ON CERTAIN LIPSCHITZIAN INVOLUTIONS IN BANACH SPACES

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1. Introduction

In [3], [4], K. Goebel and E. Zlotkiewicz investigated conditions under which lipschitzian involutions or lipschitzian maps with nonexpansive square of a closed bounded convex subset X of a Banach space B have fixed points. A map \( T : X \rightarrow X \) is called an involution if \( T^2 = I \), where \( I \) denotes the identity map, and a \( k \)-lipschitzian if \( \|Tx - Ty\| \leq k\|x - y\| \) holds for all \( x, y \in X \). A 1-lipschitzian map is said to be nonexpansive.

In the present paper, the main results of [3], [4] are so strengthened that some information concerning the geometric estimations of fixed points are given.

Our tool in this paper is the following in [7], which is a consequence of the well-known variational principle of Ekeland [1], [2] for approximate solutions of minimization problems.

**Theorem 0.** Let \( V \) be a complete metric space and \( f : V \rightarrow V \) be a map such that there exists an \( L \in [0, 1) \) satisfying

\[
d(fx, f^2x) \leq Ld(x, fx)
\]

for any \( x \in V \).

If \( F(x) = d(x, fx) \) on \( V \) is l. s. c., then

1. \( \lim f^nx = p \) exists for any \( x \in V \),

\[
d(f^nix, p) \leq \frac{L^n}{1-L}d(x, fx),
\]

and \( p \) is a fixed point of \( f \), and

2. for any \( u \in V \) and \( \varepsilon > 0 \) satisfying

\[
F(u) \leq (1-L)\varepsilon,
\]

\( f \) has a fixed point in \( \bar{B}(u, \varepsilon) \). Further, if \( f \) is a quasi-lipschitzian with

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217
constant $k$, then either $u$ is a fixed point of $f$ or $f$ has a fixed point in $\overline{B}(u, \varepsilon) \setminus B(u, s)$ where $s = d(u, fu) (1 + k)^{-1}$.

Note that $\overline{B}(u, \varepsilon)$ denotes the closed ball with center $u$ and radius $\varepsilon$, and $B(u, \varepsilon)$ the corresponding open ball.

A map $f : V \rightarrow V$ is called a quasi-lipschitzian with constant $k$ if $\|fx - fp\| \leq k\|x - p\|$ holds for all $x \in V$ and for every fixed point $p$ of $f$.

## 2. Main Results

The modulus of convexity of the space $B$ is the function $\delta : [0, 2] \rightarrow [0, 1]$ defined by the following formula

$$
\delta(\varepsilon) = \inf \{1 - \frac{\|x + y\|}{2} : x, y \in \overline{B}(0, 1), \|x - y\| \geq \varepsilon\}.
$$

Note that the function $\delta(\varepsilon)$ is nonincreasing and convex.

Moreover, for any $x, y \in \overline{B}(0, r)$ and any $a$ such that $0 \leq a \leq 2r$ and $\|x - y\| \geq a$, we have

$$
\frac{\|x + y\|}{2} \leq (1 - \delta(\frac{a}{r})) r
$$

Now we have our first result:

**Theorem 1.** Let $X$ be a closed convex subset of a Banach space $B$ and $T : X \rightarrow X$ a $k$-lipschitzian involution. If $L = k(1 - \delta(2/k)) / 2 < 1$, then for any $u \in X$ and $\varepsilon > 0$ satisfying

$$
\|u - Tu\| \leq (1 - L) \varepsilon,
$$

either $u$ is a fixed point of $T$ or there is a fixed point of $T$ in $\overline{B}(u, \varepsilon/2) \cap X \setminus B(u, s)$ where $s = \|u - Tu\| (k + 3)^{-1}$.

