proof

Obviously the biquad of the highest Q produces the largest maximum value. Therefore we first examine the functions of the highest Q role pair, i.e. $p_{n/2}$.

Since $Q_{n/2} > \alpha_{(n/2)j}$ we use (5a) to write

$$|t_{(n/2)j}|_{\min} = (\omega_{zj}/\omega_{p(n/2)})^2$$

$$|t_{(n/2)j}|_{\max} = \frac{\omega_{zj}^2\left[1-\left(\frac{\omega_m}{\omega_{zj}}\right)^2\right]}{\left[(\omega_{p(n/2)}^2-\omega_m^2)^2+\left(\frac{\omega_{p(n/2)}}{Q_{n/2}}\omega_m\right)^2\right]^{1/2}}$$

Noting that $\omega_{p(n/2)} \approx 1$, we now obtain the flatness by

$$f_{(n/2)j} = \frac{|t_{(n/2)j}|_{\min}}{|t_{(n/2)j}|_{\max}} = \frac{1}{\left[1-\left(\frac{1}{\omega_{zj}}\right)^2\right]Q_{n/2}}$$

where ω_{zj} is closest to unity when $j=n/2$, making the flatness largest. Thus $p_{n/2}$ should be paired with $z_{n/2}$. A similar argument can be followed for the remaining poles

$$p_{(n/2)-1},\ p_{(n/2)-2},\ \cdots,\ p_1$$

The above theorem confirms the results previously obtained. This development differs from Lueder's in the sense that the F matrix is introduced to the special case of elliptic filter functions. A simultaneous method is not applied for special functions such as elliptic functions.

III. Sequencing of biquads of lowpass notch type

For cascading biquads of lowpass notch type, there are $(n/2)!$ possible sequences. The objective in this section is to find the particular sequence that yields the flattest possible magnitude response from the filter input to the various outputs of the lowpass notch type biquads in succession. Let the ith biquad be denoted by $t_i(s)$. Implementing Theorem 1, we let $j=1$ in (3) and (4) to rewrite

$$|t_i| = \frac{\omega_{zi}^2-\omega^2}{\left[(\omega_{pi}^2-\omega^2)^2+\left(\frac{\omega_{pi}}{Q_i}\right)^2\omega^2\right]^{1/2}} \qquad (8a)$$

$$\omega_{mi} = \left[\frac{\left(\frac{\omega_{zi}}{\omega_{pi}}\right)^2\left(1-\frac{1}{2Q_i^2}\right)-1}{\left(\frac{\omega_{zi}}{\omega_{pi}}\right)^2+\frac{1}{2Q_i^2}-1}\right]^{1/2}\omega_{pi} \qquad (8b)$$

$$\alpha_i = \frac{\omega_{pi}}{(\omega_{zi}^2-1)\left[\left(\frac{\omega_{pi}}{\omega_{zi}}\right)^4-\left(\frac{\omega_{pi}^2-1}{\omega_{zi}^2-1}\right)^2\right]^{1/2}} \qquad (8c)$$

where ω_{mi} is the frequency at which $|t_i|$ becomes maximum, and α_i is a criterion such that if $Q_i \geqslant \alpha_i$, $|t_i|_{\min}$ occurs at $\omega=0$.

In order to develop a theorem for the cascading sequence of the notch type biquads, let us first assume that the order n is even, and further classify into two cases (for $k=1, 2, \cdots$)

Case (a) $n=4k$
Case (b) $n=4k-2$

Theorem 2

Let an elliptic function of even order n be expressed as a product of $n/2$ biquads as

$$T(s) = K \prod_{i=1}^{n/2} t_i(s)$$

where the subscript is in the order of increasing Q. The sequence of biquads for the optimum dynamic range in cascade realization is:

Case (*a*)

$$T(s) = Kt_{n/4} \cdot t_{(n/4)+1} \cdot t_{(n/4)-1} \cdot t_{(n/4)+2} \cdot t_{(n/4)-2} \cdots$$
$$\text{for } n = 4k, \quad k = 1, 2, \ldots \qquad (9a)$$

