Lebesgue-Stieltjes Measures and Differentiation of Measures

by

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Abstract

The thery of measure is significant in that we extend from it to the theory of integration.

As specific metric outer measures we can take Hausdorff outer measure and Lebesgue-Stieltjes outer measure connecting measure with monotone functions. ([12])

The purpose of this paper is to find some properties of Lebesgue-Stieltjes measure by extending it from R^1 to $R^n(n \geq 1)$ (§3) and differentiation of the integral defined by Borel measure (§4).

If in detail, as follows.

We proved that if ${}_n\lambda_f^*$ is Lebesgue-Stieltjes outer measure defined on a finite monotone increasing function $f: R \longrightarrow R$ with the right continuity, then ${}_n\lambda_f^*(I) = \prod_{j=1}^{n}(f(b_j)-f(a_j))$, where $I=\{(x_1,\ldots,x_n) | a_j < x_j \leq b_j,\ j=1,\ldots,n\}$. (Theorem 3.6).

We've reached the conclusion of an extension of Lebesgue Differentiation Theorem in the course of proving that the class of continuous function on R^n with compact support is dense in $L^p(d\mu)$ $(1 \leq p < \infty)$ (Proposition 2.4).

That is, if f is locally μ-integrable on R^n, then

$$\lim_{h\to 0} \frac{1}{\mu(Q_x(h))} \int_{Q_x(h)} f\, d\mu = f(x) \quad \text{a.e. } (\mu).$$

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1. Introduction

The theory of measure is significant in that we extend from it to the theory of integration.

In this sense, P. Halmos defined the measure on Hausdorff space ([5]) and L. Schwartz developed the theory of integration from the measure defined on the local convex linear space ([11]).

In [7] and [14], we can also find Lebesgue integral of the function with its value on Banach space. As the specific metric outer measures we can take Hasudorff outer measure and Lebesgue-Stieltjes outer measure connecting measures with monotone functions([12]).

The purpose of this paper is to find some properties of Lebesgue-Stieltjes measure by extending it from $\boldsymbol{R}^1$ to $\boldsymbol{R}^n(n\geq 1)$ (§3) and differentiation of the integral defined by Borel measure (§4).

If we describe the contents of this paper, it is as follows:
In §2, we described the definition of some terms and basic properties to help understand the theory of §3 and §4.

We proved that the class of continuous function on $\boldsymbol{R}^n$ with compact support is dense in $L^p(d\mu)(1\leq p<\infty)$ (Proposition 2.4).

In §3, we evolved Lebesgue-Stieltjes measure on $\boldsymbol{R}^n$ (Definition 3.1) and we proved that if ${}_n\lambda_f^*$ is Lebesgue-Stieltjes outher measure defined on a finite monotone increasing function f: $\boldsymbol{R}\longrightarrow\boldsymbol{R}$ with the right continuity, then ${}_n\lambda_f^*(I)=\prod_{j=1}^{n}(f(b_j)-f(a_j))$, where $I=\{x_1,\ldots,x_n \mid a_j<x_j\leq b_j,\ j=1,\ldots,n\}$. (Theorem 3.6).

We also proved that every Borel measure μ on $\boldsymbol{R}^n$ such that its value is finite on each bounded Borel set is regular (Lemma 4.2).

Finally we proved in the Theorem 4.4 that if f is locally μ-integrable on $\boldsymbol{R}^n$, then

$$\lim_{h\to 0}\frac{1}{\mu(Q_x(h))}\int_{Q_x(h)} f\,d\mu=f(x) \qquad a.e.\,(\mu).$$

2. Preliminaries

Let $\mathscr{S}$ be an arbitrary set and $\mathscr{P}(\mathscr{S})$ be the collection of all subsets of $\mathscr{S}$. A function Γ: $\mathscr{P}(\mathscr{S})\longrightarrow\boldsymbol{R}$($\boldsymbol{R}$: reals) is called an *outer measure* if it satisfies the following conditions:

(a) $\Gamma(\phi)=0$, (ϕ: null set).

(b) $\Gamma(A_1)\leq\Gamma(A_2)$ if $A_1\subset A_2$ *in* $\mathscr{P}(\mathscr{S})$.

(c) $\Gamma(\bigcup A_k)\leq\sum\Gamma(A_k)$ for any countable collection of sets $\{A_k\}$ in $\mathscr{P}(\mathscr{S})$.

For an outer measure Γ: $\mathscr{P}(\mathscr{S})\longrightarrow \boldsymbol{R}$, $E\in\mathscr{P}(\mathscr{S})$ is said to be *Γ-measurable*, or simply *measurable*, if for every $A\in\mathscr{P}(\mathscr{S})$

$$\Gamma(A)=\Gamma(A\cap E)+\Gamma(A-E).$$

If E is a measurable set, then $\Gamma(E)$is called its *Γ-measure* or simply *measure*. The following properties have been already proved ([1], [5], [12], [13], [15]).

Property 2.1. Let Γ: $\mathscr{P}(\mathscr{S})\longrightarrow \boldsymbol{R}$ be an outer measure.

(i) The family of all Γ-measurable subsets forms a σ-algebra.

(ii) If $\{E_k\}$ is a countable collection of disjoint Γ-measurable subsets, then $\Gamma(\bigcup F_k)=\sum\Gamma(E_k)$. ///

Let us assume that $\mathscr{S}$ is a metric space with metric d. The distance between A_1 and A_2 in $\mathscr{P}(\mathscr{S})$ is defined by

$$d(A_1,\ A_2)=\inf\{d(x_1,\ x_2)\,|\,x_1\in A_1,\ x_2\in A_2\}.$$

Let Γ: $\mathscr{P}(\mathscr{S})\longrightarrow \boldsymbol{R}$ be an outer measure. Then Γ is called a *Carathéodory outer measure* or a *metric outer measure* if

$$\Gamma(A_1\cup A_2)=\Gamma(A_1)+\Gamma(A_2)\ \text{whenever}\ d(A_1,\ A_2)>0.$$

We have the following ([4], [8], [10], [14]).

Property 2.2. Let Γ be a metric outer measure on a metric space $\mathscr{S}$. Then

(i) Every Borel subset of $\mathscr{S}$ is Γ-measurable.

(ii) Every semi-continuous function defined on $\mathscr{S}$ is Γ-measurable (i.e., for every real number α the set $\{x\in\mathscr{S}\,|\,f(x)\geq\alpha\}$ is Γ-measurable). ///

We again assume that $\mathscr{S}$ is an arbitrary set and consider an algebra $\mathscr{A}$ in $\mathscr{P}(\mathscr{S})$.

For an algebra $\mathscr{A}$ a function λ: $\mathscr{A}\longrightarrow \boldsymbol{R}$ is called a *measure* if it satisfies the conditions:

(a) $\forall A\in\mathscr{A}$ $\lambda(A)\geq 0$, $\lambda(\phi)=0$.

(b) If $\{A_k\}$ is a countable collection of disjoint sets in $\mathscr{A}$ and $\bigcup A_k\in\mathscr{A}$, then

$$\lambda(\bigcup A_k)=\sum\lambda(A_k).$$

Note that if an algebra $\mathscr{A}$ is a σ-algebra, then a measure defined on $\mathscr{A}$ is a usual measure.

