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A Class of Singular Quadratic Control Problem With Nonstandard Boundary Conditions

by

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Abstract

A class of singular quadratic control problem is considered. The state is governed by a higher order system of ordinary linear differential equations and very general nonstandard boundary conditions. These conditions in many important cases reduce to standard boundary conditions and because of the conditions the usual controllability condition is not needed. In the special case where the coefficient matrix of the control variable in the cost functional is a time-independent singular matrix, the corresponding optimal control law as well as the optimal controller are computed. The method of investigation is based on the theory of least-squares solutions of multi-valued operator equations.

1. Introduction

In many classical control problems associated with a first order system of differential equations, it is usually required that the state go through an initial target and a terminal target. But if the system is not completely controllable, then this is not always possible. In some physical problems. due to the possible error involved in the measuring the targets, it may be sufficient to require that the system be "nearest" possible to the targets, a concept which is somewhat similar to the least-squares solutions of a matrix equation. A similar idea has been used in the literature. For example, Minamide and Nakamura [19, 20] considered the system which is required to go through one given target, but to be within a limit from a second target. Because of the uniqueness theorem of a linear differential equation, this system generates only

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single responses, and the control spaces must be restricted to get the desired effect. But the controllability condition is not needed. There is another idea used in this paper. It is to require that the system be simultaneously "closest" possible to two given targets without insisting that the system go through any of its targets. In this case also the controllability condition is not needed and the admissible controllers become cheap (they cover the whole space under consideration). But each controllers may generate infinitely many responses. The idea of this nonstandard boundary condition was first used in [13] to study a regular quadratic control problem generated by a first order system. It was shown there that in many important cases this condition becomes a classical boundary condition.

There are many physical problems where a state is governed by a higher order scalar equation ([3]). The usual way of studying this is to convert the equation into a first order sustem by arbitrary choices of change of variables. These changes often obscure the essential of the problem and are inconvenient. Thus in this paper we will consider singular quadratic control problems associated directly with higher order systems with very general nonstandard boundary conditions. This is investigated by a new method based on multi-valued operator theory. This paper generalizes and at the same time simplifies the corresponding results of [13]. Some of the results in this paper is announced in [14]. The paper is organized as follows: In §2, we summarize some known results on the least-squares solutions and the generalized inverses of multi-valued linear operator equations. In §3, we discuss differential operators generated by higher order systems which will be needed in the next section in deriving the adjoint equations for optimal controllers. §4 is the main part of this paper. The main results are Lemma 4.1(V) for the description of the dynamical system. Theorem 4.8 for feedback-like optimal controllers and Theorem 4.9 for the descriptions of the optimal controllers by state equations and adjoint equations. Finally the feedback laws and explicit forms of optimal controllers are given for a generalized two-point problem. In particular, our method of computing the optimal controllers is much simpler than the complicated method used in [4] and work for nonautonomous cases as well.

We now fix some notations. The field of complex numbers is denoted by $\boldsymbol{C}$. The n-dimensional complex Euclidean space $\boldsymbol{C}^n$ will be identified as $n\times 1$ complex matrices. If M is a complex matrix, then M^*, M^t and $\bar{M}$ will denote the transpose conjugate, the transpose and the conjugate of M, respectively. If M is a densely defined linear operator, its adjoint will also be denoted by M^*. Finally for $\alpha\in\boldsymbol{C}^n$, the Euclidean norm $(\alpha^*\alpha)^{1/2}$ of α is denoted by $|\alpha|$.

2. Least-squares solutions

In this section we will discuss some recent known results on the least-sqares solutions of multi-valued linear operator equations which will be needed later. If H is a Hilbert space (over the complex field $\boldsymbol{C}$). then its norm and inner product will be denoted by $\|\cdot\|$ and $<,>$ without indicating their dependence on H. suppose that H_1 and H_2 are Hilbert spaces over $\boldsymbol{C}$. Then $H_1 \oplus H_2$ will denote the direct-sum Hilbert space of all ordered pairs $\{x, y\}$ with $x \in H_1$, $y \in H_2$ with the inner product $<,>$ and the norm $\|\cdot\|$ defined by $\|\{x,y\}\| = [\|x\|^2 + \|y\|^2]^{1/2}$,

$$<\{x,y\}, \{p,q\}> := <x,p> + <y,q>.$$

Let M be a *subset* of $H_1 \oplus H_2$. Then

$M^\perp \equiv$ the orthogonal complement of M

$:= \{x \in H_1 \oplus H_2 : <m,x> = 0, \text{ all } m \in M\}$,

$M^{-1} \equiv$ the graph inverse of M

$:= \{\{x,y\} \in H_2 \oplus H_1 : \{y,x\} \in M\}$,

$M^* \equiv$ the adjoint of M

$:= \{\{y,-x\} \in H_2 \oplus H_1 : \{x,y\} \in M^\perp\}$,

Dom $M := \{x : \{x,y\} \in M \text{ for some } y\}$,

Range $M := \{y : \{x,y\} \in M \text{ for some } x\}$

Null $M := \{x : \{x,0\} \in M\}$

$M(u) := \{y : \{u,y\} \in M\}$.

It is well-known that when M is the graph a densely defined linear operator, M^* is the graph of the adjoint operator of M. In any case, M^* is always closed vector subspace of $H_2 \oplus H_1$.

suppose now that M is a linear manifold (called also a vector space, or a multi-valued linear operator). Then we say that $u \in M$ is a least-squares solution (briefly LSS) of an inclusion $g \in M(x)$ (g is a given element of H_2) if $u \in$ Dom M and for some $y \in M(u)$.

$$\|y-g\| = \min \{\|z-g\| : z \in \text{Range } M\}.$$

The following can be found in [16].

Theorem 2.1 Let $M \subset H_1 \oplus H_2$ be linear and $g \in H_2$. Then

(i) $g \in M(x)$ has a LSS iff

$$g \in \text{Range } M + (\text{Range } M)^{\perp}.$$

If Range M is closed, then

$$\text{Range } M + (\text{Range } M)^{\perp} = H_2.$$

(ii) u is a LSS of $g \in M(x)$ if and only if $u \in$ Dom M and $g \in M(u) + \text{Null } M^*$.

In order to find all LSSS of $g \in M(x)$, we need the concept of the orthogonal generalized inverse of a vector space.

Assume further that M is a closed vector subspace of $H_1 \oplus H_2$. Let $\mathscr{P}$ be the orthogonal projector of H_1 onto null M (*i.e.*. $\mathscr{P}$ is a bounded selfadjoint linear operator in H_1 such that $\mathscr{P}^2 = \mathscr{P}$ and Range $\mathscr{P} =$ Null M). Let $\mathscr{P}^+$ be the orthogonal projector of H_2 onto Null M^*. Dejine a vectos subspace $M^{\#} \subset H_2 \oplus H_1$ (called the orthogonal generalized inverse of M) by

$$M^{\#} := \{\{x, (I - \mathscr{P})(z)\} : x \in H_2,\ z \in H_1,\ \{z - (I - \mathscr{P}^+)(x) \in M\}.$$

The following can be found in [18] (see also [16]).

Theorem 2.2 Let $M \subset H_1 \oplus H_2$ be a closed vector subspace. Then

(i) $M^{\#} = R + (\text{Null } M^* \oplus \{0\})$ (orthogonal direct sum).

where

$$R := \{\{x, (I - \mathscr{P})(y) : \{y, x\} \in M\}.$$

In particular, $M^{\#}$ is the graph of a closed linear operator and satisfies

$$\text{Dom} M^{\#} = \text{Range } M + \text{Null } M^*,$$
$$\text{Range } M^{\#} = \text{Dom } M + (\text{Null } M)^{\perp}.$$

(ii) Dom $M^{\#} = H_2$ if and only if Range M is closed.

(iii) $\{y, g\} \in M$ if and only if $g \in$ Range M and

$y = M^{\#}(g) + k$ for some $k \in$ null M.

Remark 2.1 It is shown in [18] that R is the unique closed operato (the graph of) such that M^{-1} is the orthogonal direct sum of R and $\{0\} \oplus$ null M. R is called the orthogonal operator part of M^{-1}.

By part (i), $M^{*}(x)=y$ if and only if $x=z+k$ and $y=R(z)$ for some $k\in$Null M^{*} and $z\in$Range M.

Remark 2.2 By the above theorem.

$$\text{Null } M^{*}=\text{Null } M^{*}+M(0).$$

For, $x\in$Null M^{*} if and ohly if $\{x,0\}\in M^{*}$. This is equivalent by the above theorem to $x=y+k$ for some $k\in$Null M^{*}, $y\in$Null R.