**Proof.** For any $x \in X$,

$$
\|T\left(\frac{x + Tx}{2}\right) - x\| = \|T\left(\frac{x + Tx}{2}\right) - T^2x\| \\
\leq k\left|\frac{x + Tx}{2} - Tx\right| \\
= \frac{k}{2}\|x - Tx\| \\
\|T\left(\frac{x + Tx}{2}\right) - Tx\| \leq k\left|\frac{x + Tx}{2} - x\right| \\
= \frac{k}{2}\|x - Tx\|.
$$
Thus, by the property of modulus of convexity, we have
\[
\| \frac{x + T^2 x}{2} - T \left( \frac{x + T x}{2} \right) \| \leq \left( 1 - \delta \left( \frac{2}{k} \right) \right) \frac{k}{2} \| x - T x \|.
\]
Now if we put \( G = \frac{1}{2} (I + T) \), then
\[
\| G x - G^2 x \| = \| \frac{G x + T G x}{2} - G x \|
\]
\[
= \frac{1}{2} \| T G x - G x \|
\]
\[
\leq \frac{1}{2} \left( 1 - \delta \left( \frac{2}{k} \right) \right) \frac{k}{2} \| x - T x \|
\]
\[
= \left( 1 - \delta \left( \frac{2}{k} \right) \right) \frac{k}{2} \| x - G x \|
\]
\[
= L \| x - G x \|.
\]
Therefore, by Theorem 0(1), \( \lim G^\infty x = p \) exists for \( x \in X \), and \( p \in \text{Fix } T \), the fixed point set. Since \( T \) is a \( k \)-lipschitzian, \( G \) is a \((k + 1)/2\)-lipschitzian and quasi-lipschitzian. Therefore, by Theorem 0(2), for any \( u \in X \) with \( \| u - T u \| \leq (1 - L) \varepsilon \), we have \( \| u - G u \| = \| u - T u \| / 2 \leq (1 - L) \varepsilon / 2 \). Hence, \( u \) is a fixed point of \( G \) or there is a fixed point of \( G \) in \( B(u, \varepsilon/2) \cap X \setminus B(u, s) \) where \( s = \| u - G u \| / (1 + (k+1)/2) = \| u - T u \| / (k+3) \). This completes our proof.

**Corollary 1.** [3, Theorem 1]. Let \( X \) be a closed convex subset of a Banach space \( B \) and \( T : X \to X \) a \( k \)-lipschitzian involution such that \( k(1 - \delta(2/k))/2 < 1 \). Then \( T \) has at least one fixed point.

**Corollary 2.** Let \( X \) be a closed convex subset of a Banach space \( B \) and \( T : X \to X \) a \( k \)-lipschitzian involution. If \( 0 \leq k < 2 \), then for any \( u \in X \) and \( \varepsilon > 0 \) satisfying
\[
\| u - T u \| \leq \left( 1 - \frac{k}{2} \right) \varepsilon
\]
the conclusion of Theorem 1 holds.

**Proof.** Let \( L = k/2 \) and \( G = \frac{1}{2} (I + T) \). Then \( L < 1 \), and
\[
\| G x - G^2 x \| \leq \left( 1 - \delta \left( \frac{2}{k} \right) \right) \frac{k}{2} \| x - G x \|
\]
\[
\leq \frac{k}{2} \| x - G x \|
\]
\[
= L \| x - G x \|.
\]
Thus, by Theorem 1, we have the same conclusion to Theorem 1.
Corollary 2 improves [4, Theorem 1].

The characteristic of convexity of the space $B$ is the number $\varepsilon_0 = \sup \{\varepsilon : \delta(\varepsilon) = 0\}$.

Some of Banach spaces can be fully characterized by the number $\varepsilon_0$ and the modulus of convexity. The following facts are known [3]

1. If $\varepsilon_0 < 1$, then $B$ has normal structure,
2. $B$ is uniformly non-square iff $\varepsilon_0 < 2$, and
3. $B$ is strictly convex iff $\delta(2) = 1$.

**Theorem 2.** Let $X$ be a closed convex bounded subset of a Banach space $B$ with $\varepsilon_0 < 1$ and $\delta(2) = 1$, and $T : X \rightarrow X$ a $k$-lipschitzian map such that $T^2$ is nonexpansive. If $L = k(1 - \delta(2)/k)/2 < 1$, then the conclusion of Theorem 1 holds.