A measure λ on $\mathscr{A}$ is said to be *σ-finite* with respect to $\mathscr{A}$ if $\mathscr{S}$ can be written as $\mathscr{S}=\bigcup S_k$ with $S_k\in\mathscr{A}$ and $\lambda(S_k)<\infty$. If λ is a measure on $\mathscr{A}$ and μ a measure on a σ-algebra Σ containing $\mathscr{A}$ such that for every $A\in\mathscr{A}$ $\mu(A)=\lambda(A)$, then μ is said to be an extension of λ to Σ.

Let λ be a measure on an algebra $\mathscr{A}$ defined on $\mathscr{S}$. Then, since $\mathscr{S}\in\mathscr{A}$, every subset A of $\mathscr{S}$ is covered by a countable collection $\{A_k\}$ in $\mathscr{A}$.

We define λ^*: $\mathscr{P}(\mathscr{S})\longrightarrow \boldsymbol{R}$ by

$$\text{(2—1)}\qquad \lambda^*(A)=\inf\{\sum\lambda(A_k)\mid A_k\in\mathscr{A},\ A\subset\bigcup A_k\},$$

for each $A\in\mathscr{P}(\mathscr{S})$. Then the following holds ([8], [12]).

Property 2.3. Let λ be a measure on $\mathscr{A}$ in $\mathscr{P}(\mathscr{S})$ and λ^* a set function defined by (2—1). Then

(i) λ^* is an outer measure such that $\lambda(A)=\lambda^*(A)$ for all $A\in\mathscr{A}$. Moreover, every $A\in\mathscr{A}$ is λ^*-measurable.

(ii) Let $\mathscr{A}^*$ be the σ-algebra of λ^*-measurable sets. If λ is σ-finite with respect to $\mathscr{A}$, and if Σ is any σ-algebra with $\mathscr{A}\subset\Sigma\subset\mathscr{A}^*$, then λ^* is the only measure on Σ which is an extension of λ (Carathéodory-Hahn Extension Theorem). ///

Let Σ be the σ-algebra consisting of all Borel sets in $\boldsymbol{R}^n$. A measure defined on Σ is called a *Borel measure* on $\boldsymbol{R}^n$. Let μ: $\Sigma\longrightarrow\boldsymbol{R}$ be a Borel measure on $\boldsymbol{R}^n$. μ is said to be *regular* if for every Borel set $E\subset\boldsymbol{R}^n$

$$\mu(E)=\inf\{\mu(G)\mid G\text{: open in }\boldsymbol{R}^n,\ E\subset G\}.$$

This means that given $\varepsilon>0$ there exists an open set G such that $\mu(G)<\mu(E)+\varepsilon$ or $\mu(G-E)<\varepsilon$.

For a μ-measurable function f: $\boldsymbol{R}^n\longrightarrow\boldsymbol{R}$ if

$$\int_{\boldsymbol{R}^n}|f|^p d\mu<+\infty \quad (0<p<+\infty),$$

then f is said to be $\mu(p)$-*integrable.*

Let us put

$$L^p(d\mu)=\{f: \boldsymbol{R}^n \longrightarrow \boldsymbol{R} \mid f \text{ is } \mu(p)\text{-integrable}\}.$$

For $f\in L^p(d\mu)$ the $\mu(p)$-norm $\|f\|_p$ of f is defined by

$$\|f\|_p=\{\int_{\boldsymbol{R}^n}|f|^p d\mu\}^{\frac{1}{p}}.$$

Then, as is well-known, $L^p(d\mu)$ is a Banach space with $\mu(p)$-norm. In particular, for f, $g\in L^p(d\mu)$ with $1\leq p<\infty$

$$\text{(2—2)} \qquad \|f+g\|_p\leq\|f\|_p+\|g\|_p \qquad ([3],[7],[9],[14]).$$

Proposition 2.4. Let μ be a regular Borel measure on $\boldsymbol{R}^n$, and let $C_0(\boldsymbol{R}^n)$ be the class of continuous functions on $\boldsymbol{R}^n$ with compact support. Then, $C_0(\boldsymbol{R}^n)$ is dense in] $L^p(d\mu)$, $1\leq p<\infty$.

Proof. We want to prove that for each $f\in L^p(d\mu)$ there exists a sequence $\{c_k\}$ in $C_0(\boldsymbol{R}^n)$ such that

$$\int_{\boldsymbol{R}^n}|f-c_k|^p d\mu \longrightarrow 0 \quad \text{as} \quad k\longrightarrow\infty.$$

In this case we say that f satisfies the property $\mathscr{D}$, Our proof is devided into three steps.

Step I. When f_1 and f_2 in $L^p(d\mu)$ satisfy the property $\mathscr{D}$, so do f_1+f_2 and α_1, where α is a constant. Assume that

$$\int_{\boldsymbol{R}^n}|f_1-c_k|^p \longrightarrow 0, \int_{\boldsymbol{R}^n}|f_2-c_k|^p \longrightarrow 0 \quad \text{as} \quad k\longrightarrow\infty,$$

where $\{c_k\}$ and $\{c_k'\}$ are sequences in $C_0(\boldsymbol{R}^n)$.

Since c_k and c_k' are in $L^p(d\mu)$ for $k=1,2,\dots,$ f_1-c_k and f_2-c_k' are in $L^p(d\mu)$. Thus

$$\{\int_{\boldsymbol{R}^n}|(f_1+f_2)-(c_k+c_k')|^p\}^{\frac{1}{p}}\leq\{\int_{\boldsymbol{R}^n}|f_1-c_k|^p\}^{\frac{1}{p}}+\{\int_{\boldsymbol{R}^n}|f_2-c_k'|^p\}^{\frac{1}{p}}.$$

Therefore we have the following:

$$\lim_{k\to\infty}\int_{R^n}|(f_1+f_2)-(c_k+c_k')|^p\leq\lim_{k\to\infty}\{(\int_{R^n}|f_1-c_k|^p)^{\frac{1}{p}}+(\int_{R^n}|f_2-c_k'|^p)^{\frac{1}{p}}\}^p$$

$$=\{(\lim_{k\to\infty}\int_{R^n}|f_1-c_k|^p)^{\frac{1}{p}}+(\lim_{k\to\infty}\int_{R^n}|f_2-c_k'|^p)^{\frac{1}{p}}\}^p$$

$$=0$$

which means that f_1+f_2 satisfies the property $\mathscr{P}$, because that c_k+c_k' is in $C_0(R^n)$.

For a constant α if

$$\int_{R^n}|f_1-c_k|^p\longrightarrow 0,$$

then

$$\int_{R^n}|\alpha f_1-\alpha c_k|^p=|\alpha|^p\int_{R^n}|f_1-c_k|^p\longrightarrow 0.$$

Therefore αf_1 satisfies the property $\mathscr{P}$.

