FINITENESS OF INTEGRAL DIFFERENTIAL FORMS

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1. Preliminaries

Let $A$ be a commutative ring with unit, and $M$ an $A$-module. Suppose that $T(M)$ is a tensor algebra of $M$ and $T(M)^*$ the dual module of $T(M)$. Let

$$K_n(M) = \{ f : f \in T(M)^*, f | T_k(M) = 0 \text{ for all } k \neq n \},$$

and

$$K(M) = \sum_{n=0}^{\infty} K_n(M).$$

Then $K(M)$ is a submodule of $T(M)^*$ and the sum is direct. $K(M)$ can be made into an algebra by defining multiplication as follows: Let $N$ and $P$ be $A$-modules, and $N^*$ and $P^*$ dual modules. For $f \in N^*$, $g \in P^*$, define $f \ast g \in (N \otimes P)^*$ by $(f \ast g)(a \otimes b) = f(a)g(b)$, $a \in N$, $b \in P$.

To define a multiplication in $K(M)$, let

$$\gamma_{n,m} : T_n(M) \otimes T_m(M) \rightarrow T_{n+m}(M)$$

be the canonical isomorphism, i.e., $x \otimes y \mapsto xy$, $x \in T_n(M)$, $y \in T_m(M)$. Let $\eta_{n,m} = \gamma_{n,m}^{-1}$, and $\eta_{n,m}^*$ be the dual homomorphism. For $f, g \in K(M)$, we define a product by

$$fg = \sum_{n,m} \eta_{n,m}^* ((f \circ j_n) \ast (g \circ j_m)) \circ p_{n+m},$$

where $j_n : T_n(M) \rightarrow T(M)$, $j_m : T_m(M) \rightarrow T(M)$ are natural injections and $p_{n+m} : T(M) \rightarrow T_{n+m}(M)$ is the $(n+m)$ th projection. $K(M)$ is called the algebra of multilinear forms on $M$.

Let $R$ be an integral domain, $K$ a field containing $R$. $\mathcal{D}_{K/R}$ will denote the $K$-module of all $R$-derivations $D : K \rightarrow K$. The algebra of multilinear forms on $\mathcal{D}_{K/R}$ is called the algebra of differential forms on $K/M$. 

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(on \( K \)), and will be denoted by \( D(K/R) \), and the submodule of homogeneous elements of degree \( k \) by \( D_k(K/R) \). The elements of \( D(K/R) \) are called differential forms (on \( K \)). Let \( d : K \rightarrow D_1(K/R) \) be an \( R \)-derivation defined by \( d(a)(D) = D(a) \) for \( D \in \Omega_{K/R} (= T_1(\Omega_{K/R})) \) and \( d(a) |_{T_k(\Omega_{K/R})} = 0 \) for \( n \geq 1 \). A differential form \( x \in D(K/R) \) will be said to be integral if \( x \in \sum S(dS)^k \) for all valuation rings \( S \) in \( K \) containing \( R \). Here \( S(dS)^k = \{ \sum s_0s_1...dS_k | s_i \in S \} \). Integral differential forms in \( D_k(K/R) \) are called homogeneous differential forms of degree \( k \).

2. Finiteness

In this section, \( R \) will denote a Noetherian integrally closed domain; \( P=R[x_1,...,x_n] \) a polynomial ring over \( R \) in \( x_1,...,x_n \); \( K_0 \) a field of quotients of \( P \); and \( K \) a finite separable algebraic extension of \( K_0 \). It is well known that \( D_1(K/R) \), in this case, is a vector space over \( K \) with \( \{ dx_1,...,dx_n \} \) as basis, where \( d : K \rightarrow D_1(K/R) \) is an \( R \)-derivation as is defined in Section 1; and \( D(K/R) \) is isomorphic to a tensor algebra of \( D_1(K/R) \). Hence a homogeneous differential form of degree \( k \) is uniquely expressed in the form

\[ x = \sum a_i dx_{i_1}...dx_{i_k}, \quad a_i \in K, \quad 1 \leq i_1,...,i_k \leq n. \]

The main result in this section is that the \( R \)-module of homogeneous differential forms of degree \( k \) is finitely generated.

