SUFFICIENT CONDITIONS FOR LOCAL SOLVABILITY OF NONSOLVABLE PSEUDODIFFERENTIAL OPERATORS

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Introduction

Let $D$ be a pseudodifferential operator acting upon $C^\infty(Q; H^{\pm \infty}(R^n))$. In this paper we consider the existence of local solution of the equation

$$D_u = f$$

for a given $f \in C^\infty(Q \times R^n)$, i.e., we investigate what conditions should be imposed on the function $f$ for the existence of a $C^1$ solution $u$ of the equation (0.1) in some neighborhood of the origin. We consider only the case when $D = \partial_t + \partial_x B(t, D_x)$ is a nonsolvable operator in a neighborhood of the origin. The main result of this paper is given in Prop. 3.1.

1. Preliminaries

Let $R^\nu$ (resp. $R^n$) be a $\nu$-dimensional (resp. $n$-dimensional) Euclidean space. Throughout this paper, we shall denote by $Q$ an open subset of $R^n$ and by $R_n$ the dual of $R^n$. For any real number we denote by $H^s = H^s(R^n)$ the standard Sobolev space on $R^n$, i.e., the space of tempered distributions $u$ in $R^n$ whose Fourier transform $\hat{u}$ is a measurable function in $R^n$, satisfying

$$\|u\|_s = (2\pi)^{-\frac{n}{2}} \left( \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < + \infty.$$ 

Starting with $H^s$ we build the following spaces

$$H^{-\infty} = \bigcup_{s \in \mathbb{R}} H^s, \quad H^{+\infty} = \bigcap_{s \in \mathbb{R}} H^s.$$ 

For any real $s$, let $E^s$ denote the subspace consisting of the generalized functions $u$ whose Fourier transform $\hat{u}$ is a measurable function in $R_n$ satisfying

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As with the Sobolev spaces, we form the union and intersection of the space \( E_s \), but for \( s \) going to zero:

\[
E^0_0 = \bigcup _{s>0} E_s, \quad E^0_0 = \bigcap _{s>0} E^{-s}.
\]

Let \( t = (t_1, \ldots, t_n) \) denote the variable point in an open set \( \Omega \subset \mathbb{R}^n \). Let \( E \) be any one of the spaces \( H^{\pm \infty}, E^{0\pm} \). If \( p \) is any integer such that \( 0 \leq p \leq \nu \), we denote by \( A^pC^\infty(\Omega; E) \) the spaces of \( C^\infty \) \( p \)-form valued in the space \( E \). Thus to say that \( u \) belong to \( A^pC^\infty(\Omega; E) \) is the same as the saying that

\[
u(t, x) = \sum_{|J|=p} u_J dt_J
\]

where \( J \) is an ordered multi-index \( (j_1, \ldots, j_p) \) of integers such that \( 1 \leq j_1 < \cdots < j_p \leq \nu \), \( |J| \) its length, here equal to \( p \), \( dt = dt_1 dt_2 \ldots \ldots dt_\nu \), and \( u_J \) are \( C^\infty \) functions from \( \Omega \) to \( E \).

Now we consider a \( C^\infty \) one form in \( \Omega \), depending on the parameter \( \xi \) of \( R^n \):

\[
b(t, \xi) = \sum_{j=1}^n b_j(t, \xi) dt_j.
\]

We assume that the one form \( b(t, \xi) \) is exact in \( \Omega \). Thus there exists a primitive \( B \) of \( b \) such that \( b(t, \xi) = dt B(t, \xi) \). We also assume that

1. \( B(t, \xi) \) is real valued and positive homogeneous of degree one with respect to \( \xi \), and
2. \( B(t, \xi) \) is a \( C^\infty \) function of \( t \) in \( \Omega \) with values in \( C^1(R^n \setminus 0) \).

Under these assumptions \( b_j(t, D_x) \) defines naturally a pseudodifferential operator

\[
b_j(t, D_x) u(t, x) = (2\pi)^{-n} \int e^{ix \cdot \xi} b_j(t, \xi) \hat{u}(t, \xi) d\xi.
\]

Here \( \hat{u} \) denotes the Fourier transform with respect to \( x \). We form a pseudodifferential operator

\[
D = d_x + b(t, D_x) A.
\]

For each \( \rho = 0, 1, \ldots, \nu-1 \), it defines a linear operator

\[
D^\rho : A^\rho C^\infty(\Omega; E) \rightarrow A^{\rho+1}C^\infty(\Omega; E).
\]

We set \( D^0 = 0 \). Then obviously we have

\[
D^2 = D^{\rho+1} \circ D^\rho = 0
\]
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for any \( p = 0, 1, \ldots, \nu - 1 \). We note that \( \hat{D} = e^{-B(t, \xi)} d_t e^{B(t, \xi)} \). It is evident that \( \hat{D} \), hence also \( D \), generates a complex.

