FLEXIBLE MALCEV-ADMISSIBLE ALGEBRAS WITH NIL CARTAN SUBALGEBRAS

YOUNGSO KO, HYO CHUL MYUNG AND DONG SUN SHIN

1. Flexible Malcev-admissible algebras

Let $A$ denote an (nonassociative) algebra over a field $F$ of characteristic $\neq 2$ with multiplication denoted by juxtaposition $xy$. Denote by $A^-$ the algebra with the commutator product $[x, y] = xy - yx$ defined on the vector space $A$. We call $A$ Malcev-admissible if $A^-$ is a Malcev algebra; that is, $A^-$ satisfies the Malcev identity

$$[[x, y], [x, z]] = [[[x, y]z, x]] + [[[y, z], x], x]] + [[[z, x], x], y]].$$

An algebra $A$ is said to be Lie-admissible if $A^-$ is a Lie algebra; namely $A^-$ satisfies the Jacobi identity

$$[[x, y], x]] + [[[y, z], x], x]] + [[[x, z], x], y]] = 0.$$  

It is well known that any Lie-admissible algebra is Malcev-admissible but not conversely (Myung [10, 12]).

A useful identity in the study of Malcev-admissible algebras $A$ is the well known flexible identity $(xy)x = x(yx)$, which is equivalent to the property that the adjoint map $ad_x : A \rightarrow A$ defined by $ad_x(y) = [x, y]$ is a derivation of the commutative algebra $A^+$ with multiplication $x \circ y = \frac{1}{2}(xy + yx)$ defined on $A$ for all $x \in A$. Thus, $A$ is a flexible algebra over $F$ if and only if the identity

$$(1) \quad [x, y \circ z] = y \circ [x, z] + [x, y] \circ z$$

holds for all $x, y, z \in A$ ([8, 10]). It can be shown that an algebra $A$ is flexible Lie-admissible if and only if $ad_x$ is a derivation of $A$, or equivalently, the identity

---89---
Youngso Ko, Hyo Chul Myung and Dong Sun Shin

(2) \([x, yz] = y[x, z] + [x, y]z\)
is satisfied for all \(x, y, z \in A\) ([8]). Basic examples of flexible Malcev-admissible algebras which are not Lie-admissible are octonion algebras.

The structure of finite-dimensional flexible Malcev-admissible algebras of characteristic 0 has been studied in light of earlier investigations of flexible Lie-admissible algebras (see Myung [10, 12], for example). Representation theory of Lie algebras and Cartan subalgebras play main roles for the structure of these algebras. Very little is known for the structure of flexible Malcev-admissible algebras of arbitrary dimension.

Let \(A\) be a flexible Malcev-admissible algebra over \(F\), and suppose that \(A\) has a Cartan decomposition relative to a Cartan subalgebra \(H\) of \(A^\cdot\):

\[ A = H + \sum_{\alpha \neq 0} A_{\alpha}, \]

where \(A_{\alpha} = \{x \in A | (ad_{h} - \alpha(h)I)^{n}(x) = 0 \text{ for some } n = n(h), h \in H\}\) is the root space corresponding to root \(\alpha\). Using the flexible identity (1), it can be shown that

\[ A_{\alpha} \cdot A_{\beta} \subseteq A_{\alpha + \beta} \text{ for all roots } \alpha, \beta, \]

\[ A_{\alpha} A_{\beta} \subseteq A_{\alpha + \beta} \text{ for } \alpha \neq \beta, \]

\[ A_{\alpha} A_{\beta} \subseteq A_{2\alpha + A_{-\alpha}}. \]

