EPIMORPHISMS OF NONCOMMUTATIVE $C^*$-ALGEBRAS

TAE GEUN CHO AND JAE CHUL RHO

1. Introduction

It was proved in Esterle [4] that every epimorphism from $C(X)$ onto a Banach algebra is continuous. Here $C(X)$ denotes the commutative $C^*$-algebra of continuous functions on a compact Hausdorff space $X$. Then Laursen has shown in [6] that an epimorphism from a $C^*$-algebra onto a commutative Banach algebra is necessarily continuous. However, it remains unknown whether every epimorphism of a $C^*$-algebra with noncommutative range is continuous.

In this note we study some properties of epimorphisms of $C^*$-algebras and give sufficient conditions for continuity of epimorphisms with noncommutative range. Let $A$ be a $C^*$-algebra and $B$ a Banach algebra. We show that for every epimorphism $\theta : A \to B$ the image of the closure of the kernel of $\theta$ coincides with the radical of $B$. Thus an epimorphism $\theta : A \to B$ is continuous if the radical of $B$ is commutative. This result slightly generalizes the Laursen's result. Also it is shown that an epimorphism $\theta : A \to B$ is continuous if the radical $R$ of $B$ satisfies the condition $\cap (R^n) = \{0\}$. This result generalizes corollaries 3.4 and 3.5 in [2] for epimorphisms of $C^*$-algebras.

2. Preliminaries

Let $A$ and $B$ be Banach algebras. By a homomorphism $\theta : A \to B$ we mean a multiplicative linear map which maps $A$ into $B$, and an epimorphism means a surjective homomorphism. For a homomorphism $\theta : A \to B$ the separating space of $\theta$ is a linear subspace of $B$ defined by

$\mathcal{S}(\theta) = \{ b \in B : \text{there is a sequence } a_n \to 0 \text{ in } A \text{ with } \theta(a_n) \to b \}.$

A homomorphism $\theta : A \to B$ is continuous if and only if $\mathcal{S}(\theta) = \{0\}$ by

Received March 27, 1986.
This research is supported by MOE grant 1985.
the closed graph theorem, and it can be easily shown that \( \mathfrak{d}(\theta) \) is a closed two-sided ideal of \( B \) if the range of \( \theta \) is dense in \( B \) [10].

The radical of a Banach algebra is the intersection of all maximal modular left ideals of the algebra, and a Banach algebra is called semi-simple if the radical of the algebra contains only the zero element. If a Banach algebra has no maximal modular left ideal, the radical of the algebra is defined to be the algebra itself, in this case the Banach algebra is called a radical algebra. The radical of a Banach algebra is a closed two-sided ideal, and it is itself a radical algebra since the radical of a closed two-sided ideal of a Banach algebra is the intersection of the ideal and the algebra [1: Coro. 24. 20].

An element \( a \) of a \( C^* \)-algebra is called self-adjoint if \( a=a^{*} \), and a subalgebra is called self-adjoint if it is closed under involution. Every closed two-sided ideal of a \( C^* \)-algebra is known to be selfadjoint, and the quotient algebra of a closed two-sided ideal is also self-adjoint hence such a quotient algebra is itself a \( C^* \)-algebra with the quotient norm. It is well-known that a \( C^* \)-algebra is semisimple.

The most important positive result on the continuity of epimorphisms on which our results depend is the following theorem due to Johnson [5].

**Theorem (Johnson)** Every epimorphism from a Banach algebra onto a semisimple Banach algebra is continuous.

### 3. Epimorphisms

**Lemma 1.** Let \( A \) and \( B \) be the Banach algebras and \( \theta : A \to B \) be an epimorphism. Then for each maximal modular left ideal \( M \) of \( B \) the inverse image of \( M \) is a maximal modular left ideal of \( A \) containing the kernel of \( \theta \).

**Proof.** Let \( M \) be a maximal modular left ideal of \( B \). Then the inverse image \( \theta^{-1}(M) \) is clearly a modular left ideal of \( B \) containing the kernel of \( \theta \). Let \( u \) be a right modular unit for \( \theta^{-1}(M) \) and let \( M' \) be a proper left ideal of \( A \) containing \( \theta^{-1}(M) \). Then \( \theta(M') \) is a modular left ideal of \( B \) containing \( M \). Suppose that \( \theta(M') = B \). Then \( \theta(u) \in \theta(M') \) and hence there is an element \( a \in M' \) with \( \theta(u) = \theta(a) \). Hence \( u \) must belong to \( M' \), which is impossible since it is a right modular unit for \( M' \). Thus we have \( \theta(M') \neq B \), hence by maximality of \( M, \theta(M') = M \) and thus \( M' = \theta^{-1}(M) \). Therefore \( \theta^{-1}(M) \) is maximal.
The context of the following lemma is included in the proof of Proposition 25.10 of [1]. But we include another proof of the lemma since it is an easy consequence of Lemma 1.

