Strong Homotopy Monads and Iterated Loop Spaces

by

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1. Introduction

A topological space X is called a loop space if there exists a space Y and a weak homotopy equivalence $X \longrightarrow \Omega Y$; such a space Y is called a classifying space for X. Here the symbol ΩY denotes the set of continuous basepoint preserving maps from S^1, the 1-sphere, into Y topologized with the compact-open topology.

We consider the problem of whether a given space X is an n-fold. loop space; i.e., whether there is a space Y and a weak homotopy equivalence $X \longrightarrow \Omega^n Y$. In this case Y is called an n-th classifying space for X.

In category theory there is a concept of a functor being a monad. Beck. [2] showed that if the monad $\Omega^n S^n$ acts on a space X in a certain manner, then an n-fold classifying space could be constructed. Although this theorem gives a procedure for identifying an iterated loop space, the trouble is that there are few spaces on which $\Omega^n S^n$ acts properly.

May [11] generalize this result to monads that look like $\Omega^n S^n$ (Theorem 4.9). One point that is missing in the theorem is homotopy invariance.

In this dissertation we generalize May's results up to homotopy.

In Section 2 we define the notion of a monad acting on a space up to strong homotopy.

Section 3 deals with a discussion of the homotopy invariance property.

In Section 4 May's theorem are stated and the terminology needed to discuss it is developed.

Section 5 contains the main result of this paper: given a space on which an appropriate monad acts up to strong homotopy, the space is an n-fold loop space. This

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is proved by showing that such a space is a retract of another space on which the same monad acts in the sense of May. Then we apply May's theorem to this second space to construct its n-th classifying space and thus obtain the desired classifying space for the original space.

Throughout this work whenever mention is made of a category of topological spaces, it should be taken to mean specifically the category of compactly generated spaces with base points. It will be denoted by the symbol $\mathscr{T}$.

2. Strong Homotopy Algebras over Monads.

We begin with a discussion of the concept of monad in a category. Next, we generalize this definition to an action up to strong homotopy.

Definition 2.1 A *monad* (D,ε,η) in any category $\mathscr{C}$ consists of a covariant functor D: $\mathscr{C}\longrightarrow\mathscr{C}$ together with natural transformations of functors ε: $D^2\longrightarrow D$ and η: $1\longrightarrow D$ such that the following diagrams are commutative for all objects X in $\mathscr{C}$:

$$\begin{array}{ccccc} DX & \xrightarrow{D\eta X} & D^2X & \xleftarrow{\eta DX} & DX \\ & \searrow\!\!\!\!\!\!= & \downarrow{\scriptstyle \varepsilon X} & =\!\!\!\!\!\!\swarrow & \\ & & DX & & \end{array} \qquad \begin{array}{ccc} D^3X & \xrightarrow{\varepsilon DX} & D^2X \\ \downarrow{\scriptstyle D\varepsilon X} & & \downarrow{\scriptstyle \varepsilon X} \\ D^2X & \xrightarrow{\varepsilon X} & DX \end{array}$$

Lemma 2.2 ([10] p.134) If $\mathscr{C}$ and $\mathscr{D}$ are categories and S: $\mathscr{C}\longrightarrow\mathscr{D}$, L: $\mathscr{D}\longrightarrow\mathscr{C}$ are adjoint functors, then their composite LS: $\mathscr{C}\longrightarrow\mathscr{C}$ gives a monad.

In particular if $\mathscr{C}=\mathscr{D}$ is a category of topological spaces, we may let S be the suspension functor and L be the loop functor, usually denoted by Ω. Then ΩS is a monad.