**Proof.** Since $\varepsilon_0 < 1$, $B$ is uniformly non-square and in view of [5], it is reflexive, and moreover it has normal structure. Since $T^2$ is nonexpansive, by Kirk's fixed point theorem [6], the set $C^* = \{x : T^2x = x\}$ is nonempty. $\delta(2) = 1$ means the strict convexity of $B$ and implies that $C^*$ is convex. Obviously we have $T(C^*) = C^*$ and $T^2 = I$ on $C^*$. Hence, using Corollary 1 for the restriction of $T$ on $C^*$, we can apply Theorem 1.

Theorem 2 improves [3, Theorem 2].

**Theorem 3.** Let $X$ be a closed convex subset of a uniformly convex Banach space $B$ and $T : X \rightarrow X$ a $k$-lipschitzian involution. If $L = k\delta^{-1}(1-1/k)/4 < 1$, then for any $u \in X$ and $\varepsilon > 0$ satisfying $\|u - Tu\| < (1 - L)\varepsilon$, either $u$ is a fixed point of $T$ or $T$ has a fixed point in $B(u, \varepsilon/2) \cap X \setminus B(u, s)$ where $s = \|u - Tu\|(k + 3)^{-1}$.

**Proof.** Let $G = (I + T)/2$ and let for $x \in X$, $y = Gx$ and $z = Ty$. Then

$$
\|x - Ty - x\| = \|T^2 - x\| \\
\leq k\|y - Tx\| = k\|Gx - Tx\| \\
= k\left\|\frac{x + Tx}{2} - Tx\right\| = \frac{k}{2}\|x - Tx\| \\
\|T^2 - y - x\| = \|2Gx - T^2 - x\| = \|x + Tx - Ty - x\| \\
= \|Tx - Ty\| \leq k\|x - y\| = k\|x - Gx\| \\
= k\left\|\frac{x + Tx}{2} - x\right\| = \frac{k}{2}\|x - Tx\|
$$
On certain lipschitzian involutions in Banach spaces

and

\[ \| \frac{z + (2y - z)}{2} - x \| \leq \| y - x \| = \| x - Gx \| = \frac{1}{2} \| x - Tx \|. \]

Thus, by the property of modulus of convexity, we have

\[ \| z - (2y - z) \| \leq \frac{k}{2} \delta^{-1} \left( 1 - \frac{1}{k} \right) \| x - Tx \|. \]

But since

\[ \| z - (2y - z) \| = 2 \| y - z \| = 2 \| Gx - Ty \| = 4 \| Gx - G^2x \| \]

and

\[ \| x - Tx \| = 2 \| x - Gx \| , \]

we have

\[ 4 \| Gx - G^2x \| \leq k \delta^{-1} \left( 1 - \frac{1}{k} \right) \| x - Tx \| \]

i.e.,

\[ \| Gx - G^2x \| \leq \frac{k}{4} \delta^{-1} \left( 1 - \frac{1}{k} \right) \| x - Gx \| = L \| x - Gx \| , \]

and

\[ \| x - Gx \| \leq (1 - L) \varepsilon / 2. \]

Thus \( G \) satisfies all the hypothesis of Theorem 0, and we conclude our result.

Theorem 3 strengthens \cite[Theorem 2]{4}.

**Theorem 4.** Let \( X \) be a closed bounded convex subset of a uniformly convex Banach space \( B \), and \( T : X \to X \) a \( k \)-lipschitzian map such that \( T^2 \) is nonexpansive. If \( L : = k \delta^{-1} (1 - 1/k) / 4 < 1 \), then the conclusion of Theorem 3 holds.

**Proof.** Since every uniformly convex Banach space is strictly convex, reflexive and has normal structure, by Theorem 2 and Theorem 3, we obtain our result.

Theorem 4 strengthens \cite[Theorem 3]{4}.

**References**

3. K. Goebel, *Convexity of balls and fixed point theorems for mappings with*


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