By the definition of R,

$$\begin{aligned}\text{Null } R&=\{x:\ (I-\mathscr{P})(y)=0 \text{ for some } y \text{ such that } \{y,x\}\in M\}\\&=\{x:\ y\in\text{Null } M,\ \{y,x\}\in M\}\\&=\{x:\ \{0,x\}\in M\}=M(0).\end{aligned}$$

Thus

$$\text{Null } M^{*}=\text{Null } M^{*}+M(0).$$

Remarks 2.3 In the case when M is the graph of densely defined closed linear operator, M^{*} coincides with the graph of the Moore-penrose generalized inverse of M. Now any $m\times n$ complex matrix D can and will be in this paper identified as a linear operator from C^{n} into C^{m}. Then D^{*} is precisely the $n\times m$ Moore-penrose generalized inverse matrix of D.

Using generalized inverses we can find all Lsss of an inclusion in the following theorem. Its proof can be found in [17].

Theorem 2.3 Let $M\subset H_1\oplus H_2$ be a closed vector subspace.

Assume that $g\in$Range $M+$Null M^{*}. Then $M^{*}(g)+$Null M is the set of all Lsss of $g\in M(x)$. In particular, $M^{*}(g)\in(\text{Null } M)^{+}$ and is the unique LSS of $g\in M(x)$ with the smallest H_1-norm.

3. Differential operator

In this section we will summarize some known facts and prove some results on differential operators which will be needed later. All the notations used in this section will be used in the next section. Let $[t_0, t_1]$ be a compact interval. Let γ and γ^{+}(the formal adjoint of γ) be the lth order formal expression:

$$\gamma\, x = x^{(l)} + \sum_{0}^{l-1} P_i(t) x^{(i)}, \quad t_0 \leq t \leq t_1,$$

$$\gamma^+ x = (-1)^s x^{(s)} + \sum_{0}^{s-1} (-1)^i (P_i^*(t) x)^{(i)}, \quad t_0 \leq t \leq t_1,$$

where x is an $n\times 1$ vector, and $x^{(i)}$ denotes the ith derivative of x. For each i, P_i is an $n\times n$ matrix-valued function of t whose ith derivatives (taken componentwise) exist and are continuous on $[t_0, t_1]$. Let X_n (similar for X_m) be the Hilbert space over C of all $n\times 1$ vector-valued functions f defined on $[t_0, t_1]$ such that $\|f\|$ $[\int_{t_0}^{t_1} |f(t)|^2 dt]^{1/2} < \infty$. The inner product $<,>$ of this space is defined by

$$\langle f, g\rangle := \int_{t_0}^{t_1} f^t(t) \overline{g(t)}\, dt.$$

In the case when f and g are $n\times k_1$ and $n\times k_2$ matrix-valued functions, respectivly, for convenience, $\langle f, g\rangle$ will denote the $k_1\times k_2$ matrix $\int_{t_0}^{t_1} f^t(t)\, \bar{g}(t) dt$, provided that the integrals exist. We now define our basic differential operators. Let T_0 and T_1 be the differential operators (also are called the minimal and maximal differential operators, respectively) in X_n defined by:

Dom $T_1 = \{x \in X_n:\ x^{(l-1)}$ is absolutely continuous on $[t_0, t_1]$, $x^{(l)} \in X_n\}$,
$T_1 x = \gamma x$, $x \in$ Dom T_1:
Dom $T_0 = \{x \in$ Dom T_1: $x^{(i)}\ (t_0) = x^{(i)}(t_1) = 0,\ 0 \leq i \leq l-1\}$,
$T_0 x = \gamma x$, $x \in$ Dom T_1,

The following is well-known. Sse, for example, [12], [7].

Theorem 3.1 $T_0 \subset T_1$, $T_1^* \subset T_0^*$ and T_0, T_1 are densely defined and are closed. Moreover.

Dom $T_1^* =$ Dom T_0, Dom $T_1 =$ Dom T_0^*,
$T_0^* x = \gamma^+ x$, $x \in$ Dom T_0^*.

For $g \in$ Dom T_1, let $\tilde{g}$ be the $nl\times 1$ matrix-valued function of t defined by

$$\tilde{g} := \begin{pmatrix} g \\ g^{(1)} \\ \vdots \\ g^{(l-1)} \end{pmatrix}$$

Then there exists a $nl\times nl$ continuous, invertible matrix-valued function $E(t)$ such that the Green's formula holds:

$$(3.1)\ \int_{t_0}^{t_1}[(\gamma x)^t\ \bar{y}-x^t\overline{(\gamma^+y)}](t)dt$$
$$=\tilde{x}^t(t_1)E(t_1)\bar{\tilde{y}}(t_1)-\tilde{x}^t(t_0)E(t_0)\bar{\tilde{y}}(t_0)$$

for all $x, y\in \mathrm{Dom}\ T_1$.

Throughout this paper ϕ will denote an arbitrary but fixed $n\times nl$ matrix solution satisfying

$$(3.2)\ \gamma\phi(t)=0_{n\times nl},\ \ \det\tilde{\phi}(t_0)\neq 0.$$

Thus $\phi(t)$ is a fundamental matrix solution and $\tilde{\phi}(t)$ is invertible for all $t\in[t_0, t_1]$. Let us partition its inverse as

$$(3.3)\ \tilde{\phi}^{-1}(t)=[R(t),\ S(t)]$$

where $R(t)$ is $nl\times n(l-1)$ and $S(t)$ is $nl\times n$. Clearly if $l=1$. then $\phi^{-1}=S$. Using the method similar to the one used in p. 87, of [6] we have the following variation-of-constant formula. We state it for completeness.

Theorem 3.2 Let $f\in X_n$. Then $\gamma x=f$ if and only if

$$x(t)=\phi(t)\alpha+\phi(t)\int_{t_0}^{t}(Sf)(s)ds,\ \ t_0\leq t\leq t_1 \text{ for}$$

some $\alpha\in C^{nl}$.

Remark 3.1 We can also write the last term as

$$\phi(t)\int_{t_0}^{t}\tilde{\phi}^{-1}(s)\ [{}^on_{l_n}(l-1)\times n]f(s)\ ds.$$

The following, especially (ii), will play an important role in the next section in deriving the integro-differential equations for optimal pairs. It appears that the result is new in the literatue.

Theorem 3.3 Assume further that $P_0, P_1\cdots P_{l-1}$ are $(l-1)$ times differentiable on

$[t_0, t_1]$. Then

(i) S^* is a $n\times nl$ fundamental matrix solution of $\gamma^+ x=0$.

(ii) For $f\in X_n$, let

$$u(t):=S^*(t)\int_t^{t_1}(\phi^* f)(s)ds,\ t_0\leq t\leq t_1.$$

Then

$$\tilde{u}(t_1)=0.$$
$$\gamma^+ u=f\ a.a.t\in[t_0, t_1].$$

Proof. Define a $nl\times nl$ matrix A by

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix},$$

Where

$$A_{11}=0_{n(l-1)\times n},\quad A_{12}=I_{n(l-1)},$$
$$A_{21}=-P_0,\qquad -A_{22}=[P_1\cdots P_{l-1}].$$

Then $\gamma x=0$ is equivalent to $\dot{\tilde{x}}=A\tilde{x}(\dot{\tilde{x}}\equiv d\tilde{x}/dt)$.

Moreover, $\tilde{\phi}$ is a $nl\times nl$ fundamental matrix solution of $\dot{\tilde{x}}=A\tilde{x}$. It is clear that $\tilde{\phi}^{-1}=[R,S]$ is differentiable,

and

(1) $\frac{d}{dt}[R,S]=-[R,S]A.$

since each P_i is $(l-1)$-times differentiable. A is $(l-1)$-times differentiable. Thus it follows from (1) that (R,S) and hence S are l-times differentiable, Since

(2) $\tilde{\phi}[R,S]=I_{nl}$,

we see that

(3) $\phi^{(l-1)}S=I_n,\ \phi^{(i)}S=0,\ (0\leq i\leq l-2).$

Differentiating the second equality

(4) $\phi^{(i)}S^{(1)}=0,\ (0\leq i\leq l-3).$

Since $\phi^{(l-2)}S=0$, differentiating this

(5) $\phi^{(l-2)}S,=-\phi^{(l-1)}S=-I_n,$

Differentiating (4),

(6) $\phi^{(i)}S^{(2)}=0 \ (0\leq i\leq l-4)$.

since $\phi^{(l-3)}S^{(1)}=0$. differentiating and then using (5),

(7) $\phi^{(l-3)}S^{(2)}=(-1)^2I_n$.