We concern the equation

\[
Du = f,
\]

where \( f \in \mathcal{A}^{p+1}(Q; \mathcal{E}) \). By the Fourier transform with respect to \( x \) we see that (1.1) is equivalent to

\[
d_t (e^{B(t)} f) = e^{B(t)} f \quad \text{(for a.e. } \xi \text{ in } \mathbb{R}^n)\]

We denote by \( \mathcal{S}_p \mathcal{C}^\infty(Q; \mathcal{E}) \) the space of elements \( f \) of \( \mathcal{A}^{p+1}(Q; \mathcal{E}) \) which satisfy the compatibility condition; namely,

\[
(1.3) \quad \text{for a.e. } \xi \text{ in } \mathbb{R}^n, \text{ the } p\text{-form } e^{B(t, \xi)} f(t, \xi) \text{ is a coboundary}
\]

2. Property (\( \psi \)) and the statement of the solvability results.

F. Treves [3] found the necessary and sufficient condition for the solvability of the equation (1.1), which is a natural generalization of the condition (\( P \)) for a single linear partial differential operator. We will state the property (\( \psi \)) and the solvability results in F. Treves [3].

We consider the complex

\[
\cdots \rightarrow \mathcal{A}^p \mathcal{C}^\infty(Q; \mathcal{E}) \xrightarrow{d_p} \mathcal{A}^{p+1}(Q; \mathcal{E}) \rightarrow \cdots.
\]

Let \( Q' \) be a nonempty open subset of \( Q \), possibly equal to \( Q \). Let \( U, V \) be two open subsets of \( Q \) such that \( U \subset Q' \cap V \). For any \( \xi \in \mathbb{R}^n \) and any real \( r \) we write

\[
U(\xi, r) = \{ t \in U; B(t, \xi) < r \}
\]

and similarly with \( V \) substituted for \( U \).

We consider the natural homomorphisms

\[
(2.1) \quad H_p(U(\xi, r)) \xrightarrow{i_p} H_p(Q') \xleftarrow{j_p} H_p(V(\xi, r))
\]

where \( H_p \)'s stand for the \( p \)-th homology groups.

DEFINITION 2.1. We say that the system \( D \) has property (\( \psi \)), in dimension \( p \), in \( Q' \) relative to \( Q \), if to every open subset \( U \subset Q' \) there exists an open set \( V \subset Q \) containing \( U \) such that, given any \( \xi \) in \( \mathbb{R}^n \) and real number \( r \),

\[
(2.2) \quad \ker i_p \subset \ker j_p
\]

We say that \( D \) has property (\( \psi \)), in dimension \( p \) in \( Q \) if it has it in
THEOREM 2.1. Suppose that $D$ does not have property $(\phi)$, in dimension $p$, in $\Omega'$ relative to $\Omega$. Then there is an element $f$ of $\mathcal{E}_{D}^{p+1}C^{\infty}(\Omega'; H^{+\infty})$ and a relatively compact subset $U$ of $\Omega'$ such that

$$Du=f \text{ in } U$$

has no solution $u$ in $A^{p}C^{\infty}(U, H^{-\infty})$.

THEOREM 2.2. Suppose that the system $D$ has property $(\phi)$, in dimension $p$, in $\Omega'$ relative to $\Omega$. Let $E$ be any one of $H^{\pm\infty}, E^{0\pm}$. Then, given any relatively compact open subset $U$ of $\Omega'$ and any element $f$ in $\mathcal{E}_{D}^{p+1}C^{\infty}(\Omega; E)$, the equation

$$Du=f \text{ in } U$$

has a solution $u$ in $A^{p}C^{\infty}(U; E)$.

For the proofs of Theorem 2.1 and Theorem 2.2 see Section II.3 in [3].

REMARK 2.1. When $\Omega'$ is homologically trivial in dimension $p$, the property (2.2) can be stated in a simpler manner. Thus assume that $H_{0}(\Omega')=\mathbb{C}$ and $H_{p}(\Omega')=0$ ($p>0$). Then (2.2) has the following meaning: If $p=0$,

(2.3) any two points in $U(\xi, r)$ can be joined by a continuous path contained in $V(\xi, r)$, whereas, if $p>0$,

(2.4) every $p$-cycle in $U(\xi, r)$ is homologous to zero in $V(\xi, r)$.