Step II. If for a sequence $\{f_k\}$ such that $f_k\in L^p(d\mu)$ satisfying the property $\mathscr{P}$ $(k=1,2,\ldots)$,

$$\int_{R^n}|f-f_k|^p\longrightarrow 0 \quad \text{as} \quad k\longrightarrow\infty,$$

then f satisfies the property $\mathscr{P}$. Since

$$\|f\|_p\leq\|f-f_k\|_p+\|f_k\|_p$$

and $\|f_k\|_p$, $\|f-f_k\|_p<\infty$ we have

$$\int_{R^n}|f|^p d\mu=\|f\|_p^p<\infty.$$

That is, $f\in L^p(d\mu)$. Take a function f_{k_0} in $\{f_k\}$ such that $\|f-f_{k_0}\|_p<\dfrac{\varepsilon^{\frac{1}{p}}}{2}$ for arbitrary $\varepsilon>0$. Then, by our assumption there exists $c\in C_0(R^n)$
such that

$$\|f_{k_0}-c\|_p<\frac{\varepsilon^{\frac{1}{p}}}{2}.$$

Hence

$$\|f-c\|_p^p \leq (\|f-f_{k_0}\|_p + \|f_{k_0}-c\|_p)^p$$
$$< \left(\frac{\varepsilon^{\frac{1}{p}}}{2} + \frac{\varepsilon^{\frac{1}{p}}}{2}\right)^p$$
$$= \varepsilon,$$

which implies that f satisfies the property $\mathscr{D}$.

Step Ⅲ. For $f \in L^p(d\mu)$, writting $f = f^+ - f^-$, we may assume that $f \geq 0$. Since every function can be written as the limit of a sequence $\{f_k\}$ of simple functions, we may regard f as a non-negative simple function which is μ-integrable. By definition of simple function and Step Ⅰ we may also assume that f is a characteristic function χ_E with $\mu(E) < \infty$ $(E \subset \boldsymbol{R}^n)$. By our hypothesis, since μ is regular, there exists an open set $G \supset \boldsymbol{R}^n$ such that $\mu(G-E) < \varepsilon^{\frac{1}{p}}$ for an arbitrary positive number ε. Then

$$\int_{\boldsymbol{R}^n} |\chi_G - \chi_E|^p d\mu = \{\mu(G-E)\}^p < \varepsilon,$$

so, in order to prove our Proposition it suffices to show that for an open set G with $\mu(G) < \infty$, $f = \chi_G$ satisfies the property $\mathscr{D}$. We have to note that there is a covering $G = \bigcup I_k$ of half-open intervals in $\boldsymbol{R}^n$ which are disjoint.

Let us f_N be the characteristic function of $\bigcup_{k=1}^{N} I_k$.

Since $\mu(G) = \sum_{k=1}^{\infty} \mu(I_k) < \infty$,

$$\int_{\boldsymbol{R}^n} |f - f_N|^p d\mu = \sum_{k=N+1}^{\infty} (\mu(I_k))^p \longrightarrow 0.$$

Hence, by Step Ⅱ it is enough to prove that f_N satisfies the property $\mathscr{D}$. Thus it suffices to prove that the characteristic function χ_I of an half-open interval I satisfies the proper ty $\mathscr{D}$. By the regularity of μ there exists an open interval I^* such that the closure of I is contained in I^* (i.e., $\bar{I} \subset I^*$) and $\mu(I^* - I) < \varepsilon^{\frac{1}{p}}$. By Urysohn's Lemma, there is a continuous function c with $0 \leq c \leq 1$ such that

$$c(x) = \begin{cases} 1 & x \in I \\ 0 & x \in CI^* \end{cases}$$

In consequence,

$$\int_{\boldsymbol{R}^n} |\chi_I - c|^p d\mu \leq (\mu(I^* - I))^p < \varepsilon. \qquad ///$$

3. Lebesgue-Stieltjes Measure

In the future of this paper, by a half-open interval I in $\boldsymbol{R}^n$ we mean a left open interval, i.e.,

$$I=\{(x_1,\ldots,x_n)\in\boldsymbol{R}^n \mid a_j<x_j\leq b_j,\ j=1,\ldots,n\}.$$

Let f: $\boldsymbol{R}\longrightarrow\boldsymbol{R}$ be a finite monotone increasing function. For a half-open interval I we put

$$\lambda(I)=\lambda_f(I)=\prod_{j=1}^{n}(f(b_j)-f(a_j)).$$

Note that $\boldsymbol{R}^n$ is covered by a countable union of half-open intervals in $\boldsymbol{R}^n$.

Definition 3.1. The *Lebesgue-Stieltjes outer measure* ${}_n\lambda_f^*$ corresponding to f in $\boldsymbol{R}^n$ is defined by

$${}_n\lambda_f^*(A)=\inf\sum\lambda(I_k)$$

for $A\subset\boldsymbol{R}^n$, where the inf is taken over all countable collections of half-open intervals I_k which covers A.

Further define

$${}_n\lambda_f^*(\phi)=0.$$

Proposition 3.2. ${}_n\lambda_f^*$ is a metric outer measure.

Proof. By (i) of property 2.3 ${}_n\lambda_f^*$ is an outer measure, but we have another proof as follows.

It is clear that for each $A\subset\boldsymbol{R}^n$ ${}_n\lambda_f^*(A)\geq 0$.

Since for A_1 and A_2 in $\boldsymbol{R}^n$ with $A_1\subset A_2$, $A_2\subset\bigcup I_k$ implies that $A_1\subset\bigcup I_k$, by Definition 3.1

$${}_n\lambda_f^*(A_1)\leq\inf\sum\lambda(I_k),$$

where $A_2\subset\bigcup I_k$ and $I_k=\{(x_1,\ldots,x_n)|a_j^k<x_j\leq b_j^k,\ j=1,\ldots,n\}$. Thus ${}_n\lambda_f^*(A_1)\leq{}_n\lambda_f^*(A_2)$.

To show that ${}_n\lambda_f^*$ is subadditive, let $\{A_i\}$ be a collection of nonempty subsets of $\boldsymbol{R}^n$ and $A=\bigcup A_i$.

Choose half-open intervals $\{I_k{}^l\}$ such that

$$A_l \subset \bigcup_k^{\infty} I_k{}^l,\ \sum_{k=1}^{\infty} \lambda(I_k{}^l) \leq {}_n\lambda_f^*(A_l) + \varepsilon 2^{-l}.$$

Nothing $A \subset \bigcup_{k,l} I_k{}^l$ it follows that

$${}_n\lambda_f^*(A) \leq \sum_{k,l} \lambda(I_k{}^l) \leq \sum_l ({}_n\lambda_f^*(A_l) + \varepsilon 2^{-l}) = \sum_l {}_n\lambda_f^*(A_l) + \varepsilon,$$

where $I_k{}^l = \{(x_1, \ldots, x_n) \mid a_j{}^{k,l} < x_j \leq b_j{}^{k,l},\ j = 1, 2, \ldots, n\}$ $(k = 1, 2, \ldots)$.
Since ε is arbitrary

$${}_n\lambda_f^*(A) \leq \sum_l {}_n\lambda_f^*(A_l)$$

and thus ${}_n\lambda_f^*$ is an outer measure.

In order to prove that ${}_n\lambda_f^*$ is a metric outer measure, let $d(A_1, A_2) = \delta > 0$ for A_1 and A_2 in $\boldsymbol{R}^n$.