Case (*b*)

$$T(s) = Kt_{[(n+2)/4]} \cdot t_{[(n+2)/4]-1} \cdot t_{[(n+2)/4]+1} \cdot t_{[(n+2)/4]-2} \cdot t_{[(n+2)/4]+2} \cdots$$
$$\text{for } n = 4k-2, \quad k = 1, 2, \ldots \qquad (9b)$$

Proof

Owing to the lowpass notch characteristic of the biquad, the maximum magnitude occurs in the passband, and the passband minimum occurs at the edges, namely at ω=0 or ω=1. Let us now normalize the magnitude of each biquad as shown in Fig. 3, where Q_i of $t_i(s)$ is such that

$$\underbrace{Q_1 < Q_2 \cdots < Q_{n/4}}_{\text{first group}} < \underbrace{Q_{(n/4)+1} < \cdots < Q_{n/2}}_{\text{second group}} \qquad (10)$$

and

$$0 < \frac{|t_i|}{|t_i|_{max}} \leqslant 1 \quad \text{for } 0 \leqslant \omega \leqslant 1 \qquad (11)$$

For $Q_i \geqslant \alpha_i (i = (n/4)+1, \ldots n/2)$ the flatness f_i is determined by (14a) or closely approximated by (14*b*). In (14*b*), $(\omega_{pi}/\omega_{zi})$ is smaller than unity owing to the lowpass notch characteristic, and it decreases as *i* increases. Furthermore, since Q_i increases as *i* increases, it is obvious that the flatness f_i decreases as *i* increases. For $Q_i < \alpha_i$ (i=1,2, ... $n/4$), we use (15) to show that the flatness increases with increasing Q as shown in Fig. 3 until the maximum flatness is attained with the $Q_{n/4}$ which is located at the midpoint in (10). For the optimal cascading sequence, therefore, we start with the biquad $t_{n/4}$ corresponding to $Q_{n/4}$ from the first group which yields the minimum value at ω = 1. The next biquad in the cascade must be $t_{(n/4)+1}$ corresponding to $Q_{(n/4)} + 1$ from the second group which produces the

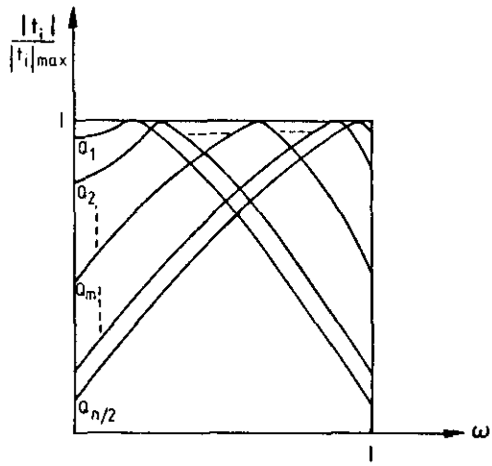


Fig. 3. Normalized magnitude of the biquads in which Q_m corresponding to midpoint Q exhibits the highest flatness. In case (*a*) m=n/4, and in case (*b*) m=(n+2)/4.

minimum at ω = 0. It can be shown that the magnitude $|t_{n/4} \cdot t_{(n/4)+1}|$ is the flattest among the magnitudes of the product of any two biquads, and the minimum value occurs at ω=0. This is turn necessitates the choice of the next closest biquad $t_{(n/4)-1}$ from the first group as the third biquad in the cascade. This process of choosing the biquads based on the normalized magnitude of biquads versus pole Q completes the sequence as shown in (9*a*). It is also to be noted that the minimum of the magnitude of the product of any number of biquads occurs at the frequency where the last biquad in the chain becomes minimum.