If \(A\) is flexible Lie-admissible, then the last relation of (4) becomes \(A_{\alpha} A_{\beta} \subseteq A_{2\alpha}\). We note that the Cartan subalgebra \(H\) is a subalgebra of \(A\). A special case of interest in this paper is when the map \(ad_{h}\) acts diagonally on each root space \(A_{\alpha}\) for all \(h \in H\) and when \(H\) is abelian; i.e., \([H, H] = 0\). As is well known, the classes of finite-dimensional split semisimple Malcev algebras of characteristic 0 and of classical Lie algebras of Seligman [14] satisfy this property. An important class of infinite dimensional Lie algebras having this property is the class of Kac-Moody algebras [3]. It is the purpose of this note to investigate the structure of flexible Malcev-admissible algebras \(A\) which have a Cartan decomposition relative to an abelian Cartan subalgebra \(H\) of \(A^\cdot\) such that all \(ad_{h}\) act diagonally on each root space \(A_{\alpha}\) under the assumption that each element of \(H\) is nilpotent in \(A\).

In a power-associative algebra \(B\), an element \(x\) is called nilpotent if \(x^{n} = 0\) for some positive integer \(n\). If every element of \(B\) is nilpotent, then \(B\) is said to be nil. If there is a positive integer \(m\) such that \(x^{m} = 0\) for all \(x \in B\), then the least such integer is called the nil-index of \(B\).
Every finite-dimensional nil algebra \( B \) has the nil-index \( \leq \dim B + 1 \).

### 2. Cartan subalgebras with nil-basis

In this section we give a condition that a finite-dimensional flexible power-associative Malcev-admissible algebra \( A \) is a nil algebra in terms of a nil-basis of a Cartan subalgebra of \( A^- \). A nil-basis of a power-associative algebra is a basis consisting of nilpotent elements. Generalizing the result in Myung [9] for Lie-admissible algebras, we have

**Theorem 1.** Let \( A \) be a finite-dimensional power-associative Malcev-admissible algebra over a field \( F \) of characteristic 0 or greater than 5 and \( \dim A \). Then, \( A \) is a nil algebra if and only if \( A^- \) has a Cartan subalgebra with a nil-basis.

**Proof.** The assumption on the characteristic guarantees the existence of a Cartan subalgebra (Malek [7]). Let \( H \) denote a Cartan subalgebra of \( A^- \) with a nil-basis. By a result on flexible power-associative algebras (Myung [9]), it suffices to verify that \( A \) has a nil-basis. By a scalar extension argument, we may assume that \( F \) is algebraically closed. Thus, \( A^- \) has a Cartan decomposition as in (3) relative to \( H \). For a root \( \alpha \), if \( A_{\alpha}^n \) denotes the linear span of Jordan products \( x_1 \circ x_2 \circ \cdots \circ x_n \) of any \( n \) elements \( x_1, \ldots, x_n \) of \( A_{\alpha} \) in all possible associations, then it follows from the first relation of (4) that \( A_{\alpha}^n \subseteq A_{\alpha^n} \). Since there are only finitely many roots, it must be that \( A_{\alpha}^n = 0 \) for some \( n > 0 \). This in particular implies that \( A_{\alpha} \) has a nil-basis for all roots \( \alpha \) and hence \( A \) has a nil-basis.

There are other conditions that force flexible power-associative algebras to be nil algebras. For example, any finite-dimensional flexible power-associative algebra \( A \) of characteristic \( \neq 2, 3 \) such that \( A^- \) is simple must be a nil algebra (Oehmke [13]).

### 3. Algebras with nil Cartan subalgebras

Let \( A \) be an algebra over \( F \) of characteristic \( \neq 2 \) with multiplication denoted by \( xy \). A bilinear form \( \phi \) on \( A \) is called invariant if it satisfies the equation

\[
\phi(xy, z) = \phi(x, yz)
\]
for all \( x, y, z \in A \). Certain invariant bilinear forms have been used for the construction of flexible Malcev-admissible algebras, and conversely those algebras satisfying certain properties arise from the construction using invariant forms (see Ko and Myung [4], Benkart [1], and Benkart and Osborn [2]). A well-known invariant form is the Killing form \( K \) on a Lie or Malcev algebra of finite dimension, which is defined by

\[
K(x, y) = \text{tr}(ad_x \ ad_y),
\]

where \( \text{tr} \) denotes the trace.