Definition 2.3 Let X be a topological space, let (D,ε,η) be a monad in $\mathscr{T}$ and let h_o: $DX\longrightarrow X$ be a continuous map in $\mathscr{T}$. The pair (X, h_o) is called a *D-space* (or D-algebra) if the following diagrams are commutative:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & DX \\ & =\!\!\!\!\!\!\searrow & \downarrow{\scriptstyle h_o} \\ & & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} D^2X & \xrightarrow{\varepsilon} & DX \\ \downarrow{\scriptstyle Dh_o} & & \downarrow{\scriptstyle h_o} \\ DX & \xrightarrow[h_o]{} & X \end{array}$$

If (X', g_*) is another D-space, then $f: X \longrightarrow X'$ is a *map of D-spaces* if the following diagram is commutative:

$$\begin{array}{ccc} DX & \xrightarrow{Df} & DX' \\ {\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ X & \xrightarrow[f]{} & X' \end{array}$$

Definition 2.4 Let $h_*: DX \longrightarrow X$ be a map in $\mathscr{T}$ and (D, ε, η) be a monad in $\mathscr{T}$. The pair $(X, \{h_q\})$ is called *a strong homotopy D-space* (s.h. D-space or s.h. D-algebra) if the homotopies $h_q: I^q \times D^{q+1}X \longrightarrow X$ satisfy the compatibility relations

$$h_q(t_1, \cdots, t_q, y) = \begin{cases} h_{q-1} \circ (1 \times D^{j-1}\varepsilon_{q-j})(t_1, \cdots, \hat{t}_j, \cdots, t_q, y) \text{ if } t_j = 0, \\ h_{j-1} \circ (1 \times D^j h_{q-j})(t_1, \cdots, \hat{t}_j, \cdots, t_q, y) \\ \stackrel{\text{def}}{=\!=} h_{j-1}(t_1, \cdots, t_{j-1}, D^j h_{q-j}(t_{j+1}, \cdots, t_q, y)) \text{ if } t_j = 1. \end{cases}$$

Here, $j = 1, \cdots, q$, $q \geq 0$, $y \in D^{q+1}X$, and $\hat{t}_j$ means delete the coordinate t_j. We also require the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & DX \\ & \searrow\!\!\!\!\searrow & \downarrow{\scriptstyle h_*} \\ & & X \end{array} .$$

The map $\varepsilon_{q-j}: D^{q-j+2}X \longrightarrow D^{q-j+1} X$ is used to denote the map $\varepsilon(D^{q-j}X)$.

In particular, a strict D-space is an s.h. D-space such that all of the higher homotopies are constant.

Remark In the definition of s.h. D-space we require that $Dh: I \times DY \longrightarrow DX$. This is not known to be true for arbitrary monads, and so we assume that our monad D has this property. Suitable monads specifically are those which are continous functors.

Definition 2.5 Let $(Y. \xi_*)$ be a strict D-space and $(X, \{h_q\})$ an s.h. D-space. A map $f: Y \longrightarrow X$ is said to be *an s.h. D-map* if there exists a collection of homotopies

$$g_n: I^n \times D^n Y \longrightarrow X$$

such that $g_0=f$ and

$$g_n(t_1,\cdots,t_n,z)=\begin{cases} g_{n-1}(t_1,\cdots,t_{n-1},D^{n-1}\xi_0(z)) & \text{if } t_n=0 \\ g_{n-1}(t_1,\cdots,\hat{t}_j,\cdots,t_n,D^{j-1}\varepsilon_{n-j-1}(z)) & \text{if } t_j=0 \\ h_{n-1}(t_1,\cdots,t_{n-1},D^n g_0(z)) & \text{if } t_n=1 \\ h_{j-1}(t_1,\cdots,t_{j-1},D^j g_{n-j}(t_{j+1},\cdots,t_n,z)) & \text{if } t_j=1. \end{cases}$$