In case (*b*). the midpoint Q is $Q_{(n+2)/4}$, and the corresponding biquad $t_{n/4}$ yields the flattest magnitude. Furthermore, for $Q \geqslant Q_{(n+2)/4}$ all biquads produce a minimum at ω= 0 and for $Q < Q_{(n+2)/4}$ at ω=1. We start with $t_{(n+2)/4}$ and follow the same development as in case (*a*) to produce the optimum sequence as given by (9*b*).

Replacing ω in (8*a*) by ω_{mi} of (8*b*), and carrying out a straightforward operation, we derive the maximum magnitude $|t_i|_{max}$ explicitly in terms of the known parameters ω_{zi}, ω_{pi} and Q_i as

$$|t_i|_{\max} = \left[\left(\frac{\omega_{zi}}{\omega_{pi}}\right)^4 - \left(2 - \frac{1}{4Q_i^2}\right)\left(\frac{\omega_{zi}}{\omega_{pi}}\right)^2 + 1\right]^{1/2} \frac{Q_i}{\left(1 - \frac{1}{4Q_i^2}\right)^{1/2}} \quad (12)$$

The minimum value $|t_i|_{\min}$, on the other hand, depends on the value of Q_i in comparison with threshold α_i defined by (8c)

$$|t_i|_{\min} = |t_i(0)| = \left(\frac{\omega_{zi}}{\omega_{pi}}\right)^2 \quad \text{if } Q_i \geq \alpha_i \quad (13a)$$

$$|t_i|_{\min} = |t_i(1)| = \frac{\omega_{zi}^2 - 1}{\left[(\omega_{pi}^2 - 1)^2 + \left(\frac{\omega_{pi}}{Q_i}\right)^2\right]^{1/2}}$$

otherwise (13b)

The flatness index is now obtained as

$$f_i = \frac{|t_i|_{\min}}{|t_i|_{\max}} = \frac{\left(1 - \frac{1}{4Q_i^2}\right)^{1/2}}{\left[1 - \left(2 - \frac{1}{4Q_i^2}\right)\left(\frac{\omega_{pi}}{\omega_{zi}}\right)^2 + \left(\frac{\omega_{pi}}{\omega_{zi}}\right)^4\right]^{1/2} Q_i}$$

$$\text{for } Q_i \geq \alpha_i \quad (14a)$$

Since $Q_i \gg 1$, we may approximate f_i by

$$f_i = \frac{|t_i|_{\min}}{|t_i|_{\max}} = \frac{1}{\left[1 - \left(\frac{\omega_{pi}}{\omega_{zi}}\right)^2\right] Q_i} \quad \text{for } Q_i \geq \alpha_i \quad (14b)$$

For $Q_i < \alpha_i$ we use (12) and (13b) to write

$$f_i = \frac{(\omega_{zi}^2 - 1)\left(1 - \frac{1}{4Q_i^2}\right)^{1/2}}{\left[\left(\frac{\omega_{zi}}{\omega_{pi}}\right)^4 - \left(2 - \frac{1}{4Q_i^2}\right)\left(\frac{\omega_{zi}}{\omega_{pi}}\right)^2 + 1\right]^{1/2} \left[(\omega_{pi}^2 - 1)^2 + \left(\frac{\omega_{pi}}{Q_i}\right)^2\right]^{1/2} Q_i} \quad (15)$$

When $n = 10$, for example, we write five biquads in the order of pole Q_s to find the cascading sequence as indicated below by bold numbers

$$T(s) = K t_1(s) \cdot t_2(s) \cdot t_3(s) \cdot t_4(s) \cdot t_5(s)$$

4 **2** **1** **3** **5**

← cascading sequence

For $n = 12$, we have

$$T(s) = K t_1(s) \cdot t_2(s) \cdot t_3(s) \cdot t_4(s) \cdot t_5(s) \cdot t_6(s)$$

5 **3** **1** **2** **4** **6**

When n is odd, the first-order function $t_0(s) = 1/(s+\sigma)$ should be properly treated in relation to biquads. If we place t_0 in front in the sequence as $t_0 \cdot t_2 \cdot t_1 \cdot t_3$, then the minimum of $|t_2|$ occurs at $\omega=0$ while the minimum of the product $|t_0 \cdot t_2|$ does not occur at $\omega=0$ but at $\omega=1$. Thus it contradicts the optimum sequencing method proposed in theorem 2. On the other hand, if we place t_0 as last entry to the sequence as $t_2 \cdot t_1 \cdot t_3 \cdot t_0$, the procedure coincides exactly as presented in theorem 2. This leads to the realization of the first order section at the last stage of the cascade.