It is enough to prove that

$${}_n\lambda_f^*(A_1 \cup A_2) = {}_n\lambda_f^*(A_1) + {}_n\lambda_f^*(A_2).$$

We see that there exists a covering $A_1 \cup A_2 \subset \bigcup I_k$ such that

(a) $I_k = \{(x_1, \ldots, x_n) \mid a_j{}^k < x_j \leq b_j{}^k,\ b_j{}^k - a_j{}^k < \frac{\delta}{n},\ j = 1, \ldots, n\}$,

(b) $\sum \lambda(I_k) \leq {}_n\lambda_f^*(A_1 \cup A_2) + \varepsilon,\quad \forall \varepsilon > 0.$

Since I_k splits into two coverings, one of A_1 and other of A_2, it follows that

$${}_n\lambda_f^*(A_1) + {}_n\lambda_f^*(A_2) \leq \sum \lambda(I_k) \leq {}_n\lambda_f^*(A_1 \cup A_2) + \varepsilon,$$

and thus

$${}_n\lambda_f^*(A_1) + {}_n\lambda_f^*(A_2) \leq {}_n\lambda_f^*(A_1 \cup A_2) \quad \text{as} \quad \varepsilon \longrightarrow 0. \qquad ///$$

We shall set

$$\mathscr{A} = \{A \subset \boldsymbol{R}^n \mid A \text{ is } {}_n\lambda_f^*\text{-measurable}\}.$$

Then, by (i) of property 2.1, $\mathscr{A}$ is a σ-algebra and ${}_n\lambda_f^* \mid \mathscr{A}$ is a measure which is called the *Lebesgue-Stieltjes measure* corresponding to f.

Propositoin 3.3. Let A be any subset of R^n contained in an open subset G of R^n, and let

$$A_k=\{x\in A \mid d(x, CG)\geq \frac{1}{k}\} \quad (k=1,2,\ldots).$$

Then $\lim_{k\to\infty} {}_n\lambda_f^*(A_k)={}_n\lambda_f^*(A)$.

Proof. Since $A_k \uparrow A$ and $\bigcup A_k=A$, by Definition 3.1 it is obvious that $\lim {}_n\lambda_f^*(A_k) \leq {}_n\lambda_f^*(A)$.

The opposite inequality is proved as follows. Let

$$D_k=A_{k+1}-A_k \quad (k=1,2,\ldots).$$

Then $A=A_k\cup D_k\cup D_{k+1}\cup\ldots$, and thus

(3—1) $${}_n\lambda_f^*(A)\leq {}_n\lambda_f^*(A_k)+{}_n\lambda_f^*(D_k)+{}_n\lambda_f^*(D_{k+1})+\ldots .$$

If $\sum {}_n\lambda_f^*(D_j)<+\infty$, then $\sum_{j\geq k} {}_n\lambda_f^*(D_j)\longrightarrow 0$ as $k\longrightarrow\infty$. Therefore, by (3—1) we get our desired result:

$${}_n\lambda_f^*(A)\leq \lim_{k\to\infty} {}_n\lambda_f^*(A_k).$$

If $\sum {}_n\lambda_f^*(D_j)=+\infty$, then at least one of $\sum {}_n\lambda_f^*(D_{2j})$ and $\sum {}_n\lambda_f^*(D_{2j+1})$ is infinite. Thus, we can choose N such that

$${}_n\lambda_f^*(D_N)+{}_n\lambda_f^*(D_{N-2})+{}_n\lambda_f^*(D_{N-4})+\ldots$$

is arbitrary large. Since

$$d(D_{k-1}, D_{k+1})\geq \frac{1}{k(k+1)}>0,$$

by (ii) of property 2.1

$${}_n\lambda_f^*(D_N\cup D_{N-2}\cup\ldots)={}_n\lambda_f^*(D_N)+{}_n\lambda_f^*(D_{N-2})+\ldots$$

However, the fact that $D_N\cup D_{N-2}\cup\ldots\subset A_{N+1}\subset A$ implies that

$$\begin{aligned}{}_n\lambda_f^*(A_{N+1}) &\geq {}_n\lambda_f^*(D_N\cup D_{N-2}\cup\ldots)\\ &={}_n\lambda_f^*(D_N)+{}_n\lambda_f^*(D_{N-2})+\ldots\\ &>M\end{aligned}$$

for any sufficiently large number $M>0$. That is,

$$\lim_{k\to\infty} {}_n\lambda_f^*(A_k)=\infty={}_n\lambda_f^*(A).$$ ///

Proposition 3.4. (i) Every Borel set in $\boldsymbol{R}^n$ is ${}_n\lambda_f^*$-measurable.

(ii) ${}_n\lambda_f^*$ is a regular outer measure.

Proof. (i) It is clear that (i) is true by (i) of property 2.2, but we have another proof as follows. It will suffice to prove that every closed subset of $\boldsymbol{R}^n$ is ${}_n\lambda_f^*$-measurable, because that every closed subset is contained in $\mathscr{A}$ means that the Borel σ-algebra Σ is contained in $\mathscr{A}$.

Let F be a closed subset of $\boldsymbol{R}^n$. We have to prove that

$${}_n\lambda_f^*(A\cup B)={}_n\lambda_f^*(A)+{}_n\lambda_f^*(B)$$

for all $A\subset CF$ and $B\subset F$. Put

$$A_k=\{x\in A\,|\,d(x,F)\geq\frac{1}{k}\},$$

then $d(B,A_k)\geq\frac{1}{k}>0$. Hence we have

$${}_n\lambda_f^*(A\cup B)\geq{}_n\lambda_f^*(A_k\cup B)={}_n\lambda_f^*(A_k)+{}_n\lambda_f^*(B).$$

By proposition 3.3

$$\begin{aligned}{}_n\lambda_f^*(A\cup B)&\geq\lim_{k\to\infty}{}_n\lambda_f^*(A_k)+{}_n\lambda_f^*(B)\\&={}_n\lambda_f^*(A)+{}_n\lambda_f^*(B).\end{aligned}$$

Since it is ovious that ${}_n\lambda_f^*(A\cup B)\leq{}_n\lambda_f^*(A)+{}_n\lambda_f^*(B)$, we get

$${}_n\lambda_f^*(A\cup B)={}_n\lambda_f^*(A)+{}_n\lambda_f^*(B).$$

(ii) We want to prove that for each subset A of $\boldsymbol{R}^n$ there exists a Borel set B in $\boldsymbol{R}^n$ such that $A\subset B$ and ${}_n\lambda_f^*(A)={}_n\lambda_f^*(B)$. Take a covering $A\subset\bigcup_k I_k{}'$ such that for a given $\varepsilon>0$

$$\sum_k\lambda(I_k{}')\leq{}_n\lambda_f^*(A)+\varepsilon.$$

Let us put

$$B_l=\bigcup_k I_k{}^l \text{ and } B=\bigcap_l B_l,$$

then $A\subset B$, ${}_n\lambda_f^*(B_l)\leq\sum_k\lambda(I_k{}^l)\leq{}_n\lambda_f^*(A)+\varepsilon$

and $${}_n\lambda_f^*(B)\leq{}_n\lambda_f^*(B_l)\leq{}_n\lambda_f^*(A)+\varepsilon.$$

That is, $${}_n\lambda_f^*(B)={}_n\lambda_f^*(A).$$ ///

Lemma 3.5. If f is a finite monotone increasing function which is continuous from the right, then

$${}_1\lambda_f^*((a,b])=f(b)-f(a)$$

and $${}_1\lambda_f^*(\{a\})=f(a)-f(a-),$$

where $f(a-)=\lim_{h\to 0} f(a-h)$ $(h>0)$ and $-\infty<a<+\infty$.