Let \( A \) be any algebra over \( F \), and assume that \( c_1, \ldots, c_n \) are linearly independent elements such that \([c_i, A] = 0\) for \( i = 1, \ldots, n \). Let \( \phi_i(i = 1, \ldots, n) \) be symmetric bilinear forms on \( A \). We define a multiplication \( "*" \) on the vector space \( A \) by

\[
x \ast y = \frac{1}{2}[x, y] + \sum_{i=1}^{n} \phi_i(x, y)c_i
\]

for \( x, y \in A \). By symmetry of \( \phi_i \), we have \((A, \ast)^{-} = A^{-}\), and the Jordan product in \((A, \ast)\) is given by

\[
x \circ y = \frac{1}{2}(x \ast y + y \ast x) = \sum_{i=1}^{n} \phi_i(x, y)c_i
\]

for all \( x, y \in A \).

**Lemma 2.** Let \((A, \ast)\) be the algebra with multiplication defined by (5). Then, \((A, \ast)\) is flexible if and only if the symmetric bilinear forms \( \phi_1, \ldots, \phi_n \) are invariant on \( A^{-} \) or on \((A, \ast)^{-}\).

**Proof.** Note first that \((A, \ast)\) is flexible if and only if relation (1) holds for \((A, \ast)\). Hence, if \((A, \ast)\) is flexible, then by (6)

\[
0 = [x, y \circ z] = y \circ [x, z] + [x, y] \circ z = \sum_{i=1}^{n} [\phi_i(y, [x, z]) + \phi_i([x, y], z)]c_i,
\]

which implies that \( \phi_i([x, y], z) = \phi_i(x, [y, z]) \) for \( x, y, z \in A \) and \( i = 1, \ldots, n \), since \( c_1, \ldots, c_n \) are linearly independent. The converse is immediate.

The product \( "*" \) of the form (5) has arisen from the classification of finite-dimensional flexible Malcev-admissible algebras \( A \) over an algebraically closed field of characteristic 0 such that \( A^{-} \) is reductive. In this case, the forms \( \phi_i \) are multiples of the Killing form (Myung[10]). We extend this to a class of flexible Malcev-admissible algebras of arbitrary
Flexible Malcev-admissible algebras with nil Cartan subalgebras

dimension. For this, the following results are instrumental.

**Lemma 3.** Let $A$ be a flexible algebra over $F$ of characteristic $\neq 2$.

(i) If $h$ is a power-associative element of $A$ and $X \in A$ is a common eigenvector of $ad_h$ and $ad_h^2$, then $[x, h^3] = [x, h^4] = 0$ imply $[x, h^2] = 0$.

(ii) If $h$ is a power-associative element of $A$ and $x$ is a common eigenvector of $ad_h$, $ad_h^2$, $R_h$, and $R_h^2$, then $[x, h^4] = [x, h^5] = 0$ imply $[x, h^3] = 0$. Here, $R_h$ denotes the right multiplication in $A$ by $h$.

**Lemma 4.** Let $A$ be a flexible Malcev-admissible algebra with multiplication denoted by $xy$ (not necessarily finite-dimensional) over a field $F$ of characteristic $\neq 2$, and let $H$ be an abelian Cartan subalgebra of $A^-$. Assume that $A^-$ has a Cartan decomposition relative to $H$ such that each $ad_h (h \in H)$ diagonally acts on root space $A_\alpha$ for all roots $\alpha$; i.e., $[h, x] = \alpha(h)x$ for all $x \in A_\alpha$ and $h \in H$.

(i) If $h \in H$ and $x \in A_\alpha$ for $\alpha \neq 0$, then $hx$ and $xh$ are multiples of $x$.

(ii) If $x \in A_\alpha$, $y \in A_\beta$ and $\alpha \neq -\beta$ for $\alpha, \beta \neq 0$, then $xy$ is a multiple of $[x, y]$.