IV. Illustrative examples

Example 1

Find the pole-zero pairing, cascading sequence and the gain distribution for the optimum dynamic range realization of the elliptic function under the specification $K_p = 1$ dB, $K_s \geq 40$ dB, and $\omega_s = 1.1$.

From the standard table we find $n = 6$, which yields a pole-zero location as shown in Fig.4(a).

For the purpose of confirmation let us construct the F matrix.

	z_1 $j2.970935$	z_2 $j1.309230$	z_3 $j1.115061$
p_1 $-0.315089+j0.409244$ ($Q_1=0.8195$)	f_{11} 0.2433	f_{12} 0.1171	f_{13} 0.0610
p_2 $-0.118730+j0.874514$ ($Q_2=3.7165$)	f_{21} 0.2913	f_{22} 0.4821	f_{23} 0.3105
p_3 $-0.023927+j0.999416$ ($Q_3=20.8908$)	f_{31} 0.0540	f_{32} 0.1143	f_{33} 0.2425

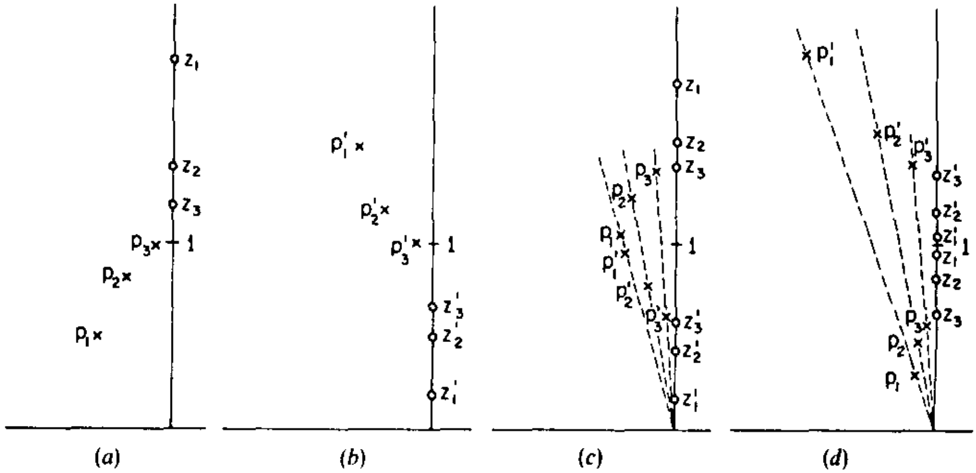


Fig. 4. Schematic pole-zero locations:
(*a*) Elliptic low-pass function of order 6.
(*b*) Elliptic high-pass function of order 6.
(*c*) Elliptic band-pass function of order 12.
(*d*) Elliptic band-stop function of order 12.

The determinant of the above matrix consists of six terms, namely $f_{11}\cdot f_{22}\cdot f_{33}$, $f_{13}\cdot f_{21}\cdot f_{32}$, $f_{12}\cdot f_{23}\cdot f_{31}\cdot$ $f_{13}\cdot f_{22}\cdot f_{31}$, $f_{11}\cdot f_{23}\cdot f_{32}$ and $f_{12}\cdot f_{21}\cdot f_{33}$, where f_{ij} is computed by (6*c*), (3*c*) and (5). The minimum elements in each of the six terms are $f_{33} = 0.2425, f_{13} = 0.0610$, $f_{31} = 0.0540$, $f_{32} = 0.1143$, and $f_{12} = 0.1171$. The largest among the six elements is f_{33}, and we therefore take the term containing f_{33}, i.e. $f_{11}\cdot f_{22}\cdot f_{33}$.
Subsequently the three biquads are identified as