PROPOSITION 2.1. The system $D$ has property $(\phi)$, in dimension $\nu-1$, in $\Omega'$ relative to $\Omega$ if and only if any one of the following equivalent properties holds, for any $\xi$ in $\mathbb{R}_{n}$ and any $r$ in $\mathbb{R}$:

(2.5) The natural homomorphism $H_{\nu-1}(\Omega'(\xi, r)) \rightarrow H_{\nu-1}(\Omega')$ is injective.

(2.6) The natural homomorphism $H^{\nu-1}(\Omega') \rightarrow H^{\nu-1}(\Omega'(\xi, r))$ is surjective.

(2.7) No connected component of $\Omega' \backslash \Omega'(\xi, r)$ is compact

For the proof see [3].

EXAMPLE 2.1. Let $B(t, \xi) = -(t_{1}^{2}+t_{2}^{2})|\xi|$, where $(t_{1}, t_{2}) \in \Omega = (-T_{1}, T_{1}) \times (-T_{2}, T_{2})$ and $\xi \in \mathbb{R}_{n}$. Then $D=d_{x}+d_{t}B(t, D_{x})A$ does not satisfy the property (2.3) when $p=0$ and hence $D$ is a nonsolvable operator in dimension 0, in $\Omega$. Also $D$ does not satisfy the property (2.7) when $p=1$, and therefore $D$ is a nonsolvable operator in dimen-
Remark 2.2. When \( \nu = 1 \), there is only one case: \( p = 0 = \nu - 1 \), and (2.2) is equivalent to (2.7). Let us take \( \Omega' = \Omega \) to be an interval. Then (2.2) is equivalent to (2.3). When \( \nu = 1 \), one deals with a single operator \( D = \partial / \partial t + b(t, D_2) \), where \( b(t, \xi) = \partial B(t, \xi) / \partial t \). It is seen at once that the validity of (2.5) in \( \Omega \) (in dimension zero) is equivalent to the following property:

(2.8) For all \( \xi \in \mathbb{R}^n \) if \( b(t^0, \xi) > 0 \) for some \( t^0 \) in \( \Omega \), then \( b(t, \xi) \leq 0 \) for every \( t \in \Omega, t > t^0 \).

Example 2.2. Let \( B(t, \xi) = -t^2 |\xi|^2 \) and hence \( D = \partial / \partial t + b(t, D_2) \). Let \( \Omega \) be an interval containing the origin. Then \( b(t, \xi) \) does not satisfy condition (2.8) and hence \( D \) is a nonsolvable operator.

3. Sufficient conditions for the solvability of nonsolvable operators

In this section we concern nonsolvable operators. First we deal a single operator \( D = \partial / \partial t + b(t, D_2) \), where \( b(t, \xi) = \partial B(t, \xi) / \partial t \). We will make the assumptions for \( B(t, \xi) \) as follows:

(3.1) For some fixed \( \xi \in \mathbb{R}^n \setminus \{0\} \), \( B(t, \xi) \) has a local maximum at \( t = 0 \), in which case the function \( B(t, \xi) \) of \( t \) is decreasing in \( (0, T) \) and increasing in \( (-T, 0) \). Let \( V_1 = \{ \xi \in \mathbb{R}^n \setminus \{0\} : B(t, \xi) \) is decreasing in \( (0, T) \) and increasing in \( (-T, 0) \) with respect to \( t \} \). Then \( V_1 \) is a cone since \( B(t, \xi) \) is positive homogeneous of degree 1 with respect to \( \xi \).

(3.2) For any fixed \( \xi \in \mathbb{R}^n \setminus V_1 \), \( B(t, \xi) \) is a monotone function of \( t \) in \( (-T, T) \) or it has a local minimum at \( t = 0 \), in which case \( B(t, \xi) \) is increasing in \( (0, T) \) and decreasing in \( (-T, 0) \).

If \( B(t, \xi) \) satisfies the condition (3.1), then, from Remark 2.2, we see that the operator \( \partial / \partial t + b(t, D_2) \), where \( \partial B(t, \xi) / \partial t = b(t, \xi) \), is a nonsolvable operator.