(iii) If $HH = 0$, then $xy = -\frac{1}{2}[x, y]$ for all $x, y \in A$, and hence $A$ is a Malcev algebra isomorphic to $A^-$. 

(iv) If the center of $A^-$ is zero and $H$ is a nil algebra under the product $xy$ with bounded nil index, then $HH = 0$ and hence $A$ is a Malcev algebra.

A proof of Lemma 3 may be found in Myung [8]. Several different versions of Lemma 4 have been proved. Lemma 4 was first proved by Myung [8] when $A$ is finite-dimensional and $\dim A_\alpha = 1$ for $\alpha \neq 0$. Benkart [1] proved the present form for Lie-admissible algebras. A proof of Lemma 4 for the finite-dimensional case has been given by Malek [6], and the present form of the lemma is due to Ko and Myung [5].

For the case of characteristic $p > 0$, two well known classes of algebras satisfying the conditions of Lemma 4 are the classical Lie algebras (Seligman [14]) and generalized Witt algebras. Among infinite-dimensional Lie algebras of characteristic 0 which satisfy the hypotheses of Lemma 4 are the Virasoro algebra which arises in relativistic string dual model theory, and the Kac-Moody algebras which are best understood infinite-
dimensional Lie algebras (Kac [3]). Examples are given in Myung [8] and Malek [6] to show that the hypotheses that the center of $A^-$ is 0 and the Cartan subalgebra $H$ is nil in $A$ are necessary in Lemma 4(iv).

We note that Kac-Moody algebras have in general nonzero center, but there exist Kac-Moody algebras without center; for example, certain affine Kac-Moody algebras. Our principal result in this section is to determine flexible Malcev-admissible algebras $A$ satisfying the conditions of Lemma 4, when the Cartan subalgebra $H$ is a nil subalgebra of $A$ of nil-index $\leq 4$.

**Theorem 5.** Let $A$ be a flexible Malcev-admissible algebra over $F$ of characteristic $\neq 2$, which satisfies the hypotheses of Lemma 4. Assume that the center $Z$ of $A^-$ is finite-dimensional and $\{e_1, ..., e_n\}$ is a basis of $Z$. If the Cartan subalgebra $H$ of $A^-$ is nil in $A$ and of nil-index $\leq 4$, then the multiplication $"*"$ in $A$ is given by relation (5) for some symmetric invariant forms $\phi_1, ..., \phi_n$ on $A^-$ satisfying the property

$$\phi_i(h^2, h) = \phi_i(h^3, h^2) = 0, \quad i = 1, ..., n$$

for all $h \in H$.

**Proof.** By the assumption, Lemma 4(i) and Lemma 3 (ii) we note that $h^3$ lies in the center $Z$ for all $h \in H$, since $h^3 = h^5 = 0$. In view of Lemma 3 (i), this in turn implies that $h^2 \in Z$ for all $h \in H$. Since $H$ is abelian, we have $h_1 * h_2 = \frac{1}{2} \left[ (h_1 + h_2)^2 - h_1^2 - h_2^2 \right]$ and hence $H * H \subseteq Z$. Thus, we can let

$$h_1 * h_2 = \phi_1(h_1, h_2) e_1 + \cdots + \phi_n(h_1, h_2) e_n$$

for some symmetric bilinear forms $\phi_1, ..., \phi_n$ on $H$. Let $x \in A_\alpha$ for any nonzero root $\alpha$. If $h, h' \in H$, then since $H * H \subseteq Z$, by (1)

$$0 = [h \circ h', x] = [h, x] \circ h' + h \circ [h', x]$$

$$= \alpha(h) x \circ h' + \alpha(h') h \circ x,$$

which shows that $A_\alpha \circ H = 0$, and hence $h \circ x = -x \circ h = \frac{1}{2} \alpha(h) x = \frac{1}{2} [h, x]$ for $h \in H$, $x \in A_\alpha$, $\alpha \neq 0$. In particular, we have $A_\alpha * Z = Z * A_\alpha = 0$ for $\alpha \neq 0$.