**Lemma 2.** Let $A$ and $B$ be Banach algebras and $\theta : A \to B$ be an epimorphism with the kernel $K$. Then $\theta(K)$ is contained in the radical $R$ of $B$.

**Proof.** For each maximal modular left ideal $M$ of $B$ the closure $K$ of the kernel is contained in $\theta^{-1}(M)$ since $\theta^{-1}(M)$ is closed and contains the kernel $K$. Hence we have

$$K \subseteq \bigcap \{\theta^{-1}(M) : M \text{ is a maximal modular left ideal of } B\}.$$ 

Since $\theta(\bigcap \theta^{-1}(M)) = \bigcap \theta(\theta^{-1}(M))$ and $\theta(\theta^{-1}(M)) = M$, we have

$$\theta(K) \subseteq \theta(\bigcap \theta^{-1}(M)) \subseteq \bigcap M = R$$

where intersection is taken over all maximal modular left ideals $M$ of $B$.

In the following lemma we prove that for every epimorphism $\theta$ of a C*-algebra with the kernel $K$, $\theta(K)$ is closed without assuming the continuity of $\theta$.

**Lemma 3.** Let $A$ be a C*-algebra and $B$ a Banach algebra. Then for each epimorphism $\theta : A \to B$ the image of the closure of the kernel of $\theta$ is closed in $B$.

**Proof.** Let $K$ denote the closure of the kernel of $\theta$ and let $\pi : A \to A/K$ be the quotient map. Now, we define a map

$$\bar{\theta} : B \to A/K \text{ by } \bar{\theta}(b) = a + K$$

where $a$ is an element of $A$ with $\theta(a) = b$. The map $\bar{\theta}$ is well-defined and clearly it is an epimorphism from $B$ onto the C*-algebra $A/K$. By Johnson's theorem the epimorphism $\bar{\theta}$ is continuous and it has the closed kernel. To complete the proof it is enough to show that $\theta(K) = \ker(\bar{\theta})$ where $\ker(\bar{\theta})$ denotes the kernel of $\bar{\theta}$.

If $b \in \theta(K)$ there is an $a \in K$ with $\theta(a) = b$. Thus $\bar{\theta}(b) = a + K = K$, which implies that $b$ belongs to $\ker(\bar{\theta})$. Conversely, let $b$ be an element of $\ker(\bar{\theta})$, then $\bar{\theta}(b) = K$. Let $a \in A$ with $\theta(a) = b$. By the definition of $\bar{\theta}$ we have $a + K = \theta(b) = K$. Hence $a \in K$ and $b \in \theta(K)$.

**Theorem 4.** Let $A$ be a C*-algebra and $B$ a Banach algebra with the radical $R$. For every epimorphism $\theta : A \to B$ we have $\theta(K) = R$. 

Proof. It is enough to show that $R \subseteq \theta(\overline{K})$ since we have already shown that $\theta(\overline{K}) \subseteq R$. Since $\theta(\overline{K})$ is a closed two-sided ideal of $B$ the quotient algebra $B/\theta(\overline{K})$ is a Banach algebra. Let $\overline{\theta} : B \to A/\overline{K}$ be the epimorphism defined in Lemma 3 and let $\phi : B \to B/\theta(\overline{K})$ be the quotient map. Since the kernel of $\overline{\theta}$ and $\theta(\overline{K})$ coincide there exists a continuous isomorphism

$$\hat{\theta} : B/\theta(\overline{K}) \to A/\overline{K}$$

such that $\overline{\theta} = \hat{\theta} \circ \phi$. Since $\hat{\theta}$ maps $B/\theta(\overline{K})$ onto the $C^*$-algebra $A/\overline{K}$ the Banach algebra $B/\theta(\overline{K})$ is semisimple.