Continuing this process, we obtain

(8) $\phi^{(i)}S^{(j)}=0 \ (0\leq i\leq l-2-j,\ 0\leq j\leq l-2)$.

(9) $\phi S^{(l-1)}=(-1)^{l-1}I_n$.

We now show that $\gamma^+S^*=0$.

Put

$$R^*=\begin{bmatrix}\tilde{R}_1\\ \vdots\\ \tilde{R}_{l-1}\end{bmatrix}$$

Then it follows from (1) that

(10) $\dot{\tilde{R}}_1=P_0S^*$,

$\dot{\tilde{R}}_i=-\tilde{R}_{i-1}+P_{i-1}S^*,\ (2\leq i\leq l-1)$,

$\dot{S}^*=-\tilde{R}_{l-1}+P_{l-1}S^*$,

Thus, for $1\leq i\leq l-1$,

(11) $S^{*(i)}=(P_{l-1}S^*)^{(i-1)}+(-1)(P_{l-1}S^*)^{(i-2)}$

$+\cdots+(-1)^{i-1}(P_{l-i}S^*)+(-1)^i\tilde{R}_{l-i}$.

Taking $i=l-1$, and then differentiating,

(12) $(S^*)^{(l)}=(P_{l-1}S^*)^{(l-1}{}_{(}-(P_{l-1}S^*)^{(l-3)}$

$+\cdots+(-1)^{l-1}P_0S^*$.

The last identity implies that $\gamma^+S^*=0$. We now show that S^* is a $n\times nl$ fundamental matrix solution. Suppose that $S^*\alpha=S^{*(1)}\alpha=\cdots=S^{*(l-1)}\alpha=0$ for some constant vector α.

Then it follows from (11) that

$$S^{*(i)}(\alpha)=(-1)^i\tilde{R}_{l-i}\alpha=0 \ (1\leq i\leq l-1).$$

Thus

$$\begin{bmatrix}R^*\\ S^*\end{bmatrix}\alpha=0. \text{ and so } \alpha=0$$

as $[R,S]$ is invertible. Thus S^* is a fundamental matrix solution of $\gamma^+x=0$.

Now let u be as in the theorm. Then using (8)

(13) $u^{(i)}=(S^*)^{(i)}(t)\int_t^{t_1}(\phi^*f)(s)ds \ (0\leq i\leq l-1)$.

Differentiating $u^{(l-1)}$ and using (9),

(14) $u^{(l)}=(S^*)^{(l)}q+(-1)^l f$.

where

$$q:=\int_t^{t_1}(\phi^* f)(s)ds.$$

Using (8) and (13),

(15) $(P_i^* u)^{(i)}=(P_i^* S^*)^{(i)}q \ (0\leq i\leq l-1)$.

Thus using (14) and (15), and $\gamma^+S^*=0$,

we see that

$$\gamma^+ u=\sum_0^{l-1}(-1)^i(P_i^* S^*)^{(i)}q+(-1)^l[S^{*(l)}q+(-1)^l f]$$
$$=(\gamma^+S^*)q+f=f.$$

It is clear that $\tilde{u}(t_1)=0$. This completes the proof.

Remark 3.2 As is well-known in the scalar case,

$$g(t)\equiv\phi(t)\int_{t_0}^t(Sf)ds,\ t_0\leq t\leq t_i,$$

is the unique solution of

$$\gamma g=f,\ \tilde{g}(t_0)=0_{nl\times 1}.$$

One of the main features of this paper is a completely general nature of boundary operators involved. Thus we will discuss it. By Theorem 3.1, the (maximal) operator T_1 is closed. Thus its domain becomes a Hilbert space with the T_1-norm defined by

$$||x||_{T_1}:=(||x||^2+||T_1x||^2)^{1/2}.$$

The following is well-known, but we state it for completeness. Its proof follows immediately from the Riez-Fisher respresentation theorem for a linear functional.

Theorem 3.4 F is a T_1-continuous linear operator from Dom T_1 into C^d if and only if there exist $n\times d$ matrix-valued fuctions Ω_1,Ω_2 of $t\in[t_0,t_1]$ such that their column vectors are in X_n and

$$F(x)=\int_{t_0}^{t_1}(\Omega_2^*\ r\ x+\Omega_1^* x)dt$$

for all $x\in$Dom T_1.

Under certain condition on Ω_2,Ω_1, this boundary operator is reduced to a familiar evaluation operator. This is shown in the following.

Corollary 3.5 Assume that F has the same representation as in the above theorem. Then there exist $d\times nl$ constant matrices M and N such that $F(x)=M\tilde{x}(t_0)+N\tilde{x}(t_1)$ for all $x\in\text{Dom } T_1$ if and only if the columns of Ω_2 are in Dom T_1 and $\gamma^+\Omega_2=-\Omega_1$ almost all t.

$$M=-\tilde{\Omega}_2^*(t_0)\, E^t(t_0),\quad N=\tilde{\Omega}_2^*(t_1)\, E^t(t_1).$$

Proof. Assume that F has the representation as in the above.

Then for all $x\in\text{Dom } T_0$,

$$<\gamma x,\, \Omega_2>=<x,\, -\Omega_1>.$$

Hence $\Omega_2\in\text{Dom } T_0^*$ columnwise and $T_0^*\Omega_2=-\Omega_1$ columnwise. Thus by Theorem 3.1, $\Omega_2\in\text{Dom } T_1$ columnwise and $\gamma^+\Omega_2=-\Omega_1$ almost everywhere. Returning to the representation and using the Green's formula (3.1).

$$\begin{aligned}M\tilde{x}(t_0)+N\tilde{x}(t_1)&=[<\gamma x,\, \Omega_2>-<x, \gamma^+\Omega_2>]^t\\&=\tilde{\Omega}_2^*(t_1)\, E^t(t_1)\, \tilde{x}(t_1)-\tilde{\Omega}_2^*(t_0)\, E^t(t_0)\, \tilde{x}(t_0)\end{aligned}$$

for all $x\in\text{Dom } T_0$.

Thus

$$M=-\tilde{\Omega}_2^*(t_0),\quad N=\tilde{\Omega}_2^*(t_1)\, E^t(t_1).$$

This proves the "only if" part. The "if" part can be proved easily using the Green's formula. This completes the proof.

4. Singular Control Problem

We will consider in this section the problem of minimizing

(4.1) $J(u,x)=\int_{t_0}^{t_1}(|Uu|^2+|W(x-x_0)|^2)dt+|G(x)|^2,$

subject to

(4.2) $u\in X_m,$

(4.3) $x\in\text{Dom } T_1,\ T_1x=Bu,$

(4.4) $|F(x)-\gamma|=\min\{|F(y)-\gamma|: y\in \text{Dom } T_1, T_1y=Bu\}$,

Here (i) U, W and B are $m\times m$, $n\times n$ and $n\times m$ matrix-valued functions of $t\in[t_0, t_1]$, respectively, whose entries are Lebesque integrable and bounded almost everywhere in $[t_0, t_1]$,

(ii) x_0 is a given element of X_n and γ as a given $d\times 1$ vector in C^d.

(iii) G is a T_1-continuous lineas operator from Dom T_1 into C^{d_1}, and F is a T_1-continuous linear operator from Dom T_1 into C^d.

Since U can vanish partially or totally on $[t_0, t_1]$, the problem is called singular.

When u and x satisfy (4.2)-(4.4), u is called a control and x is called a response corresponding to u. *The set of all ordered pairs $\{u, x\}$ satisfying these conditions will be denoted by $\mathscr{D}$.* When $\{u, x\}$ minimizes J over $\mathscr{D}$, we will call it an optimal pair, and u and x are called an optimal controller and an optimal response corresponding to u, respectively.

As we will see later in the following lemma, the condition (4.4) becomes the "standard" exact conditions $F(x)=\gamma$. It is shown in p.8o of [7] that these exact conditions often reduce to stieltjes conditions. This type of generalized boundary conditions arise naturally, for example, from diffusion processes, nuclear reactors, vibrating strings in magnetic fields and economic modelling ([11]). Optimal control problems associated with Stieltjes conditions have been investigated in [8], [9].