Definition 2.6 Let $(Y,\ \xi_0)$ be a strict D-space and $(X,\ \{h_q\})$ be an s.h. D-space. A map $f\colon X\longrightarrow Y$ is said to be *an s.h. D-map* if there exists a collection of homotopies

$$g_n\colon I^n\times D^nX\longrightarrow Y$$

such that $g_0=f$ and

$$g_n(t_0,t_1,\cdots,t_{n-1},z)=\begin{cases} \xi_0\circ Dg_{n-1}(t_1,\cdots,t_{n-1},z) & \text{if } t_0=0 \\ g_{n-1}(t_0,\cdots,\hat{t}_j,\cdots,t_{n-1},D^{j-1}\varepsilon_{n-j-1}(z)) & \text{if } t_j=0 \\ g_0\circ h_{n-1}(t_1,\cdots,t_{n-1},z) & \text{if } t_0=1 \\ g_j(t_0,\cdots,t_{j-1},D^j h_{n-j-1}(t_{j+1},\cdots,t_{n-1},z)) & \text{if } t_j=1. \end{cases}$$

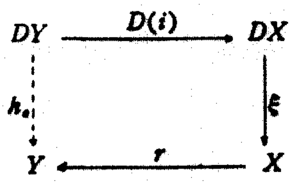
3. The Homotopy Invariant Properties

In this chapter, given the map $\eta;\ Y\longrightarrow DY$ we require the pair $(DY,\ Y)$ to have the homotopy extension property. For example, the pair $(DY,\ Y)$ has this property if it is an NDR pair in the category $\mathcal{T}$.

Theorem 3.1 Let $(X,\ \xi)$ be a strict D-space and assume that the space Y is homotopy equivalent to X. Then Y is a strong homotopy D-space.

Proof: Let $(X,\ \xi)$ be a strict D-space and assume that the homotopy equivalence between X and Y is given by the maps $i\colon Y\longrightarrow X$ and $r\colon X\longrightarrow Y$ and a homotopy $k\colon I\times X\longrightarrow X$ such that $k|_0=$identity on X and $k|_1=i\circ r$. We also have that $r\circ i$ is homotopic to the identity on Y.

Define a map $h_0\colon DY\longrightarrow Y$ by the composition



i.e., $h_o = r\circ\xi\circ D(i)$.

Note that $h_o\circ\eta$ is the identity map on Y. In fact, if $y\in Y$, then

$$\begin{aligned} h_o\circ\eta(y) &= r\circ\xi\circ D(i)\circ\eta(y) \\ &= r\circ\xi\circ\eta\circ i(y) \text{ (by the naturality of } \eta) \\ &= r\circ i(y) \text{ (since } \xi\circ\eta = id). \end{aligned}$$

But $r\circ i$ is homotopic to the identity on Y and if the pair (DY, Y) has the homotopy extension property, we have that $r\circ i\ (y) = y$.

We now need to construct a homotopy $h_1 : I\times D^2Y \longrightarrow Y$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} D^2Y & \xrightarrow{\ \varepsilon\ } & DY \\ {\scriptstyle Dh_o}\downarrow & & \downarrow{\scriptstyle h_o} \\ DY & \xrightarrow{\ h_o\ } & Y \end{array} .$$

Define

$$h_1 = r\circ\xi\circ D(k)\circ D(\xi)\circ D^2(i)$$

where $D(k) : I\times DX \longrightarrow DX$ is defined in the manner described earlier.

To see that h_1 is the correct homotopy, note that

$$\begin{aligned} h_1|_o &= r\circ\xi\circ D(k)|_o\circ D(\xi)\circ D^2(i) \\ &= r\circ\xi\circ D(\xi)\circ D^2(i) \text{ (since } D(k)|_o = id) \\ &= r\circ\xi\circ\varepsilon\circ D^2(i) \text{ (since } X \text{ is strict D-space)} \\ &= r\circ\xi\circ D(i)\circ\varepsilon \text{ (since } \varepsilon \text{ is natural)} \\ &= h_o\circ\varepsilon \end{aligned}$$

and that

$$\begin{aligned} h_1|_1 &= r\circ\xi\circ D(k)|_1\circ D(\xi)\circ D^2(i) \\ &= (r\circ\xi\circ D(i))\circ(D(r)\circ D(\xi)\circ D^2(i)) \\ &= h_o\circ Dh_o. \end{aligned}$$