Proof. It is clear that ${}_1\lambda_f^*((a,b])\leq\lambda((a,b])=f(b)-f(a)$ by Definition 3.1. In order to prove the opposite inequality, suppose that

$$(a,b]\subset\bigcup[a_k,b_k].$$

Given $\varepsilon>0$ use the right continuity of f to choose $\{b_k'\}$ such that

$$b_k<b_k' \text{ and } f(b_k)>f(b_k')-\frac{\varepsilon}{2^k}.$$

Take a' svch that $a<a'<b$, then

$$[a',b]\subset\bigcup(a_k,b_k').$$

By the compactness of $[a',b]$ there exists a positive integer N such that

$$[a',b]\subset\bigcup_{k=1}^{N}(a_k,b_k').$$

By discarding unnecessary (a_k,b_k') and reindexing the rest, we can assume that $a_{k+1}<b_k'$, $k=1,\ldots,N-1$, $a_1<a'$ and $b<b_N'$. Thus we have

$$f(a_1)\leq f(a'),\ f(b)\leq f(b_N') \text{ and so on.}$$

In this case,

$$\sum\lambda((a_k,b_k])\geq\sum_{k=1}^{N}\lambda((a_k,b_k])=\sum_{k=1}^{N}(f(b_k)-f(a_k))$$

$$=f(b_N)-f(a_1)+\sum_{k=1}^{N-1}(f(b_k)-f(a_{k+1})),$$

$$f(b_N)-f(a_1)=(f(b_N)-f(b_N'))+(f(b_N')-f(a_1))$$
$$>-\varepsilon 2^{-N}+(f(b)-f(a'))$$
$$>-\varepsilon+f(b)-f(a').$$

Since $f(b_k')-f(a_{k+1})\geq 0$,

$$\sum_{k=1}^{N-1}(f(b_k')-f(a_{k+1}))=\sum_{k=1}^{N-1}(f(b_k)-f(b_k'))+\sum_{k=1}^{N-1}(f(b_k')-f(a_{k+1}))$$

$$\geq\sum_{k=1}^{\infty}(-\frac{\varepsilon}{2^k})=-\varepsilon.$$

Thus we have

$$\sum_{k=1}^{\infty}\lambda((a_k,b_k])\geq -2\varepsilon+f(b)-f(a').$$

Letting $\varepsilon\longrightarrow 0$ and $a'\longrightarrow a$ it follows that

$$\sum\lambda((a_k,b_k])\geq f(b)-f(a')\geq f(b)-f(a)=\lambda((a,b]).$$

This means that ${}_1\lambda_f^*((a,b])\geq\lambda((a,b])$ by Definition 3.1.

The second statement is proved by setting

$$\{a\}\subset(a-h,a]\quad(h>0),$$

That is, ${}_1\lambda_f^*(\{a\})=\inf_{h>0}\lambda((a-h,a])$

$$=\inf_{h>0}(f(a)-f(a-h))$$
$$=\lim_{h>0}(f(a)-f(a-h))$$
$$=f(a)-f(a-).$$

Since $(-\infty,a]$ is covered by a countable union of disjoint half-open intervals, we have

$$(-\infty,a]=\bigcup(a_k,b_k]$$

and

$$ {}_1\lambda_f^*((-\infty,a])=\sum\lambda((a_k,b_k])=\sum(f(b_k)-f(a_k)). $$

We shall prove the following theorem by mathematical induction.

Theorem 3.6. Let $f: \boldsymbol{R}\longrightarrow\boldsymbol{R}$ be a finite monotone increasing function with the right continuity. Then

$$ {}_n\lambda_f^*(I)=\lambda(I) \text{ and } {}_n\lambda_f^*(\{a\})=\prod_{j=1}^{n}(f(a_j)-f(a_j-)), $$

where I is a bounded half-open interval in $\boldsymbol{R}^n$ and $a=(a_1,\dots,a_n)$.

Proof. We have already proved that our assertion is true in $\boldsymbol{R}^1$(Lemma 3.5).

We assume that for all half-open intervals I^r in $\boldsymbol{R}^r(r=1,\dots,n-1)$ our theorem holds.

Take a half-open interval

$$ I^n=\{(x_1,\dots,x_n)\,|\,a_j<x_j\leq b_j,\ j=1,\dots,n\} $$

and consider a covering of I^n such that

$$ I^n\subset\bigcup_{k=1}^{\infty}I_k{}^n\subset J. $$

where

$$ I_k{}^n=\{(x_1,\dots,x_n)\,|\,a_j{}^k<x_j\leq b_j{}^k,\ j=1,\dots,n\} $$

and J is a bounded half-open interval. Since I^n covers itself,

$$ {}_n\lambda_f^*(I^n)=\inf\sum\lambda(I_k{}^n)\leq\lambda(I^n). $$

In order to prove the opposite inequality, we shall put

$$ \mathscr{S}=\{I_k{}^n\,|\,k=1,2\dots\} $$

and

$$ \mathscr{S}_1=\{I_k{}^n\in\mathscr{S}\,|\,b_n{}^k\geq b_n\}. $$

We also put

$$I^{n-1}(b_n)=\{(x_1,\dots,x_n)\,|\,a_j<x_j\leq b_j,\ x_n=b_n,\ j=1,\dots,n-1\}$$

and

$$I_k^{n-1}(b_n)=\{(x_1,\dots,x_n)\,|\,a_j^k<x_j\leq b_j^k,\ x_n=b_n,\ j=1,\dots,n-1\},$$

then it is clear that

$$I^{n-1}(b_n)\subset\bigcup_k\{I_k^{n-1}(b_n)\,|\,I_k^n\in\mathscr{S}_1\}.$$

If we regard $I^{n-1}(b_n)$ and $I_k^{n-1}(b_n)$ $(k=1,2,\dots)$ as half-open intervals in $\boldsymbol{R}^{n-1}$, then

$$\lambda(I^{n-1}(b_n))\leq\sum_k\{\lambda(I_k^{n-1}(b_n))\,|\,I_k^n\in\mathscr{S}_1\}$$

by our inductive hypothesis. (In fact, since we identify $\{(x_1,\dots,x_{n-1})\,|\,a_j<x_j\leq b_j,\ j=1,\dots,n-1\}$ with $\lambda(I^{n-1}(b_n))$,

$$\lambda(I^{n-1}(b_n))=\prod_{j=1}^{n-1}(f(b_j)-f(a_j)),$$

and also

$$\lambda(I_k^{n-1}(b_n))=\prod_{j=1}^{n-1}(f(b_j^k)-f(a_j^k)).$$

Since $\lambda(I^n)<\infty$, we may assume that

$$\sum_{I_k^n\in\mathscr{S}}\lambda(I_k^n)<\infty$$

and also that

$$\sum_k\{\lambda(I_k^{n-1}(b_n))\,|\,I_k^n\in\mathscr{S}_1\}<\infty.$$

Given $\varepsilon>0$, reindexing if necessary, we can write as

$$\lambda(I^{n-1}(b_n)\leq\sum_{k=1}^{N_1}\{\lambda(I_k^{n-1}(b_n))\,|\,I_k^n\in\mathscr{S}_1\}+\frac{\varepsilon}{M},$$

where $M>f(b_n)-f(a_n)$. Let us put

$$\mathscr{S}_2=\{I_1^n,\ldots,I^n_{N_1}\}\subset\mathscr{S}_1$$

and

$$\alpha_n^1=\min\{a_n^1,\ldots,a_n^{N_1}\}.$$

It follows that

$$\lambda(I^{n-1}(b_n))\cdot\lambda((\alpha_n^1,b_n])$$
$$\leq\sum_{k=1}^{N_1}\{\lambda(I_k^{n-1}(b_n))\cdot\lambda((\alpha_n^1,b_n^k])|I_n^k\in\mathscr{S}_2\}+\frac{\varepsilon}{M}\lambda((\alpha_n^1,b_n]).$$