Since \( K \) is a finite separable algebraic extension of \( K_0 \), let \( K=K_0(\alpha) \) and \( f(X)=X^m+a_1X^{m-1}+...+a_m, \quad a_i \in K_0, \) be the minimal polynomial of \( \alpha \). Suppose that \( S_0 \) is a discrete rank one valuation ring in \( K_0 \) containing \( P \), and let \( \mathcal{A} \) be the set of all valuation rings in \( K \) which are extensions of \( S_0 \). \( \bar{S}_0 \) will denote the integral closure of \( S_0 \) in \( K \).

**Lemma 1.** If \( \alpha \in \bar{S}_0 \),

\[ (f'(\alpha))^{3k+1} (\bigcap_{S \in \mathcal{A}} S(dS)^k) \subseteq \sum S_0[\alpha]dx_{i_1}...dx_{i_k}, \]

where \( 1 \leq i_1,...,i_k \leq n \).

**Proof.** \( \alpha \in \bar{S}_0 \) implies that \( f(X) \in S_0[X] \). Since \( f(\alpha)=0 \), and \( a_i \in S_0 \) which is a localization of \( P \),

\[ 0 = d(f(\alpha)) = f'(\alpha)d\alpha + \sum_{i=1}^{m-1} \alpha^{m-i}d(a_i). \]
where \( \alpha^n_i d(a_i) \in \sum_{i=1}^n S_0 \, dx_i. \)

It follows that

\[
(1) \quad f'(\alpha) d\alpha \in \sum_{i=1}^n S_0 dx_i.
\]

It is well known that (c.f. \([3]\))

\[
(2) \quad f'(\alpha) S_0 \subseteq S_0[\alpha]
\]

Hence, for any \( b \in S_0 \),

\[
d(f'(\alpha) b) \equiv d(S_0[\alpha]) \subseteq \sum_{i=1}^n S_0[\alpha] dx_i + S_0[\alpha] d\alpha.
\]

On the other hand,

\[
d(f'(\alpha) b) = f'(\alpha) db + bd(f'(\alpha)),
\]

where \( bd(f'(\alpha)) \in \sum_{i=1}^n S_0 dx_i + S_0 d\alpha. \)

Hence, \( f'(\alpha) db \subseteq \sum_{i=1}^n S_0 dx_i + S_0 d\alpha. \)

By using (2), it follows that

\[
(f'(\alpha))^2 dS_0 \subseteq \sum_{i=1}^n S_0[\alpha] dx_i + S_0[\alpha] d\alpha.
\]

Also,

\[
(f'(\alpha))^2k (dS_0)^k \subseteq \sum_{i} S_0[\alpha] dx_{i_1}...dx_{i_j} (d\alpha)^{k-j},
\]

and by using (1),

\[
f'(\alpha)^2k (dS_0)^k \subseteq \sum_{i} S_0[\alpha] dx_{i_1}...dx_{i_k}.
\]

It is well known that \( S \) is a localization of \( S_0 \), and therefore,

\[
S dS \subseteq S dS_0
\]

Hence,

\[
(f'(\alpha))^2k \left( \bigcap_{S \in \in \alpha} S dS \right)^k = (f'(\alpha))^2k \left( \bigcap_{S \in \in \alpha} (dS_0)^k \right)
\]

\[
\subseteq \sum_{i} \left( \bigcap_{S \in \in \alpha} S \right) dx_{i_1}...dx_{i_k}
\]

\[
= \sum_{i} S_0 dx_{i_1}...dx_{i_k}.
\]

By multiplying \( f'(\alpha) \) and using (2) again,
LEMMA 2. Let $I_k$ be the $R$-module of integral differential forms degree $k$, and $\bar{P}$ the integral closure of $P$ in $K$. If $\alpha \in \bar{P}$,

$$(f'(\alpha))^{3k+1}(\bigcap_{S \in \mathcal{S}} S(dS)^k) \subseteq \sum_i S_0[\alpha] dx_{i_1} \ldots dx_{i_k}.$$ 