In many classical control problems, a trajectory is required to go through a fixed initial target and a fixed terminal target. However, this is not always possible if the state equation is not completely controllable. One can easily construct a state equation which is not completely controllable. In some phyical problems, due to the error involved in the targets, it may be sufficient to require that the state be closest possible to the targets simultaneously. This is the main motivation of considering the general condition (4.4). Because of this, as we will see later, the complete controllability condition is not needed and the control space is the whole space. But it gives us a mathematically more interesting problem-a control generates infinitely many respones. A somewhat similar idea was used in [19, 20] where the state is required to pass through a given target, but is required to stay within a limit from the second target.

Because of the uniqueness theorem of a linear differential equation, the system in [19, 20] generetes only single responses and the control space cannot be the whole space. It is true that a higher order system of differential equations can be changed into a system of first order equations by great many different ways. But since we deal with a higher order case directly, we do not need such a change. For example, if we are interested in a higher order scalar cae, we simply take one dimensional case. Moreover, a higher order system was used in [3] in studying multivariable feedback systems. This justifies the study of a higher order system. This paper generalizes and at the same time improves the corresponding results of [13] to a higher order singular case.

Our plan in this section is: Change the control problem into a least-squares problem for a multi-valued operator equation and then derive integro-differential equations for a controller.

This is a new approach to the problem. Once adjoint equations are derived, we then derive feedback laws.

By Theorem 3.4 there exist $n\times d$ matrix-valued functions Ω_1, Ω_2 such that their columns are in Xn and

$$(4.5)\quad F(x)=\int_{t_0}^{t_1}(\Omega_2^*\ \gamma x+\Omega_1^* x)\ dt,\ \text{for all } x\in \text{Dom } T_1.$$

Define an operator $\mathscr{K}$: $X_m \longrightarrow X_n$ by

$$(4.6)\quad \mathscr{K}(u)=\phi(t)\int_{t_0}^{t}(SBu)(s)\ ds,$$

where ϕ and s are defined in (3.2) and (3.3). Notice that $\mathscr{K}(u)$ is the unique solution of $\gamma x = Bu$ satisfying $\mathscr{K}(u)(t_0)=0$. First we have

Lemma 4.1 The following are equivalent:

(i) $(u,x)\in\mathscr{D}$.

(ii) $u\in X_m$ and $x\in \text{Dom}T_1$ such that $T_1 x = Bu$,

$$F(x)-\gamma=[F(\phi)\ (F(\phi))^{\#}-1]\ [\gamma-F(\mathscr{K}(u))].$$

(iii) $u\in X_m$ and

$$x=\phi(F(\phi))^{\#}[\gamma-F(\mathscr{K}(u))]+\phi k+\mathscr{K}(u)$$

for some $k\in$ Null $F(\phi)$.

(iv) $u\in X_m$ and $x=\phi a+\mathscr{K}(u)$ for some $a\in C^{nl}$ which is a least-squares solution of the matrix equation (for u fixed)

$$F(\phi)(\beta)=\gamma-F(\mathscr{K}(u)).$$

(v) $u\in X_m$, $x\in \mathrm{Dom}T_1$, $T_1x=Bu$,

$$F(x)-\gamma\in \mathrm{Null}\ (F(\phi))^*.$$

Proof. Using the variation-of-constant formula in Theorem 3.2 and an elementary property of a matrix equation, the proof for (i) $\langle=\rangle$ (ii) $\langle=\rangle$ (iii) $\langle=\rangle$ (iv) can be carried out by the method similar to the one used to prove Lemma 3.1 and proposition 3.2 (ii) of [13]. Assume (ii) holds. Then since $F(\phi)\ (F(\phi))^*-I$ is the orthogonal projector of C^d onto Null $(F(\phi))^*$, we see that $F(x)-\gamma\in$ Null $(F(\phi))^*$. Thus (ii) $\langle=\rangle$ (v). Assume (v) holds. Write $x=\phi k+\mathscr{X}(u)$ for some $k\in C^{nl}$. Now $F(x)-\gamma=F(\mathscr{X}(u))+F(\phi)k-\gamma\in$ Null $(F(\phi))^*$.

Thus

$$\begin{aligned}F(x)-\gamma&=[F(\phi)\ (F(\phi))^*-I]\ [F(\mathscr{X}(u))+F(\phi)k-\gamma]\\&=[F(\phi)\ (F(\phi))^*-I]\ [(F\mathscr{X})(u)-\gamma].\end{aligned}$$

Thus(v)$\Longrightarrow$(ii), and so the proof is complete.

Remark 4.1 If $F(x)=\begin{bmatrix}\tilde{x}(t_0)\\ \tilde{x}(t_1)\end{bmatrix}$, then dim Null $(F(\phi))^*>0$.

Thus the condition (4.4) can *not* be reduced to $[\tilde{x}^t(t_0),\ \tilde{x}^t(t_1)]=\gamma^t$.

But it reduces to the mixed two-point conditions:

$$\tilde{\phi}^*(t_0)\ \tilde{x}(t_0)+\tilde{\phi}^*(t_1)\ \tilde{x}(t_1)=[\tilde{\phi}^*(t_0),\ \tilde{\phi}^*(t_1)]\gamma.$$

Remark 4.2 If dim Null $(F(\phi))^*=0$, equivalently Rank $F(\phi)=d$, then the condition (4.4) is reduced to $F(x)=\gamma$. For example, if M is a $d\times nl$ constant matrix of rank d and if $F(x)=M\tilde{x}(t_0)$, then the condition (4.4) becomes $M\tilde{x}(t_0)=\gamma$. This is in turn, becomes $\tilde{x}(t_0)\in M^*(\gamma)+$Null M. Thus in this particiuiar case our control problem can also be worked out by the Maximum Principle, or other existing methods.

Remark 4.3 Let

$$\hat{\mathscr{D}}=\{(u,x)\in X_m\oplus X_n:\ x\in \mathrm{Dom}T_1,\ x=Bu,\ F(x)=\gamma\}$$

Assume that $\hat{\mathscr{D}}$ is nonempty. Then it is clear that $\min\{J(u,x):\ (u,x)\in\mathscr{D}\}\leq\min\{J(u,x):\ (u,x)\in\hat{\mathscr{D}}\}$.

Remark 4.4 Dom $\mathscr{D}=X_m$. Moreover, for each control u, either there exists a unique response corresponding to u or there exist infinitely many responses corresponding to u Thus J, if considered as a function of u, becomes a multi-valued function of u. We now characterize J. By Theorem 3.4, there exist $n\times d_1$ matrix-valued functions Λ_1, Λ_2 such that their columns are in X_n and

$$(4.7)\ G(x)=\int_{t_0}^{t_1} (\Lambda_2^*\ \gamma x+\Lambda_1^* x)\ dx,\ \ x\in \text{Dom } T_1.$$

Define $\mathscr{M}_1$: $X_m \longrightarrow X_m\oplus X_n \oplus C^{d_1}$ by

$$(4.8)\ \mathscr{M}_1(u)=\{Uu,\ W[\mathscr{X}(u)-\phi F(\phi))^{\#}F(\mathscr{X}(u)],\ G(\mathscr{X}(u)-G(\phi)\ (F(\phi))^{\#}(F(\mathscr{X}(u)))\},$$

and

$$\mathscr{M}_2\text{: Null } F(\phi)\subset C^{nl}\rightarrow X_m\oplus X_n\oplus C^{d_1}$$

by

$$(4.9)\ \mathscr{M}_2(k)=\{0, W\phi k,\ G(\phi)k\}.$$

Notice that $\mathscr{M}_2$ is nondensely defined in C^{nl}, and its range is closed. $\mathscr{M}_1$ is a bounded linear operator but its range may not be closed because U may not be invertible. Let ζ be the vector on $X_m\oplus X_n\oplus C^{d_1}$ defined by

$$(4.10)\ \zeta=\{0, W[x_0-\phi F(\phi)^{\#}\gamma], -G(\phi)\ (F(\phi))^{\#}\gamma\}.$$

Let $\|,\|$ also denote the norm of $X_m\oplus X_n\oplus C^{d_1}$. Then we have the following.

Lemma 4.2

(i) For any $(u,x)\in\mathscr{D}$ with
$x=\phi(F(\phi))^{\#}[\gamma-F(\mathscr{X}(u))]+\phi k+\mathscr{X}(u)$,
where $k\in$ Null $F(\phi)$, we have
$J(u,x)=\|\mathscr{M}_1(u)+\mathscr{M}_2(k)-\zeta\|^2$.

(ii) (u^+,x^+) is an optimal pair if and only if
$\|\mathscr{M}_1(u^+)+\mathscr{M}_2(k^+)-\zeta\|\le\|\mathscr{M}_1(u)+\mathscr{M}_2(k)-\zeta\|$

for all $u\in X_m$, $k\in$ Null $F(\phi)$.