We make new half-open intervals $'I_k^n(k=1,\ldots,N_1)$ such that

$$'I_k^n=\{(x_1,\ldots,x_n)\,|\,a_j^k<x_j\leq b_j^k,\ j=1,\ldots,n-1,\ a_n^k<x_n\leq\alpha_n^1\}$$

Note that if we put

$$''I_k^n=\{(x_1,\ldots,x_n)\,|\,a_j^k<x_j\leq b_j^k,\ j=1,\ldots,n-1,\ \alpha_n^1<x_n\leq b_n^k\},$$

then

(3—2) $$I_k^n={}'I_k^n\cup{}''I_k^n \text{ and } {}'I_k^n\cap{}''I_k^n=\phi.$$

Suppose that

$$\mathscr{S}_3=(\mathscr{S}-\mathscr{S}_1)\cup\{'I_k^n\,|\,I_k^n\in\mathscr{S}_1\}$$

and

$$\mathscr{S}_4=\{I_{k_i}^n\in\mathscr{S}_3\,|\,\alpha_n^1\leq b_n^{k_i}\}.$$

For the half-open interval

$$I^{n-1}(\alpha_n^1)=\{(x_1,\ldots,x_n)\,|\,a_j<x_j\leq b_j,\ j=1,\ldots,n-1,\ x_n=\alpha_n^1\}$$

and

$$I_{k_i}^{n-1}(\alpha_n^1)=\{(x_1,\ldots,x_n)\,|\,a_j^{k_i}<x_j\leq b_j^{k_i},\ j=1,\ldots,n-1,\ x_n=\alpha_n^1\},$$
$$I^{n-1}(\alpha_n^1)\subset\bigcup\{I_{k_i}^{n-1}(\alpha_n^1)\,|\,I_{k_i}^n\in\mathscr{S}_4\}$$

and that there exists a positive integer N_2 such that

$$\lambda(I^{n-1}(\alpha_n^1))\leq\sum_{i=1}^{N_2}\{\lambda(I_{k_i}^{n-1}(\alpha_n^1))\mid I_{k_i}^n\in\mathscr{S}_4\}+\frac{\varepsilon}{M}.$$

We also put

$$\mathscr{S}_5=\{I_{k_1}^n,\dots,I^n_{k_{N_2}}\}$$

and

$$\alpha_n^2=\min\{a_n^{k_1},\dots,a_n^{k_{N_2}}\}.$$

It follows that

$$\lambda(I^{n-1}(\alpha_n^1))\cdot\lambda((\alpha_n^2,\alpha_n^1])$$
$$\leq\sum_{i=1}^{N_2}\{\lambda(I_{k_i}^{n-1}(\alpha_n^1))\cdot\lambda((\alpha_n^2,b_n^{k_i}])\mid I_{k_i}^n\in\mathscr{S}_5\}+\frac{\varepsilon}{M}\lambda((\alpha_n^2,\alpha_n^1]).$$

Note that $\lambda(I^{n-1}(\alpha_n^1))=\lambda(I^{n-1}(b_n))$

$$=\prod_{i=1}^{n-1}(f(b_i)-f(a_i)).$$

Repeating this way, there is a positive integers l and $\bar{N}$ such that

(a) $a_n=\alpha_n^l$

(b) $\lambda(I^{n-1}(\alpha_n^{l-1}))\cdot\lambda((a_n,\alpha_n^{l-1}])\leq\sum_{t=1}^{\bar{N}}\lambda(I_{h_t}^{n-1}(\alpha_n^{l-1}))\cdot\lambda((a_n,b_n^{h_t}])$

$$+\frac{\varepsilon}{M}\lambda((a_n,\alpha_n^{l-1}]),$$

where $I_{h_t}^n=\{(x_1,\dots,x_n)\mid a_j^{h_t}<x_j\leq b_j^{h_t},\ j=1,\dots,n-1,\ a_n^{h_t}<x_n\leq\alpha_n^{l-1},\ a_n\geq a_n^{h_t}\}$ for $t=1,\dots,\bar{N}$.

In consequence, we get the following:

$$\lambda(I^{n-1}(b_n))\cdot\lambda((\alpha_n^1,b_n])+\dots+\lambda(I^{n-1}(\alpha_n^{l-1}))\cdot\lambda((a_n,\alpha_n^{l-1}])$$
$$\leq\sum_{k=1}^{N_1}(I_k^{n-1}(b_n))\cdot\lambda((\alpha_n^1,b_n^k])+\dots+\sum_{t=1}^{\bar{N}}\lambda(I_{h_t}^{n-1}(\alpha_n^{l-1}))\cdot\lambda((a_n,b_n^{h_t}])$$
$$+\frac{\varepsilon}{M}(f(b_n)-f(a_n)).$$

That is

$$\lambda(I^n)\leq\sum_{I_n^k\in\mathscr{S}}\lambda(I_k^n)+\varepsilon.$$

Note that as in (3—2) if $I_k^n = 'I_k^n \cup ''I_k^n$,
then $\lambda(I_k^n) = \lambda('I_k^n) + \lambda(''I_k^n)$. Therefore,

$$I^n \subset \bigcup I_k^n \text{ means that } \lambda(I_n) \leq \sum_k \lambda(I_k^n) \leq {}_n\lambda_f^*(I^n).$$

Consequently,

$${}_n\lambda_f^*(I^n) = \lambda(I) = \prod_{j=1}^{n} (f(b_j) - f(a_j)).$$

Next set $\{a\} \subset I_k, I_k = \{(x_1, \ldots, x_n) \mid a_j - h_j < x_j \leq a_j,\ h_j > 0\}$.

Then ${}_n\lambda_f^*(\{a\}) = \inf_{h_j>0} \lambda(I_k) = \inf_{h_j>0} \prod_{j=1}^{n} (f(a_j) - f(a_j - h_j))$

$$= \lim_{h_j>0} \prod_{j}^{n} (f(a_j) - f(a_j - h_j)) = \prod_{j=1}^{n} (f(a_j) - f(a_j -)). \qquad ///$$

4. Relative Differentiation of Measures

Let Σ be the Borel σ-algebra of $\boldsymbol{R}^n$, and let $(\boldsymbol{R}^n, \Sigma, \mu)$ be a Borel measure space. A function $f: \boldsymbol{R}^n \longrightarrow \boldsymbol{R}$ is said to be *μ-integrable* on $\boldsymbol{R}^n$ if $\int_{\boldsymbol{R}^n} f\, d\mu$ is defined and its value is finite. A μ-measurable function defined on $\boldsymbol{R}^n$ is called a *locally μ-integrable function* on $\boldsymbol{R}^n$ if it is μ-integrable over every bounded Borel set in $\boldsymbol{R}^n$.