On the other hand, by Lemma 1

$$R = \cap \{ M' : M' \text{ is a maximal modular left ideal of } B \}$$

$$\subseteq \cap \{ \phi^{-1}(M) : M \text{ is a maximal modular left ideal of } B/\theta(\overline{K}) \}$$

Thus we have

$$\phi(R) \subseteq \cap \{ M : M \text{ is a maximal modular left ideal of } B/\theta(\overline{K}) \}$$

$$= \text{the radical of } B/\theta(\overline{K}) = \{ 0 \}$$

since $B/\theta(\overline{K})$ is semisimple. Hence

$$R \subseteq \ker(\phi) = \theta(\overline{K}).$$

Remark. For an epimorphism $\theta : A \to B$ from a $C^*$-algebra onto a Banach algebra $B$ it is known that $R = \mathcal{J}(\theta)$ where $\mathcal{J}(\theta)$ is the separating space of $\theta$ [8 : Thm 4.1], [3 : Coro. 2.2]. Hence we have $\theta(\overline{K}) = R = \mathcal{J}(\theta)$.

Corollary 5. Let $A$ and $B$ be as in Theorem 4. If the radical $R$ of $B$ is commutative, then an epimorphism $\theta : A \to B$ is continuous.

Proof. Let $\hat{\theta} : \overline{K} \to R$ be the restriction of an epimorphism $\theta : A \to B$ to the closure of the kernel $K$ of $\theta$. Then $\hat{\theta}$ is an epimorphism of a $C^*$-algebra with commutative range, hence it is continuous by Laursen's result. Consequently $\hat{\theta}$ has the closed kernel and $K = \overline{K}$. Therefore we have $\mathcal{J}(\theta) = \theta(\overline{K}) = \{ 0 \}$.

Theorem 6. Let $A$ be a $C^*$-algebra and $B$ a Banach algebra with the radical $R$. If $\cap (R^n)^- = \{ 0 \}$, then every epimorphism $\theta : A \to B$ is continuous. Here, $(R^n)^-$ denotes the closure of $R^n$.

Proof. Let $\overline{K}$ be the closure of the kernel of $\theta$. Since the restriction $\theta : \overline{K} \to R$ is an epimorphism of a $C^*$-algebra, to prove the continuity of $\theta$ it is enough to show that $\theta$ is continuous on each commutative
Let \( a \in \mathcal{K} \) be a self-adjoint element, \( \mathcal{C}^*(a) \) denote the \( \mathcal{C}^* \)-algebra generated by \( a \) and \( D \) be the closure of \( \theta(\mathcal{C}^*(a)) \). Since \( D \) is a closed subalgebra of the radical algebra \( R \), it is a radical algebra.

Suppose that the restricted homomorphism \( \theta : \mathcal{C}^*(a) \rightarrow D \) is discontinuous for some self-adjoint element \( a \) of \( \mathcal{K} \). Applying Theorem 4.3 of [9] we see that for every element \( x \) of \( \mathcal{C}^*(a) \theta(x) \) is not nilpotent and

\[
(\theta(x)D)^- = (\theta(x)^nD)^-
\]

for each positive integer \( n \). Suppose that there is an element in \( \mathcal{C}^*(a) \) with \( \theta(y) \neq 0 \). Since \( \theta(y) \in D \) we have

\[
\theta(y)^2 \in (\theta(y)^nD)^- \subset (R^{n+1})^-
\]

for each positive integer \( n \) and hence

\[
\theta(y)^2 \in \bigcap_{n=1}^{\infty} (R^n)^- = \{0\},
\]

which implies that \( \theta(y)^2 = 0 \). This is contrary to \( \theta(y) \) being not nilpotent, and \( \theta \) must be continuous on \( \mathcal{C}^*(a) \) for each self-adjoint element \( a \) in \( \mathcal{K} \). Therefore the epimorphism \( \theta : \mathcal{K} \rightarrow R \) is continuous and we have \( \mathcal{C}(\theta) = \{0\} \).

If \( B \) is a Banach algebra satisfying the descending chain condition for left (or right) ideals, then the radical \( R \) of \( B \) is nilpotent, that is, there is a positive integer \( n \) such that \( R^n = \{0\} \). (See e.g. [7; p.120]). Consequently we have the following corollary to Theorem 6.

**Corollary 7.** If \( A \) is a \( \mathcal{C}^* \)-algebra and \( B \) is a Banach algebra satisfying the descending chain condition for left ideals, then every epimorphism \( \theta : A \rightarrow B \) is continuous.

**References**


Sogang University
Seoul 121, Korea