Here

$$x^+=\phi(F(\phi))^{\#}\ [\gamma-F(\mathscr{X}(u^+))]+\phi k^+ +\mathscr{X}(u^+),\ \ k^+\in\text{Null } F(\phi).$$

Proof. The proof is easy and so we omit it.

By the above lemma, we see that (u^+, x^+) is an optimal pair if and only if (u^+, k^+) is a least-squares solution of the operator equation $V(u, k)=\zeta$ where $V(u, k):=\mathscr{M}_1(u)+\mathscr{M}_2(k)$, $u\in X_m$, $k\in \text{Null } F(\phi)$.

This is a single-valued operator approachto our problem. On the other hand, if we define a closed vcetor space $\mathscr{M}$ by

$$(4.11)\quad \mathscr{M}=(\text{graph } \mathscr{M}_1)\oplus(\{0\}\oplus\text{Range } \mathscr{M}_2),$$

then by Lemma 4.2 (ii) we see easily that the following is true.

Lemma 4.3. u^+ is an optimal controller if and only if $u^+\in X_m$ and there exists $y^+\in\mathscr{M}(u^+)$ such that $\|y^+-\zeta\|\le\|z-\zeta\|$ for all $z\in\text{Range } \mathscr{M}$. Equivalently, u^+ is a least-squares solution of $\zeta\in\mathscr{M}(u)$.

In this paper we will treat an optimal controller as a least-squares solution of a multi-valued operator equation as in the above lemma. The following is an abstract description of an optimal controller.

Lemma 4.4.

(i) u^+ is an optimal controller if and only if $u^+\in X_m$ and $\zeta\in\mathscr{M}(u^+)+\text{Null } \mathscr{M}^*$.

(ii) An optimal controller exists if and only if $\zeta\in\text{Range } \mathscr{M}+\text{Null } \mathscr{M}^*$.

When this holds, the set of the optimal controllers is given by the coset $\mathscr{M}^\sharp(\zeta)+\text{Null } \mathscr{M}$, and is a singleton set if $\dim \text{Null} \mathscr{M}=0$.

Moreover, $\mathscr{M}^\sharp(\zeta)$ is the unique optimal controller with smallest X_m norm,

(iii) If U is invertible on $[t_0, t_1,]$ then an optimal controller always exists.

Proof Parts (i), (ii) follows from Lemma 4.3 together with Theorems 2.1 2.3. Assume U is invertible. Then the range of $\mathscr{M}$ is closed. Since Range $\mathscr{M}_2$ is finite dimensional, it follows from (4.11) that Range $\mathscr{M}$ is closed. Hence

$$\text{Range } \mathscr{M}+\text{null } \mathscr{M}^*=X_m\oplus X_n\oplus C^{d_1}$$

Thus by (i), an optimal controller always exists. This completes the proof.

Remark 4.5. Using the definition of $\mathscr{M}$, $u\in\text{Null } \mathscr{M}$ if and only if $u\in X_m, Uu=0$,

$$W\{\mathscr{X}(u)-\phi(F(\phi))^\sharp F(\mathscr{X}(u))-\phi k\}=0,$$
$$G(\mathscr{X}(u))-G(\phi)\,F(\phi))^\sharp\ F(\mathscr{X}(u)=G(\phi)k \text{ for some } k\in\text{Null } F(\phi).$$

The rest of this section is devoted to characterize (i) of the above lemma by equations. Thus we will characterize the adjoints of $\mathscr{M}^*$.

First we note that $\mathscr{M}^* = \mathscr{M}_1^* \cap (\text{Null } \mathscr{M}_2^* \oplus X_m)$.

Thus

$$(4.12)\quad \text{Null } \mathscr{M}^* = \text{Null } \mathscr{M}_1^*. \cap \text{Null } \mathscr{M}_2^*.$$

We can show easily that

$$\text{Null } \mathscr{M}^* = \{(v, g, \alpha) \in X_m \oplus X_n \oplus C^{d_1}:\ U^*\ u + \mathscr{X}^*(W^* g) + (G\mathscr{X})^*\alpha$$
$$= (F\mathscr{X})^*((F(\phi))^*)^*[(G(\phi))^*\alpha + \overline{\langle W\phi, g\rangle}]\},$$

$$\text{Null } \mathscr{M}_2^* = \{(v, g, \alpha) \in X_m \oplus X_n \oplus C^{d_1}:$$
$$(G(\phi))^*\alpha + \overline{\langle W\phi, g\rangle} \in (\text{Null } F(\phi))^{\perp}\},$$

Thus since

$$\mathscr{X}^*(u) = B^*(t)\, S^*(t) \int_t^{t_1} \phi^*(s)\, u(s)\ ds,\ \ u \in X_m.$$
$$(F\mathscr{X})^*(\alpha) = [B^*\Omega_2 + \mathscr{X}^*(\Omega_1)](\alpha), \alpha \in C^d,$$
$$(G\mathscr{X})^*(\alpha) = [B^*\Lambda_2 + \mathscr{X}^*(\Lambda_1)](\alpha), \alpha \in^{d_1},$$

it follows froms from (4.12) that

$$(4.13)\quad \text{Null } \mathscr{M}^* = \{(v, g, \alpha) \in X_m \oplus X_n \oplus C^{d_1}:\ U^* v + \mathscr{X}^*\ [W^* g + \Lambda_1\alpha - \Omega_1((F(\phi))^*)^* q]$$
$$+ B^*[\Lambda_2(\alpha) - \Omega_2(F(\phi))^*)^* q] = 0,$$
$$q \in (\text{Null } F(\phi))^{+}\}.$$

where

$$q \equiv (G(\phi))^*\alpha + \overline{\langle W\phi, g\rangle}.$$

In the following we will give a necessary and sufficient condition for x^+ to be an optimal response corresponding to an optimal controller. First it is convenient to introduce a function $A(z)$ by

$$(4.14)\quad A(z) = (G(\phi))^* G(z) + \overline{\langle W\phi. W(z - x_o)\rangle}.\ \ z \in X_n.$$

Then we have

Lemma 4.5 Assume that u^+ is an optimal controller. Then the following is equivalent:

(i) $(u^+. x^+)$ is an optimal pair.

(ii) $x^+ = \mathscr{H}(u^+) + \phi\{k^+ + (F(\phi))^* \gamma - (F(\phi))^* F(\mathscr{H}(u^+))\}$

for some $k^+ \in$ null $F(\phi)$ which is a least-squares solution of the operator equation (for u^+. ζ fixed) $\mathscr{M}_2(k) = \zeta - \mathscr{M}_1(u^+)$.

(iii) $T_1 x^+ = Bu^+$. $F(x^+) - \gamma \in$ Null $(F(\phi))^*$. $A(x^+) \in$(Null $F(\phi))^\perp$.

Proof First we show that

(1) $\|(\mathscr{M}_2\mathscr{M}_2^* - I)\ (\zeta - \mathscr{M}_1(u^+))\| \leq \|\mathscr{M}_2(k) + \mathscr{M}_1(u) - \zeta\|$

for all $k \in$ Null $F(\phi)$. $u \in X_m$.
For. since u^+ is optimal. by Lemma 4.2. there exists
$k^+ \in$ Null $F(\phi)$ such that

(2) $\|\mathscr{M}_1(u^+) + \mathscr{M}_2(k^+) - \zeta\| \leq \|\mathscr{M}_1(u) + \mathscr{M}_2(k) - \zeta\|$

for all $k \in$ Null $F(\phi)$. $u \in X_m$. Letting $u = u^+$. we see from (2) that $\zeta - \mathscr{M}_2(k^+) \in$ Dom $\mathscr{M}_2^*$ and $k^+ = \mathscr{M}_2^*\ (\zeta - \mathscr{M}_1(u^+)) + \alpha$ for some $\alpha \in$ Null $\mathscr{M}_2$. But then

$$\mathscr{M}_1(u^+) + \mathscr{M}_2(k^+) - \zeta = \mathscr{M}_1(u^+) + \mathscr{M}_2\mathscr{M}_2^*(\zeta - \mathscr{M}_1(u^+)) - \zeta = (\mathscr{M}_2\mathscr{M}_2^* - I)(\zeta - \mathscr{M}_1(u^+)).$$

This together with (2) implies (1).
Assume (i). Then $(u^+. x^+) \in \mathscr{D}$ and

(3) $J(u^+. x^+) \leq J(u.x)$. all $(u.x) \in \mathscr{D}$.