Proposition 3.1. Let \( B(t, \xi) \) satisfy the above hypothesis (3.1) and (3.2). Let \( f \in C^0(\mathbb{R}^n \times \mathbb{R}^n) \) and

\[
Kf(x) = (2\pi)^{-n} \int_{-T}^{T} \int_{\mathbb{R}^n} e^{ix \cdot \xi - B(0, \xi) + B(t, \xi)} \chi_{V_1}(\xi) f(s, \xi) ds \, d\xi
\]

be real analytic. Then
has a $C^1$ solution.

Proof. If we take a Fourier transform (w. r. to $x$) of the equation (3.3), then we have

$$\frac{\partial \hat{u}}{\partial t} + b(t, \xi) \hat{u} = f(t, \xi).$$

We have a formal solution of the equation (3.4):

$$\hat{u}(t, \xi) = \int_{t_0}^{t} e^{-B(t, \xi) + B(s, \xi)} f(s, \xi) \, ds,$$

where $t_0$ depends upon $\xi$. Let

$V_2 = \{ \xi \in \mathbb{R}^n : B(t, \xi) \text{ is a monotone increasing function of } t \text{ in } (-T, T) \}.$

Then for $\xi \in V_2$, we have an integral representation of a solution of the equation (3.4) as follows:

$$\hat{u}(t, \xi) = \int_{-T}^{t} e^{-B(t, \xi) + B(s, \xi)} f(s, \xi) \, ds.$$

In this case $\mathcal{F}^{-1}(\chi_{V_2} \hat{u})$ is a rapidly decreasing function and $\mathcal{F}^{-1}(\chi_{V_2} \hat{u})$ is a $C^\infty$ function of $(t, x)$.

Let $V_3 = \{ \xi \in \mathbb{R}^n \setminus (V_1 \cup V_2) : B(t, \xi) \text{ is decreasing in } (-T, T) \}.$

Then for $\xi \in V_3$, we have an integral representation of a solution of the equation (3.4) as follows:

$$\hat{u}(t, \xi) = -\int_{-T}^{t} e^{-B(t, \xi) + B(s, \xi)} f(s, \xi) \, ds.$$

In this case $\mathcal{F}^{-1}(\chi_{V_3} \hat{u})$ is a rapidly decreasing function and $\mathcal{F}^{-1}(\chi_{V_3} \hat{u})$ is a $C^\infty$ function of $(t, x)$.

Let $V_4 = \{ \xi \in \mathbb{R}^n \setminus (V_1 \cup V_2 \cup V_3) : B(t, \xi) \text{ is increasing in } (0, T) \text{ and decreasing in } (-T, 0) \}.$ Then for $\xi \in V_4$, we have a solution of the equation (3.4) as follows:

$$\hat{u}(t, \xi) = \int_{0}^{t} e^{-B(t, \xi) + B(s, \xi)} f(s, \xi) \, ds.$$

In this case $\mathcal{F}^{-1}(\chi_{V_4} \hat{u})$ is a rapidly decreasing function and $\mathcal{F}^{-1}(\chi_{V_4} \hat{u})$ is a $C^\infty$ function of $(t, x)$.

Let $V_1$ be a set defined in the condition (3.1). Then for $\xi \in V_1$, if $t > 0$, then we have a solution of the equation (3.4) as follows:
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\[ \hat{u}(t, \xi) = -\int_t^T e^{-B(t, \xi) + B(s, \xi)} \hat{f}(s, \xi) \, ds. \]

and if \( t < 0 \), we have a solution of the equation (3.4) as follows:

\[ \hat{u}(t, \xi) = \int_{-T}^t e^{-B(t, \xi) + B(s, \xi)} \hat{f}(s, \xi) \, ds. \]

In this case

\[ \hat{u}(0, \xi) - \hat{u}(-0, \xi) = -\int_{-T}^T e^{-B(0, \xi) + B(s, \xi)} \hat{f}(s, \xi) \, d\xi, \]

which means that \( \hat{u}(t, \xi) \) is not continuous at \( t = 0 \) when \( \xi \in V_1 \). If we denote \( \frac{\partial}{\partial t} \) the derivative in the distribution sense and \( \left[ \frac{\partial}{\partial t} \right] \) the classical derivative in \( t = 0 \), then for \( \xi \in V_1 \)

\[ \frac{\partial u}{\partial t} = \left[ \frac{\partial u}{\partial t} \right] + \delta(t) \left( \hat{u}(0, \xi) - \hat{u}(-0, \xi) \right) \]

