HIGHER LEVEL SIGNATURES ON VALUATION RINGS

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Introduction

In [1] E. Becker develops a theory of what he calls orderings of higher level over fields. These generalize orderings of a field in such a way that one can generalize many of the usual results in formally real fields. In [8], Kleinstein and Rosenberg show that there is a natural extension of the usual Witt ring of equivalence classes of non-degenerate bilinear forms over a field to the Witt ring of higher level.

In [5], T. Craven defines the Witt ring of higher level over a semilocal ring and extends many of the results by Kleinstein and Rosenberg. In this paper we apply the results by T. Craven to valuation rings \( A \). Thus we can obtain a generalization of the result by Knebusch on the extension of a signature of \( A \) to a signature of the quotient field of \( A \). As a Corollary we obtain a result on sum of \( 2^n \)-powers problem. We also prove that the Dress' theorem [6] can be generalized to higher level case if \( A \) is a local ring with usual conditions. Finally we give two examples convincing us the necessity of the condition \( A \) being a valuation ring in our Corollary.

All of the notations and terminologies follow those of T. Craven.

Higher level signature on valuation rings

Let \( A \) be a connected semilocal ring with no residue class field having 2 elements. We denote the group of units of \( A \) by \( A^* \), write \( G_n(A) = A^*/A^{*2^n} \) for the group of units modulo \( 2^n \)-powers, and \( \langle a \rangle_n \) for \( aA^{*2^n} \) where \( a \in A^* \).

DEFINITION 1. The Witt ring of level \( n \) of \( A \), denoted \( W_n(A) \), is the integral groupring \( \mathbb{Z}[G_n(A)] \) modulo the ideal \( J_n(A) \) generated by \( \langle 1 \rangle_n + \langle -1 \rangle_n \) and all elements of the form

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whenever \( x \) and \( \lambda_1 + \lambda_2 \cdot x \) are both in \( A^* \).

Let \( C_n \) denote the group of \( 2^n \)-th roots of unity in the complex numbers and \( U_n \) denote the ring of integers of the cyclotomic field \( \mathbb{Q}(C_n) \).

If \( \sigma : A^* \longrightarrow C_n \) is a homomorphism which extends to \( \sigma : W_n(A) \longrightarrow U_n \); that is, when \( \sigma \) is extended to a homomorphism \( \mathbb{Z}[G_n(A)] \longrightarrow U_n \), its kernel contains \( J_n(A) \), we call \( \sigma \) a signature of level \( n \).

Let \( P = P(\sigma) = \ker(\sigma : A^* \longrightarrow C_n) \), so that \( A^*/P \cong C_m \) for some \( m \leq n \). In this case we shall say \( \sigma \) is a signature of exact level \( m \). In general, we shall speak of signatures of higher level without specifying \( n \). When \( A \) is a field, these definitions coincide with those of [1] and in this case \( P(\sigma) \cup \{0\} \) is an ordering of level \( n \).

**DEFINITION 2.** Let \( \sigma \) be a signature of exact level \( n \). Define \( Q(\sigma) = \{ \sum \lambda_i a_i | \lambda_i \in A, \sum \lambda_i A = A, a_i \in P(\sigma) \} \).

Note that \( Q(\sigma) = P(\sigma) \) if \( A \) is a field. Now let \( \varphi : A \longrightarrow B \) be a homomorphism between two semilocal rings which satisfy our usual conditions.

**DEFINITION 3.** Let \( \sigma, \tau \) be signatures of exact level \( n \) of \( A \) and \( B \) respectively. Let \( \varphi_* : W_n(A) \longrightarrow W_n(B) \) be the homomorphism induced by \( \varphi \). We say \( \tau \) is a faithful extension of \( \sigma \) if the homomorphisms

\[
W_n(A) \xrightarrow{\varphi_*} W_n(B) \xrightarrow{\tau} U_n \quad \text{and} \quad W_n(A) \xrightarrow{\sigma} U_n
\]

have the same kernel.

In [5], T. Craven defines \( S_n(A, B) = \{ \sum \lambda_i a_i | \lambda_i \in B, \sum \lambda_i B = B, a_i \in P(\sigma) \} \) for \( \sigma \) and \( \varphi \) as above. Then he shows the signature of exact level \( n \) can be extended faithfully to a signature of \( B \) if and only if \( 0 \in S_n(A, B) \).

**LEMMA 4.** Let \( A \) be a local ring with maximal ideal \( \mathfrak{m} \) such that \( |A/\mathfrak{m}| = 2 \). Then every element of \( Q(\sigma) \) either lies in \( P(\sigma) \) or is a sum of two elements of \( P(\sigma) \) [5].
Now let $A$ be a valuation ring with quotient field $K$, maximal ideal $\mathfrak{m}$. Assume $|A/\mathfrak{m}|=2$ and $2\in A^*$. In [11], M. Knebusch proved any signature of level 1 of $A$ can be extended to a signature of level 1 of $K$. We have the following complete generalization in our higher level case.

**Theorem 5.** Let $A$ be as above. Then each signature of higher level of $A$ can be extended faithfully to a signature of higher level of $K$.