Let us write

(4) $x^+ = \mathscr{H}(u^+) + \phi\{k^+ + (F(\phi))^* \zeta - (F(\phi))^* F(\mathscr{H}(u^+))\}$

for some $k^+ \in$ Null $F(\phi)$. Then by Lemma 4.2. we see that (2) holds for all $u \in X_m$ and $k \in$Null $F(\phi)$. Taking $u = u^+$ there, we see that k^+ is a least-squares solution of $\mathscr{M}_2(k) = \zeta - \mathscr{M}_1(u^+)$. Thus (i)$\Longrightarrow$(ii).

Assume (ii). We will show that (3) holds for all $(u.x) \in \mathscr{D}$. In view of Lemma 4.2. it is sufficient to show that (2) holds for all $u \in X_m$. $k \in$ Null $F(\phi)$. Since Range $\mathscr{M}_2$ is closed and k^+ is a least-squares solution. $k^+ = \mathscr{M}_2^*\ (\zeta - \mathscr{M}_1(u^+)) + \alpha$ for some $\alpha \in$Null $\mathscr{M}_2$. Hence using (1). $\|\mathscr{M}_1(u^+) + \mathscr{M}_2(k^+) - \zeta\| =$

$$\|\mathscr{M}_1(u^+) + \mathscr{M}_2\mathscr{M}_2^*(\zeta - \mathscr{M}_1(u^+)) - \zeta\|$$
$$= \|(\mathscr{M}_2\mathscr{M}_2^* - I)(\zeta - \mathscr{M}_1(u^+))\| \leq \|\mathscr{M}_1(u) + \mathscr{M}_2(k) - \zeta\|$$

for all $u \in X_m$. $k \in$ Null $F(\phi)$. Thus (ii) $\Longrightarrow$ (i).

Assume (ii). Then by Lemma 4.1, $T_1x^+ = Bu^+$. $F(x^+) - \gamma \in$ Null $F(\phi))^*$. since k^+ is a least-squares solution.

(5) $\zeta - \mathscr{M}_1(u^+) - \mathscr{M}_2(k^+) \in$ Null $\mathscr{M}_2^*$.

Using the definition of $\mathscr{M}_1$. $\mathscr{M}_2$ and ζ. this is equivalent to

(6) $\{Uu^+,\ W(x^+ - x_o),\ G(x^+)\} \in$ Null $\mathscr{M}_2^*$.

Thus using the description of $\mathscr{M}_2^*$ given earlier. (6) is equivalent to $A(x^+) \in$(Null $F(\phi))^\perp$. Hence (ii) $\Longrightarrow$ (iii). Assume (iii) holds. Then by Lemma 4.1, x^+ can be written as (4) for some $k^+ \in$ Null $F(\phi)$. But then $A(x^+) \in$(Null $F(\phi)^\perp$ is equivalent to (5). and so k^+ is a least-squares solution of $M_2(k) = \zeta - M_1(u^+)$. Thus (iii) $\Longrightarrow$ (ii), and so the proof is complete.

Remark 4.6 Part (ii) of the above theorem generalizes Theorem 3. 9 of [13]. Note that if U is invertible on $[t_0, t_1]$, then u^+ there becomes $M^\dagger(\zeta)$.

Remark 4.7 By part (ii). an optimal response corresponding to an optimal controller u^+ is unique if

dim $\{k \in$Null $F(\phi)$: $W\phi k = 0,\ G(\phi)k = 0\}$.

In the following we show that the optimal responses corresponding to an optimal control forms a finite dimensional convex set (a coset in this case). This is similar to optimal controllers (see Lemma 4.4). The proof follows easily from Lemma 4.5 (iii). Thus we omit the proof.

Corollary 4.6 Let $\{u^+, x^+\}$ be an optimal pair. Then $\{u^+, z^+\}$ is an optimal pair if and only if $z^+ = x^+ + \phi\ \alpha$ for some $\alpha \in C^{nl}$ such that $F(\phi)\alpha \in$ Null $(F(\phi))^*$.

$[(G(\phi))^* G(\phi) + \overline{\langle W\phi, W\phi \rangle}]\alpha \in$(Null $F(\phi))^\perp$.

In the following we will describe an optimal controller. First let us introduce a function $\eta(z)$ $(z \in X_n)$ by

$$(4.15)\quad \eta(z) = \Lambda_2(t)\,G(z) - \Omega_2(t)\,((G(\phi))^\dagger)^* A(z) + S^*(t)\int_t^{t_1} \phi^*(s)\{W^*W(z - x_0) + \Lambda_1 G(z) - \Omega_1((F(\phi))^\dagger)^* A(z)\}(s)ds.$$

where $A(z)$ is defined in (4.14).

we have the following propesty of $\eta(z)$.

Proposition 4.7 Assume further that $P_0, P_1 \cdots P_{e-2}$ are $(e-1)$-times differentiable on $[t_0, t_1]$. Then for any $z \in x_n$. $\eta(z) - \Lambda_2 G(z) + \Omega_2((G(\phi))^*)^* A(z)$ is the unique solution of the differential equation in y:

$$\gamma^+ y = W^* W(z - x_0) + \Lambda_1 G(z) - \Omega_1((F(\phi))^*)^* A(z), \tilde{y}(t_1) = 0_{ne \times 1}.$$

Proof. This follows from Theorem 3.3.

In the following we will give feedback-like controllers. To do this it will be convenient to define an operator U in two cases as follows:

Case i U is a $m \times m$ constant matrix.

We define U to be the map from C^m into C^m by

$$U(\alpha) = U^* U(\alpha). \quad \alpha \in C^m.$$

Case ii $U(t)$ depends on t.

We define U to be the bounded linear operator from X_m into X_m by

$$(Uu)(t) = U^*(t)\, U(t)\, u(t), \quad t_0 \le t \le t_1$$

for all $u \in X_m$.

In any case the (orthogonal) generalized inverse of U will be denoted by $U^\dagger$.

In the following we obtain feedback-like controllers.

Theorem 4.8 u is an optimal controller if and only if $u \in X_m$ and there exist $x \in$ Dom T_1 and f$\in$Null U such that

(i) $T_1 x = Bu$.

(ii) $F(x) - \gamma \in$ Null $(F(\phi))^*$.

(iii) $A(x) \in$(Null $F(\phi))^\perp$.

(iv) $B^* \eta(x) \in$ Range U and

$u = f - U^\dagger(B^* \eta(x))$ in X_m.

Moreover, when (i)–(iv) hold. $\{u, x\}$ is an optimal pair. and the set of all optimal responses corresponding to u is given by

$$\{x + \phi\alpha : \alpha \in C^{ne}.\ F(\phi)\alpha \in \text{Null } (F(\phi))^*.$$
$$[(G(\phi))^* G(\phi) + \overline{\langle W\phi, W\phi \rangle}]\alpha \in (\text{Null } F(\phi))^\perp\}.$$

Proof. First we will prove that u is an optimal controller if and only if $u \in X^m$ and there exists $x \in$ Dom T_1 satisfying

(i), (ii), (iii) and

(iv), $U^*Uu = -B^*\eta(x)$ a.a.t.

The idea of the proof consists of replacing Lemma 4.4 (i) by more concrete terms. Now by this lemma. u is an optimal controller if and only if

$$(1)\ \zeta \in \mathscr{M}(u) + \text{null } \mathscr{M}^*.$$

Using the definition of $\mathscr{M}$. this is equivalent to the existence of $k \in C^{n*}$ such that

$$(2)\ k \in \text{Null } F(\phi),\ \mathscr{M}_1(u) + \mathscr{M}_2(k) - \zeta \in \text{Null } \mathscr{M}^*.$$

Let us put

$$(3)\ q = k - (F(\phi))^{\sharp}\ (F(\mathscr{H}(u))) + (F(\phi))^{\sharp}\gamma.$$

Then the second statement of (2) is

$$(4)\ \{Uu.\ W[\mathscr{H}(u) - x_0 + \phi q],\ G(\mathscr{H}(u)) + G(\phi)q\} \in \text{Null } \mathscr{M}^*.$$

This can be rewritten. using 4.13, as (5) and (6) together:

$$(5)\ A(\mathscr{H}(u) + \phi q) \in (\text{Null } F(\phi))^{\perp}.$$

$$\begin{aligned}(6)\ U^*Uu + \mathscr{H}^*\{W^*W[\mathscr{H}(u) - x_0 + \phi q] \\ + \Lambda_1[G(\mathscr{H}(u) + G(\phi)q] - \Omega_1((F(\phi))^{\sharp})^* A(\mathscr{H}(u) + \phi q)\} \\ + B^*\{\Lambda_2[G(\mathscr{H}(u)) + G(\phi)q] - \Omega_2\ ((F(\phi))^{\sharp})^* A(\mathscr{H}(u) + \phi q)\} = 0.\end{aligned}$$

Let us introduce a variable x by

$$(7)\ x = H(u) + \phi q.$$

Then z is the unique solution of

$$(8)\ \gamma x = Bu, \text{ almost everywhere}$$

satisfying

$$(9)\ \tilde{x}(t_0) = q.$$

Clearly $x \in$ Dom T_1. Because of (9). we see that (3) is equivalent to

$$(10)\ \tilde{x}(t_0) + (F(\phi))^{\sharp} F(\mathscr{H}(u)) - (F(\phi))^{\sharp}\gamma \in \text{Null } F(\phi).$$

We have shown that u is optimal iff there exists $x \in$ Dom T_1 such that (8), (10), (5) and (6), with $\mathscr{H}(u) + \phi q$ being replaced by x, and q by $\tilde{x}(t_0)$.