Proof. Let $\mathcal{S}_0$ be the set of all discrete rank one valuation rings containing $P$, and $\mathcal{S}$ the set of all extension valuation rings of members of $\mathcal{S}_0$ in $K$. Since $P$ is Noetherian integrally closed domain, $P = \bigcap S_0$, and it follows that

$$(f'(\alpha))^{3k+1}I_k \subseteq (f'(\alpha))^{3k+1}(\bigcap_{S \in \mathcal{S}} S(dS)^k) \subseteq \sum_{S \in \mathcal{S}_0} S_0[\alpha] dx_{i_1} \ldots dx_{i_k} \text{ (by lemma 1)}$$

$$(f'(\alpha))^{3k+1} \subseteq \sum_i P[\alpha] dx_{i_1} \ldots dx_{i_k}.$$ 

LEMMA 3. Under the same assumption as lemma 7, in fact, there exists a natural number $r$ such that

$$(f'(\alpha))^{3k+1}I_k \subseteq \sum_i Mdx_{i_1} \ldots dx_{i_k},$$

where $M = \sum_{i \leq r} Rx_{i_1} \ldots x_{i_r}x_{i_1}^{*r+1}.$

Proof. Consider the subring $P_1 = R[x_1^{-1}, x_2, \ldots, x_n]$ of $K_0$. Let $S_1$ be the localization of $P_1$ at the prime ideal $(x_1^{-1})$ of $P_1$. Put $y_1 = x_1^{-1}$ and $y_i = x_i$ for $i = 2, \ldots, n$. Then $K_0$ is the field of quotients of $P_1 = R[y_1, \ldots, y_n].$

Let $\alpha_1 = x_1^{-h}a$ for some positive integer $h$, and as before, let

$$f(X) = X^m + a_1X^{m-1} + \cdots + a_m, a_i \in K_0,$$

be the minimal polynomial of $\alpha$ over $K_0$. Then

$$0 = f(\alpha) = x_1^{hm} g(\alpha_1),$$

where $g(X) = X^m + a_1x_1^{-h}X^{m-1} + \cdots + a_mX_1^{-hi}X^{m-i} + \cdots + a_mx_1^{-hm}.$

Actually, $g$ is the minimal polynomial of $\alpha_1$ over $K_0$, and $K = K_0(\alpha_1)$. Moreover, we can put $h$ sufficiently large so that all coefficients $a_i x_1^{-hi}$, $i = 1, \ldots, m$, of $g$ are contained in $S_1$. Then $\alpha_1 \in S_1$. Let $S_1'$ be the set of all extensions of members of $S_1$ in $K$. By using lemma 1,

$$(g'(\alpha_1))^{3k+1}(\bigcap_{S \in S_1} S(dS)^k) \subseteq \sum_i S_1[\alpha_1] dy_{i_1} \ldots dy_{i_k}.$$
Finiteness of integral differential forms

\[ \sum_{i} S_i[\alpha_i] (-x_i^{-2})^{q_i} dx_{i_1} \ldots dx_{i_k}, \]

(where \( q_i \) is the number of \( y_i \) among \( y_{i_1} \ldots y_{i_k} \))

\[ \sum_{i} S_i[\alpha] dx_{i_1} \ldots dx_{i_k}, \]

since \( S_i[\alpha_i] (x_i^{-2})^{q_i} \subseteq S_i[\alpha] \).