$$t_1(s) = \frac{(s - z_1)(s - z_1^*)}{(s - p_1)(s - p_1^*)}$$

$$= \frac{(s^2 + 8{\cdot}826455)}{(s^2 + 0{\cdot}630179s + 0{\cdot}266762)}$$

$$t_2(s) = \frac{(s - z_2)(s - z_2^*)}{(s - p_2)(s - p_2^*)}$$

$$= \frac{(s^2 + 1{\cdot}714083)}{(s^2 + 0{\cdot}237461s + 0{\cdot}778873)}$$

$$t_3(s) = \frac{(s - z_3)(s - z_3^*)}{(s - p_3)(s - p_3^*)}$$

$$= \frac{(s^2 + 1{\cdot}243362)}{(s^2 + 0{\cdot}047854s + 0{\cdot}999404)}$$

The normalized magnitude of each biquad is plotted in Fig. 2(a).

Using Theorem 2, we take the first three biquads of (9b) in the following sequence

$$\frac{V_2}{V_1} = T(s) = Kt_2(s)\cdot t_1(s)\cdot t_3(s)$$

In order to further enhance the dynamic range we may distribute the gain K to individual biquads as

$$T(s) = k_2\, t_2(s)\cdot k_1 t_1(s)\cdot k_3\, t_3(s)$$

The optimum gain distribution among the three biquad sections of the cascade will be performed so as to yield equal-magnitude maximum at the output of all intermediate transfer functions. Assuming max $|T(j\omega)| = 1$, and therefore $|T(j0)| = 0.891251$, we obtain $K = 0.0098$. To make the maximum of the first stage equal to the maximum of the overall network, we determine

$$k_2 = \frac{\max|T(j\omega)|}{\max|t_2(j\omega)|} = 0{\cdot}210712$$

$$k_1 = \frac{\max|T(j\omega)|}{\max|k_2 t_2(j\omega)t_1(j\omega)|}$$

$$= \frac{k_2 \max|t_2(j\omega)|}{k_2 \max|t_2(j\omega)t_1(j\omega)|} = 0{\cdot}060907$$

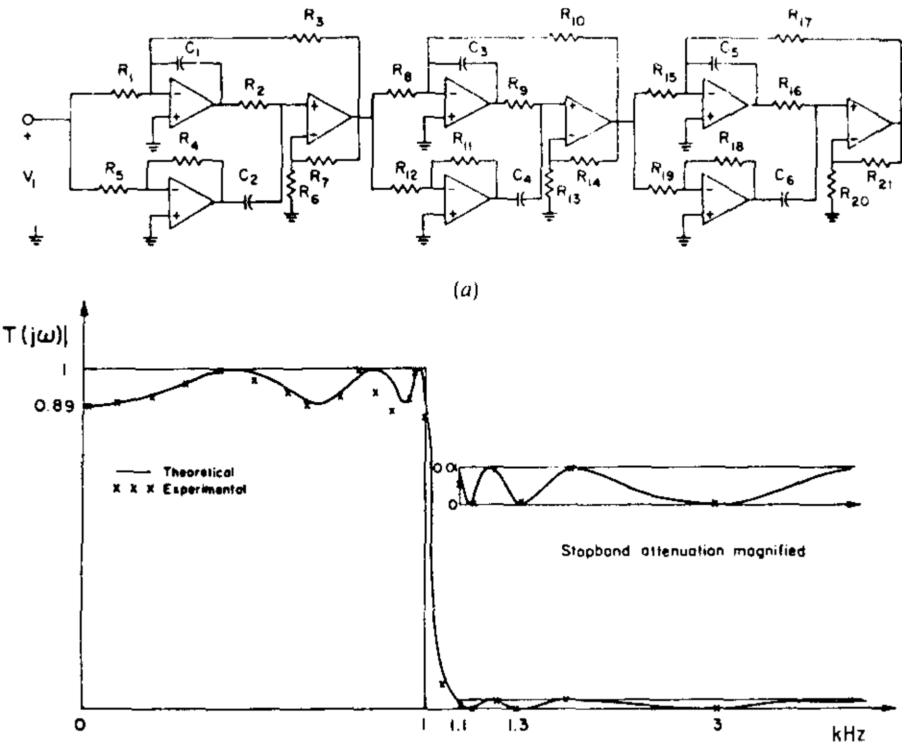
$$k_3 = \frac{K}{k_2 k_1} = 0{\cdot}762437$$