Let $\alpha, \beta$ be nonzero roots. For $x \in A_\alpha$ and $y \in A_\beta$, it follows that

$$0 = [h \circ x, y] = \beta(h) x \circ y + h \circ [x, y]$$

for all $h \in H$. The proof is completed.
for all \( h \in H \). If \( \alpha + \beta \neq 0 \), then (8) gives \( x \circ y = 0 \), since \( \beta \neq 0 \) and \( H \circ A_\gamma = 0 \) for \( \gamma \neq 0 \). Hence,

\[
x \ast y = -y \ast x = \frac{1}{2} [x, y],
\]

if \( x \in A_\alpha \) and \( y \in A_\beta \) for roots \( \alpha, \beta \) with \( \alpha + \beta \neq 0 \). Assume then that \( x \in A_\alpha \) and \( y \in A_{-\alpha} \). Since \( \alpha \neq 0 \) and \( h \circ [x, y] \in \mathbb{Z} \), by (8) we have \( x \circ y \in \mathbb{Z} \). Thus, we can write \( x \circ y = \phi_1(x, y)c_1 + \cdots + \phi_n(x, y)c_n \) where \( \phi_i(x, y) = \phi_i(y, x) \in F \) \((i = 1, \ldots, n)\) is uniquely determined by \( x \in A_\alpha \) and \( y \in A_{-\alpha} \). Extending \( \phi_1, \ldots, \phi_n \) bilinearly to \( A \) by defining \( \phi_i(A_\alpha, A_{\beta}) = 0 \) for all roots \( \alpha, \beta \) with \( \alpha + \beta \neq 0 \), we have the multiplication \( \ast \) given by relation (5), since \( x \ast y = \frac{1}{2} [x, y] + x \circ y \). Since \( A \) is flexible, the invariance of the \( \phi_i \) follows from Lemma 2. Since \( h^2 h^2 = h^5 h = 0 \) for all \( h \in H \) and \( H \ast H \subseteq \mathbb{Z} \), relation (7) is immediate from (5).

4. Application to affine Kac-Moody algebras

In this section, we give an application of Theorem 5 to the algebras \( A \) when \( A^- \) has a one-dimensional center. There are some algebras of interest where \( A^- \) has a one-dimensional center; for examples, matrix algebras, quadratic algebras including quaternion and octonion algebras, and reductive Lie or Malcev algebras with one-dimensional center.

There is an important class of infinite-dimensional Kac-Moody algebras with one-dimensional center, called (non-twisted) affine Kac-Moody algebras. For convenience, we give a realization of these algebras. Let \( L = F[t, t^{-1}] \) be the commutative associative algebra of Laurent polynomials (or equivalently, the group algebra on an infinite cyclic group). For any Laurent polynomial \( P \), the residue \( \text{Res}(P) \) of \( P \) is defined as the coefficient of \( t^{-1} \) in \( P \). Thus, \( \text{Res}(P) \) is a linear form on \( L \) defined by the relations

\[
\text{Res}(t^{-1}) = 1, \quad \text{Res}\left(\frac{dP}{dt}\right) = 0.
\]

Define the bilinear form \((,\)\) on \( L \) by

\[
(P, Q) = \text{Res}\left(\frac{dP}{dt} Q\right).
\]

Then, it is readily seen that the form \((,\)\) satisfies
\[(P, Q) = -(Q, P),\]
\[(PQ, R) + (QR, P) + (RP, Q) = 0\]
for all \(P, Q, R \in L\). In particular, we have \((t^n, t^n) = m \delta_{m, -n}\).