Therefore for all \( \xi \in \mathbb{R}^n \), \( \hat{u}(t, \xi) \) is a solution of the following equation

\[ \left( \frac{\partial}{\partial t} + b(t, \xi) \right) \hat{u} = \left[ \frac{\partial u}{\partial t} \right] + b(t, \xi) \hat{u} + \delta(t) \chi_{V_1}(\xi) \left( \hat{u}(0, \xi) - \hat{u}(-0, \xi) \right) \]

Taking an inverse Fourier transform (w.r.t. \( \xi \)) of the equation (3.5), we have

\[ \frac{\partial u}{\partial t} + b(t, D_x) u = f - \delta(t) Kf(x), \]

where

\[ Kf(x) = -\mathcal{F}^{-1}(\chi_{V_1}(\xi) \left( \hat{u}(0, \xi) - \hat{u}(-0, \xi) \right)) \]
\[ = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi - B(0, \xi) + B(s, \xi)} \chi_{V_1}(\xi) \hat{f}(s, \xi) \, ds \, d\xi. \]

Now

\[ (3.7) \quad D(H(t) Kf(x)) = \delta(t) Kf(x) + H(t) b(t, D_x) Kf(x). \]

From (3.6) and (3.7), we have

\[ D(u + H(t) Kf(x)) = f + H(t) b(t, D_x) Kf(x). \]

We define a function \( \nu(t, x) \) as follows:

\[ \nu(t, x) = - (2\pi)^{-n} \int_{\mathbb{R}^n} b(t, \xi) Kf(\xi) e^{ix \cdot \xi - B(t, \xi) + B(s, \xi)} \, ds \, d\xi \quad \text{if} \quad t \geq 0 \]

and \( \nu(t, x) = 0 \) if \( t < 0 \).

Then we have
$D(u + H(t)Kf(x)) = f$, 
and hence $w = u + H(t)Kf(x) + \nu$ is a solution of the equation (3.3). It 
is easy to show that $w$ is $C^1$. The proof is complete.

Next we consider a operator $D = \partial_t + b(t, D_x)A$, where $b(t, \xi) = d_t B(t, \xi)$ and $x \in \mathbb{R}^n$, in an open set $\Omega \subset \mathbb{R}^n$ containing the origin. If for some $\xi \in \mathbb{R}^n \setminus 0$, $B(t, \xi)$ has a local minimum at an interior point of $\Omega$, then we can not guarantee the solvability of $D$ in dimension 0, in $\Omega$. In this section we only concern the particular nonsolvable operators. Let $\Omega = (-T_1, T_1) \times \cdots \times (-T_n, T_n)$. We will make the assumptions for the nonsolvable operator $D$ as follows:

There exists $j$ $(1 \leq j \leq \nu)$ such that both (3.8) and (3.9) holds;

(3.8) For some fixed $\xi \in \mathbb{R}^n \setminus 0$, $B(t, \xi)$ has a local maximum at the origin, in which for any fixed $(t_1, \ldots, t_j, \ldots, t_\nu) B(t, \xi)$ is a decreasing function of $t_j$ in $(0, T_j)$ and increasing function of $t_j$ in $(-T_j, 0)$. Let $V = \{ \xi \in \mathbb{R}^n \setminus 0 : B(t, \xi) \text{ satisfies condition (3.8)} \}$. Then $V$ is a cone since $B(t, \xi)$ is positive homogeneous of degree 1 with respect to $\xi$.

(3.9) For any fixed $\xi \in \mathbb{R}^n \setminus V$, $B(t, \xi)$ is a monotone function of $t_j$ in $(-T_j, T_j)$ or it has a local minimum at the origin, in which case $B(t, \xi)$ is an increasing function of $t_j$ in $(0, T_j)$ and decreasing function of $t_j$ in $(-T_j, 0)$.

**Proposition 3.2.** Let $B(t, \xi)$ satisfy the conditions (3.8) and (3.9). Let $f \in \mathcal{E}D^{-1}C_{0}^{\infty}(\Omega \times \mathbb{R}^n)$ and

$$Kf(t_1, \ldots, t_j, \ldots, t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{-T_j}^{T_j} e^{ix \cdot \xi - B(t_1, \ldots, t_j, \ldots, t_\nu)} \chi_V(\xi) f(t_1, \ldots, s_j, \ldots, t_\nu, \xi) ds_j d\xi$$

be a real analytic function of $x$. Then 

$Du = f$ in $\Omega \times \mathbb{R}^n$,

**Proof.** cf. Proposition 3.1.

**References**


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