We have,

\[
(f'(\alpha))^{3k+1} I_k \subseteq \left( f'(\alpha) \right)^{3k+1} \left( \bigcap_{S \in \mathcal{A}_1} S(dS)^k \right)
= x_1^{h(m-1)(3k+1)} \left( g'(\alpha) \right)^{3k+1} \left( \bigcap_{S \in \mathcal{A}_1} S(dS)^k \right)
= x_1^{h(m-1)(3k+1)} \sum_{i} S_i[\alpha] dx_{i_1} \ldots dx_{i_k}, \text{ by (1).}
\]

Let \( r_1 = h(m-1)(3k+1) \). Then

\[
(f'(\alpha))^{3k+1} I_k \subseteq \sum_{i} x_1^{r_1} S_i[\alpha] dx_{i_1} \ldots dx_{i_k}.
\]

On the other hand, by lemma 2,

\[
(f'(\alpha))^{3k+1} I_k \subseteq \sum_{i} R[x_1, \ldots, x_n, \alpha] dx_{i_1} \ldots dx_{i_k}.
\]

Combining (2) and (3), we obtain

\[
(f'(\alpha))^{3k+1} I_k \subseteq \sum_{i} \sum_{x_i \in \alpha} R[x_2, \ldots, x_n, \alpha] x_i dx_{i_1} \ldots dx_{i_k}.
\]

Similarly, it can be shown that for each \( j=2, \ldots, n \), there exists \( r_j \) such that

\[
(f'(\alpha))^{3k+1} I_k \subseteq \sum_{i} \sum_{x_j \in \alpha_j} R[x_1, \ldots, x_{i_j}, \ldots, x_n, \alpha_j] dx_{i_1} \ldots dx_{i_k},
\]

where \( \hat{x}_j \) denotes the omission of \( x_j \).

Also, since \( \alpha \) is algebraic over \( K_0 \) and \( \alpha \in \mathcal{P} \),

\[
R[x_1, \ldots, x_n, \alpha] = \sum_{\iota_{n+1} \subseteq \deg f} R[x_1, \ldots, x_n, \alpha^{\iota_{n+1}}]
\]

Hence,

\[
(f'(\alpha))^{3k+1} I_k \subseteq \sum_{i} \sum_{x_i \in \alpha_i} R[x_1, \ldots, x_n, \alpha^{\iota_{n+1}}] dx_{i_1} \ldots dx_{i_k}.
\]

Let \( r = \max\{r_1, \ldots, r_n, \deg f\} \). Then it follows from (4), (5), and (6) that

\[
(f'(\alpha))^{3k+1} I_k \subseteq \sum_{i} Mdx_{i_1} \ldots dx_{i_k}.
\]

**Theorem 1.** Let \( R \) be an integrally closed Noetherian domain, and \( K \)
a finitely and separably generated extension field over the field of quotients of \( R \). Then the module of homogeneous integral differential forms on \( K \) of degree \( n \) is a finitely generated \( R \)-module.

Proof. Since \( K \) is finitely and separably generated over the field of quotients of \( R \), there exist \( x_1, \ldots, x_m, \alpha \in K \) such that \( P = R[x_1, \ldots, x_m] \), \( K = K_0(\alpha) \) where \( K_0 \) is the field of quotients of \( P \), and \( \alpha \) is separable algebraic over \( K_0 \). Without loss of generality, we can assume that the minimal polynomial \( f(X) \in P[X] \). In this case, \( \alpha \in P \), the integral closure of \( P \) in \( K \). We have all assumptions for lemma 2, and hence by lemma 3,

\[
I_k \subseteq \sum_i M(f'(\alpha))^{-c_k+1} dx_{i_1} \cdots dx_{i_k},
\]

Where \( M = \sum_{i_j \leq r} R x_{i_1} \cdots x_{i_n} a^{i_n+1} \).

It follows that \( I_k \) is a submodule of a finitely generated \( R \)-module. Since \( R \) is Noetherian, \( I_k \) itself is a finitely generated \( R \)-module.

Corollary 1. Let \( R \) be an integrally closed Noetherian domain, and \( K \) a finitely and separably generated extension field over the field of quotients of \( R \). Then the integral closure of \( R \) in \( K \) is a finitely generated \( R \)-module.

Proof. Integral differential forms of degree zero are elements of \( K \) which are integrally dependent on \( R \). Hence, this is a special case of Theorem 1.

References


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