Let \(G\) be a finite-dimensional Lie algebra over an algebraically closed field \(F\) of characteristic 0, and let \(K(\ , \ )\) be the Killing form on \(G\). Consider the tensor product \(L(G) = L \otimes_p G\) and define a Lie algebra product on \(L(G)\) by

\[[P \otimes x, Q \otimes y] = PQ \otimes [x, y]\]

for \(P, Q \in L\) and \(x, y \in G\). Then, \(L(G)\) is an infinite dimensional Lie algebra over \(F\), called the Loop algebra (see Kac \([3, p. 73]\)). Define a bilinear form \(\phi_0(\ , \ )\) on \(L(G)\) by

\[\phi_0(P \otimes x, Q \otimes y) = (P, Q) K(x, y)\]

for \(P, Q \in L\) and \(x, y \in G\), where \((\ , \ )\) is the bilinear form defined by (9). It easily follows from (10) that \(\phi_0\) is skew symmetric and is an \(F\)-valued 2-cocycle of the Lie algebra \(L(G)\) in the sense that the identity

\[(11) \quad \phi_0([a, b], c) + \phi_0([b, c], a) + \phi_0([c, a], b) = 0\]
holds for all \(a, b, c \in L(G)\). We now make a one-dimensional central extension of \(L(G)\) to the algebra \(\tilde{L}(G) = L(G) \oplus Fc\) with multiplication defined by

\[(12) \quad [a + \lambda c, b + \mu c] = [a, b] + \phi_0(a, b) c\]

for \(a, b \in L(G)\) and \(\lambda, \mu \in F\). Then, \(\tilde{L}(G)\) becomes an infinite dimensional Lie algebra with one-dimensional center \(Fc\) called an (non-twisted) affine Kac–Moody algebra, which satisfies the conditions of Lemma 4 holding for \(A^-\).

**Theorem 6.** Let \(A\) be a flexible Malcev-admissible algebra over \(F\) of characteristic \(\neq 2\) for which \(A^-\) satisfies the hypotheses of Lemma 4. Assume that \(A^-\) has a one-dimensional center \(Fc\). If the Cartan subalgebra \(H\) of \(A^-\) is a nil subalgebra of \(A\) of nil-index \(\leq 4\), then the multiplication \("*"\) in \(A\) is given by

\[(13) \quad x * y = \frac{1}{2} [x, y] + \phi(x, y) c\]

for \(x, y \in A\), where \(\phi\) is a symmetric invariant from on \(A^-\) satisfying
the property

$$\phi(c, A) = \phi(A_\alpha, A_\beta) = 0$$

for all roots $\alpha, \beta$ with $\alpha + \beta \neq 0$. In this case, $A$ is a nil algebra of nil-index $\leq 3$, and $A$ is a Malcev algebra if and only if $\phi = 0$. Conversely, for any prescribed Malcev algebra product $[\ , \ ]$ on $A^-$ and a symmetric invariant form $\phi$ on $A^-$, if $c$ is a fixed element of the center of $A^-$, then the product "*" defined by (13) on $A$ is flexible Malcev-admissible.

Proof. The proof is similar to that of Theorem 5, in which we have shown that $H * H \subseteq Fc$ and $h_1 * h_2 = \phi(h_1, h_2)c$ for some symmetric bilinear form $\phi$ on $H$. It follows from (1) that the center $Fc$ is a subalgebra of $A$ and hence $c^2 = 0$. Thus, for each $h \in H$, $Ph + Fc$ is a nil subalgebra of $A$ of nil-index $\leq 3$, and so is associative. This proves that $H * c = 0$ and hence by the proof of Theorem 5 $A * c = c * A = 0$. It follows from this and the proof of Theorem 5 that (13) and (14) hold for $A$. Thus, for $x \in A$, $x * x = \phi(x, x)c$ and $0 = (x * x) * x = x * (x * x)$ The remainder of the proof is immediate.