We now show that (10) is equivalent to

(11) $F(x)-\gamma\in\text{Null }(F(\phi))^*$.

For. since $x=\mathscr{K}(u)+\phi\tilde{x}(t_0)$ we see that (10) is equivalent to

$$[I-(F(\phi))^\# F(\phi)]\tilde{x}(t_0)+(F(\phi))^\#(F(x)-\gamma)\in \text{null } F(\phi).$$

Since Range $(F(\phi))^\#=(\text{Null } F(\phi))^\perp$.

$$[I-(F(\phi))^\#]\tilde{x}(t_0)\in \text{Null } F(\phi).$$

we see that the above is again equivalent to $(F(\phi))^\#(F(x)-\gamma)=0$. Equivalently. $F(x)-\gamma\in$ Null $(F(\phi))^\#=$Null $(F(\phi))^*$. This proves (11). Now using the description of $\mathscr{K}^*$. and using the definition of $\eta(x)$ and using the fact that $x=\mathscr{K}(u)+\phi\tilde{x}(t_0)$. we can write (6) as

(12) $U^*Uu=-B^*\eta(x)$ almost everywhere.

This proves the first claim.

Now $U^*Uu=-B^*\eta(x)$ in X_m if and only if $B^*\eta(x)\in$ Range U and $u=f-U^\#(B^*\eta(x))$ for some $f\in$ null U.

Thus this together with the first claim proves the first part of the theorem. Suppose now that (i)–(iv) hold. Then by the first part. u is an optimal controller and so by Lemma 4.5 together with (i)–(iii) we see that $\{u,x\}$ is an optimal pair. Hence by

Corollary 4.6, the set of all the optimal responses is given as claimed.

Remark 4.8 By the above theorem. there exists an optimal controller if and only if the functional equations

$T_1x=B\{f-U^\#(B^*\eta(x))\}$, a.a.t.,
$F(x)-\gamma\in$ Null $(F(\phi))^*$.
$A(x)\in(\text{Null } F(\phi))^\perp$,
$B^*\eta(x)\in$ Range U

has a solution $\{x,f\}$ with $x\in$ Dom T_1, $f\in$ Null U.

Remark 4.9 Assume $U(t)$ is invertible for all t.

Then $(U^\#u)(t)=(U^*(t)U(t))^{-1}u(t)$ for $u\in X_m$, Null $U^\#=\{0\}$.

Remark 4.10 If $U(t)\equiv 0$ on $[t_0, t_1]$. then $U^{\sharp}=0$. Thus any optimal controller must be a constant controller as.

$$\text{Null } U\subset C_m.$$

Remark 4.11 In Theorem 4.8, the condition (iv) can be replaced by:

$$B^*(t)(\eta(x))(t)\in \text{Range } (U^*(t)\,U(t))^{\sharp} \text{ a.a.t } \in[t_0, t_1].$$

and

$$u(t)=f(t)-(U^*(t)\,U(t))^{\sharp}B^*(t)(\eta(x))(t), \text{ a.a.t for some } f(t)\in \text{null } (U^*(t)U(t)).$$

But this approach may cause some problems if we wish solve the equation $T_1x=\mathrm{Bu}$ as it is not clear whether or not one of f and $(U^*(t)\,U(t))^{\sharp}\,B^*(t)\,(\eta(x))\,(t)$ belongs to X_m. This is the main reason why we have chosen the operator U.

In the previous theorem we have given feedback-like controllers u. This theorem is not satisfactory if we wish to solve for x as we have difficult integro-differential equations

$$\gamma x=B[f-U^{\sharp}(B^*\eta(x))]$$

(see also $\eta(x)$ in (4.15)).

Thus we will replace $\eta(x)$ in terms of a "adjoint equation" which will be crucial later in obtaining a feedback law (see the example below).

Theorem 4.9 Assume further that $P_0, P_1\cdots\ P_{l-2}$ are $(l-1)$ times differentiable. Then u is an optimal controller if and only if $u\in X_m$ and there exist $x\in$ Dom T_1, $\eta\in X_n$ and $f\in$ Null U such that

(i) $T_1x=Bu$.

(ii) $F(x)-\gamma\in$ Null $(F(\phi))^*$.

(iii) $A(x)\in(\text{Null } F(\phi))^{\perp}$.

(iv) $B^*\eta\in$ Range U and

$u=f-U^{\sharp}(B^*\eta)$.

(v) $\eta-\phi(x)\in$ Dom T_1,

(vi) $\gamma^+(\eta-\phi(x))=W^*W(x-x_0)+\Lambda_1G(x)$

$-\Omega_1[(F(\phi))^{\sharp}]^*A(x)$, a.a.t $\in[t_0, t]$.

(vii) $(\eta-\phi(x))(t_1)=0_{nl\times 1}$.

Here

$$\psi(x):=\Lambda_2G(x)-\Omega_2[(G(\phi))^{*}]^{*}A(x).$$

Moreover, when (i)–(vii) hold. $\{u,x\}$ is an optimal pair, and the set of all the optimal responses corresponding to u is given by

$$\{x+\phi\alpha:\ \alpha\in C^{nl},\ F(\phi)\alpha\in\text{Null }(F(\phi))^{*},\ [(G(\phi))^{*}G(\phi)+\overline{\langle W\phi,W\phi\rangle}]\alpha\in(\text{Null }F(\phi))^{\perp}\}.$$

Proof. In Theorem 4.8, let $\eta=\eta(x)$. Then $\eta\in X_n$ and by (4.15)

$$\eta=\psi(x)+S^{*}(t)\int_t^{t_1}\phi^{*}(s)\{W^{*}W(x-x_0)+\Lambda_1G(x)-\Omega_1((F(\phi))^{*})^{*}A(x)\}(s)ds.$$

By Proposition 4.7, we see that

$$\eta-\psi(x)\in\text{ Dom }T_1,\ \widetilde{(\eta-\psi(x))}(t_1)=0_{nl\times 1},$$
$$\tau^{+}(\eta-\psi(x))=W^{*}W(x-x_0)+\Lambda_1G(x)-\Omega_1((F(\phi))^{*})^{*}A(x).$$

Thus the theorem follows from Theorem 4.9. This completes the proof.

Remark 4.12 The above theorem generalizes and at the same time simplifies Theorem 3.12 of [13].

The above theorem can take various forms depending on F and G. In the following we will consider a very special case which is directly connected with a true two-point boundary value problem.

Corollary 4.10 Assume that $P_0, P_1\cdots P_{l-2}$ are $(l-1)$ times differentiable. Assume further that

$$F(x)=M\tilde{x}(t_0)+N\tilde{x}(t_1).$$
$$G(x)=M_1\tilde{x}(t_0)+N_1\tilde{x}(t_1)$$

for all $x\in$ Dom T_1 where M,N are $d\times nl$ constant matrices and M_1,N_1 are $d_1\times nl$ constant matrices.

Then u is an optimal controller if and only $u\in X_m$ and

$$u=f-U^{*}(B^{*}\eta)\ \text{a.a.t}$$

for some $f\in$ Null $U,\eta\in$ Dom T_1 and $x\in$ Dom T_1 such that

(i) $T_1x=Bu$.

(ii) $F(x)-\gamma\in$ Null $(F(\phi))^*$.

(iii) $(G(\phi))^*G(x)+\overline{<W\phi, W(x-x_0)>}\in(\text{Null } F(\phi))^\perp$.

(iv) $B^*\eta\in$ Range U,

(v) $\gamma^+\eta=W^*W(x-x_0)$ a. a. t.