In this section we shall prove that if f is locally μ-integrable on $\boldsymbol{R}^n$ and for $x \in \boldsymbol{R}^n$ $0 < \mu(Q_x(h))$, then

$$\lim_{h \to 0} \frac{1}{\mu(Q_x(h))} \int_{Q_x(h)} f\, d\mu = f(x) \quad a.e.(\mu),$$

where $Q_x(h)$ is an n-dimensional cube centered at x with edges parallel to the coordinate axes and with edge length h, which is an extension of the Lebesgue's Differentiation Theorem ([3], [11], [12], [15]).

Specifically, let μ and ν be two σ-finite Borel measures defined on Σ. We shall also prove some properties with respect to $\nu(Q_x(h))/\mu(Q_x(h))$ $(h>0)$ in this section. In order to do these we need the following Besicovitch Covering Lemma ([12]): For a bounded subset E of $\boldsymbol{R}^n$ and a family of cubes covering E which contains a cube Q_x with center at $x \in E$, then there exist points $\{x_k\} \subset E$ such that

(i) $E \subset \bigcup Q_{x_k}$.

(ii) $\{Q_{x_k}\}$ has bounded overlaps.

(iii) the constant c for which $\sum \chi_{Q_{x_k}} \leq c$ can be choosen to depend only on n.

Lemma 4.1 Let μ and ν be two Borel measures defined on the Borel σ-algebra Σ of $\boldsymbol{R}^n$ which are finite and equal on every half-open interval $\{x_1,\ldots,x_n \mid a_j<x_j\leq b_j,$ $j=1,\ldots,n\}$. Then for each $B\in\Sigma$ $\mu(B)=\nu(B)$.

Proof. Suppose that the algebra $\mathscr{A}$ generated by all half-open intervals in $\boldsymbol{R}^n$, i.e., $\mathscr{A}$ contains ϕ, $\boldsymbol{R}^n$ and all intervals form $\{(x_1,\ldots,x_n) \mid -\infty<x_j\leq b_j,\ j=1,\ldots,n\}$ and

$$\{(x_1,\ldots,x_n) \mid a_j<x_j<+\infty,\ j=1,\ldots,n\},$$

as well as all possible finite disjoint union of these and half-open intervals. From our assumption we see that $\mu|_{\mathscr{A}}=\nu|_{\mathscr{A}}$ and that these are σ-finite.

Recall that the smallest σ-algebra of $\boldsymbol{R}^n$ containing all half-open intervals is the Borel σ-algebra Σ of $\boldsymbol{R}^n$.

Let $\mu_{\mathscr{A}}^*$ and $\nu_{\mathscr{A}}^*$ be the corresponding outer measures to $\mu|_{\mathscr{A}}$ and $\nu|_{\mathscr{A}}$ respectively. Then, from (ii) of Property 2.3 it follows that

$$\mu=\mu_{\mathscr{A}}^*|_{\Sigma}=\nu_{\mathscr{A}}^*|_{\Sigma}=\nu. \qquad ///$$

Lemma 4.2. Let μ be a Borel measure defined on Σ such that for every bounded $B\in\Sigma$, $\mu(B)$ is finite. Then μ is a regular measure.

Proof. Consider $\mu|_{\mathscr{A}}$, where $\mathscr{A}$ is the algebra generated by all half-open intervals in $\boldsymbol{R}^n$.

Let μ^* be the outer measure corresponding to $\mu|_{\mathscr{A}}$.

The smallest σ-algebra containing $\mathscr{A}$ is the Borel σ-algebra Σ of $\boldsymbol{R}^n$. Thus, $\mathscr{A}\subset\Sigma\subset\mathscr{A}^*$, where $\mathscr{A}^*$ is the σ-algebra consisting of all μ^*-measurable sets in $\boldsymbol{R}^n$.

Since μ and μ^* are two Borel measures on Σ which agree on $\mathscr{A}$, and since μ and μ^* are finite on every bounded Borel set from our assumption, we have $\mu=\mu^*$ on Σ by Lemma 4.1. Therefore, we see that

$$\mu(B)=\inf\{\sum\mu(A_k) \mid B\subset\bigcup A_k,\ A_k\in\mathscr{A}\}$$

for every $B\in\Sigma$ by the definition of μ^* by (2—1).
This means that

$$\mu(B)=\inf\{\sum\mu(I_k) \mid B\subset\bigcup I_k\}.$$

Thus, given $\varepsilon>0$ there exists a covering of B such that $B\subset\bigcup I_k$ and

$$\sum \mu(I_k) \le \mu(B) + \varepsilon.$$

Recall that for a sequence $\{E_k\}$ of μ-measurable sets, if $E_k \downarrow E$ and $\mu(E_k) < +\infty$ then $\lim \mu(E_k) = \mu(E)$.

For a half-open interval I put

$$I_k' = \{(x_1, \ldots, x_n) \mid a_j < x_j \le b_j + \frac{1}{k}\},$$

then $\lim \mu(I_k') = \mu(I)$ since $\mu(I_k') < \infty$. (Note that I_k' is contained in a bounded half-open interval). Hence for a sufficiently small positive number ε_k there is an open subset G such that

$$B \subset G = \bigcup I_{\varepsilon_k}' \quad \text{and} \quad \mu(I_{\varepsilon_k}) \le \mu(I_k') + \frac{\varepsilon}{2^k}.$$

Therefore, we have

$$\mu(G) \le \sum \mu(I_{\varepsilon_k}') \le \sum \mu(I_k') + \varepsilon \le \sum \mu(I_k) + 2\varepsilon \le \mu(B) + 3\varepsilon. \qquad ///$$

Lemma 4.3. If μ and ν are Borel measures which are finite on every bounded Borel set, then there is a constant c depending only on n such that

(i) $\mu^*\{x \in \boldsymbol{R}^n \mid \sup_{h>0} \frac{\nu(Q_x(h))}{\mu(Q_x(h))} > \alpha\} \le \frac{c}{\alpha} \nu(\boldsymbol{R}^n)$.

(ii) $\mu^*\{x \in B \mid \lim_{j \to 0} \sup_{h \ge j} \frac{\nu(Q_x(h))}{\mu(Q_x(h))} > \alpha\} \le \frac{c}{\alpha} \nu(B)$

for every Borel set $B \subset \boldsymbol{R}^n$ and any $\alpha > 0$, where μ^* is the outer measure corresponding to $\mu|_{\mathscr{A}}$.

Proof. At first, we have to note that μ and ν are regular by Lemma 4.2 and that $\mu = \mu^*|_{\Sigma}$. Fix $\alpha > 0$.

(i) Let us put

$$S = \{x \in \boldsymbol{R}^n \mid \sup_{h>0} \frac{\nu(Q_x(h))}{\mu(Q_x(h))} > \alpha\}$$

and take a bounded Borel set B such that $S \cap B \ne \phi$. Then, for each $x \in S \cap B$ there is a cube Q_x such that $\nu(Q_x) > \alpha \mu(Q_x)$. Using the Besicovitch Covering Lemma, we can select $\{Q_{x_k}\}$ and constant c such that

$$\nu(Q_{x_k})>\alpha\mu(Q_{x_k}),\ \ S\cap B\subset\bigcup Q_{x_k},\ \ \sum\chi_{Q_{x_k}}\leq c.$$

Since

$$\mu^*(S\cap B)\leq\mu^*(\bigcup Q_{x_k})=\mu(\bigcup Q_{x_k})\leq\sum\mu(Q_{x_k})<\frac{1}{\alpha}\sum\nu(Q_{x_k}),$$

$$\sum\nu(Q_{x_k})=\sum\int_{\bigcup Q_{x_k}}\chi_{Q_{x_k}}\,d\nu\leq c\int_{\bigcup Q_{x_k}}d\nu=c\nu(\bigcup Q_{x_k}).$$

It follows that

$$\mu^*(S\cap B)\leq\frac{c}{\alpha}\nu(\bigcup Q_{x_k})\leq\frac{c}{\alpha}\nu(\boldsymbol{R}^n).$$

Letting $B\uparrow\boldsymbol{R}^n$ we obtain $\mu^*(S)\leq\frac{c}{\alpha}\nu(\boldsymbol{R}^n)$.