The multiplication defined by (13) has some relation with the construction of certain Lie-admissible algebras in terms of 2-cocycles in Lie algebras. For a Malcev algebra $M$, a skew symmetric bilinear form $\phi_0$ on $M$ is called a 2-cocycle of $M$ if $\phi_0$ satisfies

$$\phi_0([x, y], [x, z]) = \phi_0([[x, y], z], x) + \phi_0([[[y, z], x], x)$$

for all $x, y, z \in M$. Assume that $A$ is a Malcev-admissible algebra, and let $\phi$ be a bilinear form on $A$. We define a skew symmetric and symmetric bilinear forms $\phi^-$ and $\phi^+$ by

$$\phi^-(x, y) = \phi(x, y) - \phi(y, x),$$
$$\phi^+(x, y) = \frac{1}{2} [\phi(x, y) + \phi(y, x)]$$

for $x, y \in A$. Then, $\phi = \frac{1}{2} \phi^- + \phi^+$. A bilinear form $\phi$ on $A$ is called a 2-cocycle of $A$ if $\phi^-$ is a 2-cocycle of $A^-$; that is, $\phi^-$ satisfies relation (15) in $A^-$. Suppose now that $A$ denotes a flexible Malcev-admissible algebra described in Theorem 6, and let $\phi$ be a symmetric invariant form on $A^-$ in relation (13). Let $H_0$ be a subspace of $H$ complementary to $Fc$, and
denote $A_0 = H_0 + \sum_{x \neq 0} A_x$. Then, we have the vector space direct sum $A = A_0 \oplus Fc$. For $x, y \in A_0$, denote by $x \star y$ and $\phi_0(x, y)c$ the projections of $x \star y$ onto $A_0$ and $Fc$, respectively. It is readily see that $\star$ is an anticommutative product on $A_0$ and $x \star y = \frac{1}{2}[x, y]_0$ for $x, y \in A_0$, where $[x, y]_0$ denotes the projection of $[x, y]$ onto $A_0$. Since for $x, y \in A_0$,

\begin{equation}
 x \star y = x \star y + \phi_0(x, y)c,
\end{equation}

we have $x \star y = \frac{1}{2}[x, y]_0 - \frac{1}{2}\phi_0^-(x, y)c$. Since $A^-$ is a Malcev algebra, we see that $(A_0, \star) = (A_0, \frac{1}{2}[\ , \ ]_0)$ is a Malcev algebra and $\phi_0^-$ is a 2-cocycle of $(A_0, \star)$. It also follows from (13) and (16) that $\phi_0^+ = \phi$ on $A_0$ and hence $\phi_0^+$ is a symmetric invariant form $(A_0, \star)$. Thus, the multiplication $\star$ in (13) can be reformulated as

\begin{equation}
 (x + \lambda c) \star (y + \mu c) = \frac{1}{2}[x, y]_0 + \phi_0(x, y)c
 = x \star y + \phi_0(x, y)c
\end{equation}

for $x, y \in A_0$ and $\lambda, \mu \in F$. The converse of these remarks is

**Corollary 7.** Let $A_0$ be a Malcev algebra with multiplication denoted by $\star$ over a field $F$ of characteristic $\neq 2$, and let $Fc$ be a one-dimensional space over $F$. Assume that $\phi_0$ is a bilinear form on $A_0$ such that $\phi_0^-$ is a 2-cocycle of $(A_0, \star)$ and $\phi_0^+$ is an invariant form on $(A_0, \star)$. Then, the vector space direct sum $A = A_0 \oplus Fc$ with multiplication $\star$ defined by (17) is a flexible Malcev-admissible nil algebra of nil-index $\leq 3$.

The proof of Corollary 7 is straightforward. When Theorem 6 is applied to a Kac–Moody algebra with one-dimensional center, we have

**Corollary 8.** Let $A$ be a flexible Lie-admissible algebra such that $A^-$ is isomorphic to a Kac–Moody algebra with one-dimensional center and with Cartan subalgebra $H$. If $H$ is nil in $A$ and of nil-index $\leq 4$, then the multiplication in $A$ is determined by (13) or (17).

**References**

1. G.M. Benkart, *The construction of examples of Lie-admissible algebras*,
Flexible Malcev-admissible algebras with nil Cartan subalgebras


Seoul National University,
University of Northern Iowa
and
Ewha Woman University