Another approach is possible that takes the sensitivity into consideration. By selecting a pair of zeros as far apart as possible from apair of poles, it has been shown (Moschytz 1970) that the function sensitivity $S_x^{t_i(s)}$ can be reduced. Subsequently, the dynamic range will be maximized through the optimum sequencing of individual biquads and the gain distribution. It can be shown that the optimum sequencing remains the same in spite of different pole-zero pairing adopted in the second approach.

Example 2

Realize the elliptic filter of the same specifications as in Example 1. Poles and zeros are to be paired to minimize the function sensitivity, and the biquad notch functions are to be sequenced so as to render maximum dynamic tange.

The pole of highest Q will be paired with the farthest zero to generate biquads different from (16)



(a)

(b)

Fig. 5 (a) Active realization of the sixth-order elliptic filter of Example 2.
(b) Magnitude characteristics.

$$t_1(s) = \frac{s^2 + 1.243362}{s^2 + 0.630179s + 2.66762}$$

$$t_2(s) = \frac{s^2 + 1.714083}{s^2 + 0.237461s + 0.778873} \qquad (17a)$$

$$t_3(s) = \frac{s^2 + 8.826455}{s^2 + 0.047854s + 0.999404}$$

The sequencing of biquads for optimum dynamic range is conducted to produce the same result as in Example 1

$$T(s) = k_2 t_2(s) \cdot k_1 t_1(s) \cdot k_3 t_3(s) \qquad (17b)$$

The gain distribution is carried out to further enhance the dynamic range of the whole network

$K_2 = 0.210712, K_1 = 0.459223,$
$K_3 = 0.101123$

Recently an active circuit has been proposed (Moore et al. 1980) using three operational amplifiers to realize each lowpass notch section in (17). The cascaded circuit and its magnitude characteristic are shown in figs. 5(a) and (b), respectively. It has been observed that other possible sequences considerably diminish the dynamic range. The cutoff frequency is denormalized to 1000Hz. Element values are listed below. Raytheon RC4156DC type operational amplifiers are used.

C_1=C_2=0.02μF	C_3=C_4=0.02μF	C_5=C_6=9.88nF
R_1=33,512 (Ω)	R_9=12,628 (Ω)	R_{17}=336,624(Ω)
R_2=R_3=9,017	R_{10}=R_{11}=15,407	R_{18}=R_{19}=16,114
R_4=159,041	R_{12}=27,498	R_{20}=3,328,856
R_5=43,793	R_{13}=33,551	R_{21}=159,347
R_6=19,445	R_{14}=7,198	R_{22}=18,043
R_7=R_8=9,017	R_{15}=R_{16}=15,407	R_{23}=R_{24}=16,114

All resistance values are in ohms.

V. Conclusions

A simple method of pole-zero pairing has been proposed for the active cascade realization of elliptic functions. A flatness matrix has been introduced to advance a theorem which leads to the maximum dynamic range. The problem of sequencing lowpass notch type biquads has been solved, and it has been demonstrated that the generally accepted sequencing techniques do not hold for elliptic functions.

By using the nature of the frequency transformation, the techniques and results obtained for the lowpass case may be applied to the optimization of highpass and bandstop elliptic filters.

The proposed method may be extended to switched-capacitor realization of elliptic functions using the procedures like that advanced mordern technology.

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