(ii) Let us put

$$T=\{x\in B\,|\,\lim_{j\to 0}\sup_{h\geq j}\frac{\nu(Q_x(h))}{\mu(Q_x(h))}>\alpha\}$$

and assume that $\nu(B)<\infty$. By the regularity of ν there is an open set G such that

$$B\subset G\quad\text{and}\quad\nu(G)\leq\nu(B)+\varepsilon.$$

For a bounded Borel set E with $T\cap E\neq\phi$ and for each $x\in T\cap E$ there is a cube Q_x such that

$$Q_x\subset G\quad\text{and}\quad\nu(Q_x)>\alpha\mu(Q_x).$$

As above, there exists $\{Q_{x_k}\}$ with $Q_{x_k}\subset G$ such that

$$\mu^*(T\cap E)\leq\frac{c}{\alpha}\nu(\bigcup Q_{x_k}).$$

Hence

$$\mu^*(T\cap E)\leq\frac{c}{\alpha}\nu(G)\leq\frac{c}{\alpha}(\nu(B)+\varepsilon).$$

Our assertion follows by letting $\varepsilon\longrightarrow 0$ and then $E\uparrow\boldsymbol{R}^n$.

If $\nu(B)=\infty$, then there is nothing to prove. ///

Theorem 4.4. Let f be a Borel measurable function on $\boldsymbol{R}^n$ (i.e., $f:\boldsymbol{R}^n\longrightarrow\boldsymbol{R}$ and

$\{x \in R^n | f(x) > a\} \in \Sigma$) which is μ-integrable over bounded Borel set in R^n. Then

$$\lim_{h\to 0}\int_{Q_x(h)} \frac{1}{\mu(Q_x(h))} d\mu = f(x) \quad \text{a.e.}(\mu).$$

Proof. If $f \in L^1(d\mu)$, then there is a continuous function $g: R^n \longrightarrow R$ with compact support such that $\int_{R^n} |f-g| d\mu$ is arbitrary small by proposition 2.4.

In this case we have

$$\left|\frac{1}{\mu(Q_x(h))}\int_{Q_x(h)} f\, d\mu - f(x)\right|$$
$$\leq \frac{1}{\mu(Q_x(h))}\int_{Q_x(h)} |f-g| d\mu + \left|\frac{1}{\mu(Q_x(h))}\int_{Q_x(h)} g\, d\mu - f(x)\right|.$$

Noting that the last term on the right side converges to $|g(x)-f(x)|$ as $h \longrightarrow 0$, we have

(4—1) $$\lim_{h\to 0}\left|\frac{1}{\mu(Q_x(h))}\int_{Q_x(h)} f\, d\mu - f(x)\right|$$
$$\leq \sup_{h>0} \frac{1}{\mu(Q_x(h))}\int_{Q_x(h)} |f-g| d\mu + |f(x)-g(x)|.$$

Set

$$S_\varepsilon = \{x \in R^n \,|\, \left|\frac{1}{\mu(Q_x(h))}\int_{Q_x(h)} f\, d\mu - f(x)\right| > \varepsilon\}.$$

Then S_ε is contained in the union of the two sets where the corresponding terms on the right side of the inequality (4—1) exceed $\frac{\varepsilon}{2}$ respectively.

That is, if we put

$$S = \{x \in R^n \,|\, \sup_{h>0} \frac{\nu(Q_x(h))}{\mu(Q_x(h))} > \frac{\varepsilon}{2}\}$$

and

$$T = \{x \in R^n \,|\, |f(x)-g(x)| > \frac{\varepsilon}{2}\},$$

where

$\nu(Q_x(h)) = \int_{Q_x(h)} |f-g| d\mu$. Then $S_\varepsilon \subset S \cup T$.

It is easily proved that ν is a Bore measure which is finite on every bounded Borel set in $\boldsymbol{R}^n$.

Therefore, by Lemma 4.3

$$\mu^*(S) \leq \frac{2c}{\varepsilon} \int_{\boldsymbol{R}^n} |f-g| \, d\mu.$$

From the Tchebyshev's inequality ([12]) we also obtain

$$\mu^*(T) \leq \frac{2}{\varepsilon} \int_{\boldsymbol{R}^n} |f-g| \, d\mu.$$

Therefore $\quad \mu^*(S_\varepsilon) \longrightarrow 0 \quad$ as $\quad \varepsilon \longrightarrow 0.$

Let us assume that $f \notin L^1(d\mu)$. By hypothesis f is locally μ-integrable on $\boldsymbol{R}^n$. Thus, it is enough to show that our conclusion holds a.e. (μ) in every open ball. Fix a ball and replace f by zero outside this ball. Then this new function is μ-integrable over $\boldsymbol{R}^n$, i.e., in $L^1(d\mu)$. By the proof which we have just proved above, its integral is differentiable a.e. (μ), i.e., our assertion holds for this integral. That the differentiability is a local property tells us that the initial function f is differentiable a.e. (μ) in the ball.

That is,

$$\lim_{h \to 0} \frac{1}{\mu(Q_x(h))} \int_{Q_x(h)} f \, d\mu = f(x) \quad \text{a.e.} \ (\mu). \qquad ///$$

References

1. N. Bourbaki, Introduction to Measure Theory, van Nostrand, 1974.
2. ______, Intégraion sur les espaces topologiques séparés. Hermann, 1969.
3. I.M. Gelfand and N. Ya. Vilenkin, *Generalized Functions, Vol.4*. Academic Press, 1964.
4. H. Hahn, Set Functions, University of New Mexico Press, 1948.
5. P. Halmos, Measure Theory, van Nostrand, 1950.
6. E. Hewitt and K. Stromberg, Real and abstract Analysis, Springer, 1965.
7. E. Hille and R.S. Ppillips, Functional Analysis and Semigroups, Providence, 1957.
8. J. Von Neumann, Functional Operators, Princeton University, Press, 1950.
9. F. Riesz and B. Sz-Nagy, Leçons d'analyse Fonctionelle, Akadémiai. Kiadó, Budapest, 1968.
10. S. Saks, Theory of the Integral, Dover 1964.
11. L. Schwartz, Radon Measures on Arbitrary Topological Spaces and Cylinderical Measures, Oxford University Press, 1973.
12. R.L. Wheeden and A. Zygmund, Measure and Integral, Marcel Dekker, 1977.
13. H. Widom, Lectures on Measure and Integration, von Nostrand, 1969.
14. K. Yosida, Functional Analysis, Springer, 1974.
15. ______, Measues and Integrations, Iwanami, 1976.