**Proof.** Let $\sigma$ be a signature of exact level $n$ of $A$. By the remark above Lemma 4, it will suffice to show $0 \not\in S_n(A, K)$. Suppose $\lambda_1 2^n a_1 + \cdots + \lambda_k 2^n a_k = 0$ with $\lambda_i \in A$, $a_i \in P(\sigma)$ for $1 \leq i \leq k$. If $\lambda_i \in A^*$ for some $1 \leq i \leq k$, then $\lambda_1 2^n a_1 + \cdots + \lambda_k 2^n a_k \in Q(\sigma)$. Since $A$ is a local ring, we have $\lambda_1 2^n a_1 + \cdots + \lambda_k 2^n a_k$ is a sum of two elements of $P(\sigma)$ by Lemma 4. Now we have $\lambda_1 2^n a_1 + \cdots + \lambda_k 2^n a_k = a + b = 0$ for some $a, b \in P(\sigma)$, then $a = -b$, and hence $1 = \sigma(a) = \sigma(-b) = -1$, a contradiction. Therefore we may assume $\lambda_i \in A^*$ for $1 \leq i \leq k$. We denote the (additive) valuation of $A$ by $\nu$. If $\nu(\lambda_1) = \min\{\nu(\lambda_i) | 1 \leq i \leq k\}$, then $\nu(\frac{\lambda_i}{\lambda_1}) = \nu(\lambda_i) - \nu(\lambda_1) \geq 0$, i.e. $\lambda_i \in A$ for $1 \leq i \leq k$. Our equation reduces to $1 \cdot a_1 + \lambda_2 2^n a_2 + \cdots + \lambda_k 2^n a_k = 0$, $\lambda_i \in A$, $a_i \in P(\sigma)$ for $2 \leq i \leq k$. Since $1 \in A^*$, we have a contradiction. Thus $0 \not\in S_n(A, K)$, and our signature can be extended faithfully to $K$.

**Remark.** If $1 + \mathfrak{m} \subseteq P(\sigma)$, we say $\sigma$ is compatible with $A$. If this is the case, we have the following simple proof. Denote the residue field $A/\mathfrak{m}$ by $k$. Then the character $\bar{\sigma}$ defined by $\bar{\sigma}(x) = \sigma(x)$ is a well-defined signature of exact level $n$ of $k$. Since $\bar{\sigma}$ can be lifted faithfully to a signature of $K[7]$, we have proved our theorem.

Now let $\Sigma(A)$ denote the set of all elements $\sum \lambda_i 2^n$ such that $\sum \lambda_i A = A$ together with elements $x$ such that $xy = z$ where $y$ and $z$ are such sums of $2^n$-powers. If $A$ is a field then $\Sigma(A) = \Sigma A^{2^n}$. For $A$ semilocal and $n=1$, $\Sigma(A) = \Sigma A^2$ by the representation criterion for quadratic forms [10].

**Corollary 6.** Let $A$ and $K$ be as in Theorem 5. Then if a unit element $a$ of $A$ belongs to $\Sigma K^{2^n}$, $a$ is an element of $\Sigma(A)$.

In [3], Kneser, J-L Colliot-Thélène proved that $a$ is already in $\Sigma A^2$.
if $a \in A^* \cap \sum F^2$. Since $\Sigma(A) = \sum A^2$ for $n=1$, Corollary 6 is a generalization of this in higher level case.

**Proof.** Since $a \in \sum K^{2^n}, a \in \cap \{\text{orderings of level } n \text{ of } K\}$ [2]. Theorem 5 says any signature of higher level of $A$ can be extended faithfully to $K$, so that $a \in \cap P(\sigma)$ where $\sigma$ ranges over all signatures of level $n$ of $A$. By Theorem 3.7 of [5], $a \in \Sigma(A)$.

If $\sigma$ is a usual signature of level 1 of a commutative ring $A$ with $2 \in A^*$, Dress' Theorem [6] guarantees the existence of some prime ideal $p$ of $A$ such that $\sigma$ can be extended to $A_p$. We can generalize this theorem to higher level signature case for local ring $A$.

**Theorem 7.** Let $A$ be a local domain with $|A/\mathfrak{m}| \neq 2^k$ and $2 \in A^*$. If $\sigma$ is a signature of higher level of $A$, there exists a prime ideal $p$ of $A$ such that $\sigma$ can be extended faithfully to a signature of higher level of $A_p$.

**Proof.** Since $A$ is a local ring, there exists a prime ideal $p$ of $A$ such that $\sigma$ can be extended faithfully to the quotient field $A(p)$ of the integral domain $A/p$ [5]. Let $\tau$ denote the extension of $\sigma$. Since $A(p) = A_p/\mathfrak{m}A_p$, we have the following diagram.

$$
\begin{array}{ccc}
W_n(A_p) & \xrightarrow{\eta} & W_n(A_p/\mathfrak{m}A_p) \\
& & \xrightarrow{\tau} U_n \\
& & \downarrow \sigma \\
W_n(A) & &
\end{array}
$$

where $\eta$ is the natural homomorphism. Since the left triangle commutes, it is clear $\tau \eta$ is an extension of $\sigma$ on $A_p$.

Now we give two examples which show the necessity of our condition on $A$ in Corollary 6.

**Example 8.** Let $A_0 = R[x, y, z]/(x^2 + y^2 + z^2)$, $p = (x, y, z)$ and $A = (A_0)_p$. Then $A$ is a local domain of Krull dimension 2. The element $-1 \in A^*$ is a sum of two squares in $K$, but $-1$ is not a sum of squares in $A[3]$.

**Example 9.** Let $k$ be a real-closed field, and $S_0$ be the set of irred-
ucible polynomials \( s \in k[x, y] \) such that \( s \) generate a real prime ideal. Let \( S \) be the multiplicative set generated by \( S_0 \), and \( A = S^{-1}(k[x, y]) \). Then \( A \) is a PID. If \( f(x, y) = x^3 + (xy - x^2 - 1)^2 \) \( f(x, y) \) is a positive semidefinite polynomal with the property that \( f^{2r+1} \notin \sum A^2 \) for any \( r \). Now we have \( f \) is a sum of four squares in \( k(x, y) \). If \( A' = A[f^{-1}] \), then \( A' \) is a PID. The unit element \( f \notin \sum A'^{r^2} \) [4].

References

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