(vi) $E(t_1)\,\eta(t_1)=N_1{}^*G(x)$
$-N^*((G(\phi))^\#)^*\{(G(\phi))^*G(x)+\overline{<W\phi, W(x-x_0)>}\}$.

When u is defined as in the above for some x (response) and η (adjoint response), then the set of all the optimal responses corresponding to the optimal controller u is given by

$$\{x+\phi\alpha:\ \alpha\in C^{m_l},\ F(\phi)\alpha\in \text{Null } (F(\phi))^*,$$
$$[(G(\phi))^*G(\phi)+\overline{<W\phi,\ W\phi>}]\alpha\in(\text{Null } F(\phi))^\perp\}.$$

Proof In view of Theorem 4.9, it is sufficient to prove that $\eta\in$ Dom T_1, and (iii), (vi), (vii) of this theorem becomes (iii), (v), (vi) of the present theorem, respectively. Because of the special natureof F and G, we have by corollary 3.5 that Ω_2, Λ_2 belong to Dom T_1(columnwise) and

(1) $\gamma^+\Omega_2=-\Omega_1,\ \gamma^+\Lambda_2=-\Lambda_1$ a. a. t.

(2) $M=-\tilde{\Omega}^*{}_2(t_0)\,E^t(t_0),\ M_1=\tilde{\Lambda}_2{}^*(t_0)\,E^t(t_0)$,

(3) $N=\tilde{\Omega}_2{}^*(t_1)\,E^t(t_1),\ N_1=\tilde{\Lambda}_2{}^*(t_1)\,E^t(t_1)$.

In particular. (1) implies that $\psi(x)\in$ Dom T_1 and so $\eta=\psi(x)+(\eta-\psi(x))\in$ Dom T_1. Thus $W^*W(x-x_0)+\Lambda_1G(x)-\Omega_1((F(\phi))^\#)^*A(x)=\gamma^+(\eta)-\gamma^+(\psi(x))=\gamma^+(\eta)+\Lambda_1G(x)-\Omega_1((F(\phi))^\#)^*A(x)$.

Hence

$$\gamma^+(\eta)=W^*W(x-x_0).$$

Moreoyer.

$$0_{nl\times 1}=\widetilde{(\eta-\psi(x))}(t_1)=\tilde{\eta}(t_1)-\tilde{\psi}(x)(t_1)$$
$$=\tilde{\eta}(t_1)-\tilde{\Lambda}_2(t_1)\,G(x)+\tilde{\Omega}_2(t_1)\,((G(\phi))^\#)^*A(x).$$

This together with (2), (3) gives (vi) of the above theorem. This completes the proof.

Remark In this paper we have considered in essence nonstandard "two-point" boundary conditions in the sense that the state is required to be smooth. This paper can be generalized further to the case where the state is allowed to have discontinuous jumps at specified points. A special case was considered in [2].

An example. Take $l=1$, $[t_0, t_1]=[0,1]$, $G\equiv 0$, $F(x)=Mx(0)+Nx(1)$, $\gamma x=\dot{x}+P_0x$ and assume that U is a **constant** matrix. and

$$\text{rank } F(\phi)=\text{rank } [M\phi(0)+N\phi(1)]=n.$$

Then by Corollary 4.10, the problem of minimizing

$$J(u,x)=\int_0^1(|Uu|^2+|wx|^2)dt.$$

subject to $u\in X_m$. $\dot{x}+P_0x=Bu$ and

$$|Mx(0)+Nx(1)-\gamma|=\min_y\{|My(0)+Ny(1)-\gamma|:\dot{y}+P_0y=Bu\}$$

has an optimal controller if and only if there exist

$$\beta\in \text{Null } U^*U,\ x\in \text{Dom } T_1.\ \eta\in \text{Dom } T_1$$

such that

(i) $\dot{x}+P_0x=B[\beta-(U^*U)^\sharp B^*(t)\ \eta(t)]$ a.a.t.

(ii) $F(x)-\gamma\in$ Null $(F(\phi))^*$.

(iii) $B^*(t)\ \eta(t)\in$ Range U^*U a.a.t

(iv) $-\dot{\eta}+P_0{}^*\eta=W^*Wx$ a.a.t

(v) $\eta(1)=0_{n\times 1}$.

Moreover. when (i)$-$(v) hold.

(1) $u(t):=\beta-(U^*U)^\sharp B^*(t)\ \eta(t),\ 0\leq t\leq 1$

is an optimal controller and $\{u,x\}$ is an optimal pair.

Furthermore, the set of all the optimal responses corresponding to u is given by

$$\{x+\phi\alpha:\ \alpha\in C^n,\ (F(\phi))^*F(\phi)\alpha=0\}.$$

In the following we will compute the optimal pair and find a feedback law.

Assume now that (i)$-$(v) hold. To get a feed back law, let us put

(2) $\eta(t)=K(t)\ x(t)+g(t),\ 0\leq t\leq 1$

where $K(t)$ is $n\times n$, g is $n\times 1$ and are differentiable on $[0,1]$.

Then using the adjoint equation (iv) together with (i), (v) we see that

$$[\dot{k}-P_0^*\ K-KP_0-KS(U^*U)^\sharp B^*K+W^*W]x$$
$$+\dot{g}-[P_0^*+KB(U^*U)^\sharp B^*]g+KB\beta=0,$$
$$K(1)\ x(1)+g(1)=0.$$

Thus choose K and g so that

(3) $\dot{K}-KP_0-P_0^*\ K-KB(U^*U)^\sharp B^*K=0,\ K(1)=0,$

(4) $\dot{g}-(P_0^*+KB(U^*U)^\sharp B^*)g+KB\ \beta=0,\ g(1)=0.$

Then the feed back controller is given by

(5) $u(t)=-(U^*U)^\sharp B^*(t)\ K(t)\ x(t)$
$\qquad +\beta-(U^*U)^\sharp B^*(t)\ g(t),\ 0\leq t\leq 1.$

In order to compute $U(t)$, we will solve for x and η in (i)$-$(v). Note first that (i) and (iv) can be written as

(6) $$\begin{bmatrix}\dot{x}\\ \dot{\eta}\end{bmatrix}+\begin{bmatrix}P_0 & B(U^*U)^\sharp B^*\\ W^*W & -P_0^*\end{bmatrix}\begin{bmatrix}x\\ \eta\end{bmatrix}=\begin{bmatrix}B\beta\\ 0\end{bmatrix}.$$

Let $\theta(t)$ be a $2n\times 2n$ fundamental matrix solution of the homogeneous part of the above equation. Let

$$\theta(t)=\begin{bmatrix}\theta_1(t)\\ \theta_2(t)\end{bmatrix}$$

where θ_i is $n\times 2n$, and let

$$\mathrm{h}(t)=\int_0^t \theta^{-1}(s)\begin{bmatrix}B(s)\beta\\ 0_{n\times 1}\end{bmatrix}ds.$$

Then by (6).

$$x=\theta_1(t)\ (k+h(t)),\ \eta=\theta_2(t)\ (k+h(t)).$$

for some $2n\times 1$ constant matrix. Thus (ii) and (v) can be written as

$$Qk=q.$$

where

$$Q:=\begin{bmatrix}(F(\phi))^*(M\theta_1(0)+N\theta_1(1))\\ \theta_2(1)\end{bmatrix}.$$

$$q:=\begin{bmatrix}(F(\phi))^*(\gamma-N\theta_1(1)\ h(1))\\ \theta_2(1)\ h(1)\end{bmatrix}.$$

Hence since $q\in$ Range Q (this is guaranted by (ii) and (v)), we see that

$$k=k_0+Q^\sharp(q)$$

for some $k_0\in$ Null Q.

Therefore the optimal response x and the adjoint optimal response η are given by

$$(7)\quad \begin{cases}x=\theta_1 h+\theta_1(k_0+Q^\sharp(q)),\\ \eta=\theta_2 h+\theta_2(k_0+Q^\sharp(q))\end{cases}$$

Consequently the optimal controller u is given by

$$(8)\quad u=\beta-(U^*U)^\sharp B^*\{\theta_2 h+\theta_2(k_0+Q^\sharp(q))\}.$$

Remark. In [4] the optimal controller (open loop) associated with an *autonomous* state equation subject to standard two-point separated boundary conditions was computed using Drazian generalized inverses. The method used there works only when P_0. B and U are time independent.

Our method of computing the optimal controllers (see (8)) works well even if P_0 and B are time dependent and is simpler and does not make use of Drazian generalized inverses.

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