This motivation for the s.h. D-structure on Y leads us to define the requisite higher homotopies $h_q : I^q\times D^{q+1}Y \longrightarrow Y$ by $h_q = r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^j(k)\circ D^j(\xi)\circ\cdots\circ D^q(k)\circ D^q(\xi)$

$\circ D^{q+1}(i)$. Again, the symbol $D^j(k)$ should be interpreted as the map $I\times D^jX\longrightarrow D^jX$. We have also omitted the symbol "$1X$" which should preface each $D^j(k)$. Note also that we have a composition of q one-dimensional homotopies; we use each in succession on each of the q-coordinates in I^q.

For the h^q to determine a valid s.h. D-structure on Y, we have to verify the usual compatibility relations. If, for example $t_j=0$, we have

$$\begin{aligned}h_q&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi)\circ D^j(k)|_0\circ D^j(\xi)\circ\cdots\circ D^q(k)\circ D^q(\xi)\circ D^{q+1}(i)\\&=r\circ\xi\circ\cdots\circ D^{j-1}(\xi)\circ D^j(\xi)\circ D^{j+1}(k)\circ\cdots\circ D^q(k)\circ D^q(\xi)\circ D^{q+1}(i)\\&=r\circ\xi\circ\cdots\circ D^{j-1}(\xi\circ D\xi)\circ D^{j+1}(k)\circ\cdots\circ D^q(k)\circ D^q(\xi)\circ D^{q+1}(i)\\&=r\circ\xi\circ\cdots\circ D^{j-1}(\xi\circ\varepsilon)\circ D^{j+1}(k)\circ\cdots\circ D^q(k)\circ D^q(\xi)\circ D^{q+1}(i)\\&=r\circ\xi\circ\cdots\circ D^{j-1}(\xi)\circ D^j(k)\circ\cdots\circ D^{q-1}(k)\circ D^{q-1}(\xi)\circ D^q(i)\circ D^{j-1}\varepsilon_{q-j}\\&=h_{q-1}\circ D^{j-1}\varepsilon_{q-j}.\end{aligned}$$

The equality just before the last follows from the repeated application of the naturality of ε.

If, on the other hand, $t_j=1$, then

$$\begin{aligned}h_q&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi)\circ D^j(k)|_1\circ D^j(\xi)\circ\cdots\circ D^q(k)\circ D^q(\xi)\circ D^{q+1}(i)\\&=(r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi)\circ D^j(i))\circ(D^j(r)\circ D^j(\xi)\circ\cdots\circ D^q(k)\circ D^q(\xi)\circ D^{q+1}(i))\\&=h_j\circ D^jh_{q-j}.\end{aligned}$$

Thus we have proved that the $\{h_q\}$ as defined determine an s.h. D-structure for Y.///

Proposition 3.2 Let (X,ξ) be a strict D-space and assume that Y is homotopy equivalent to X. Then the maps defining this equivalence are s.h. D-maps.

Proof: We continue to use the notation from the previous theorem and first show that $r: X\longrightarrow Y$ is an s.h. D-map. We have to construct a family of homotopies g_n: $I^n\times D^nX\longrightarrow Y$ which satisfy the compatibility conditions of Definition 2.4.
Let us define $g_0=r$ and

$$g_n=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi)\circ D^j(k)\circ\cdots\circ D^{n-1}(\xi)\circ D^n(k)$$

for $n>0$. As before, $D^j(k)$ is the map $I\times D^jX\longrightarrow D^jX$.

To verify the compatibility equations, we let $t=0$. Then,

$$g_n=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(k)|_0$$

$$=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ id$$
$$=g_{n-1}\circ D^{n-1}(\xi).$$

If $t_i=0$, then

$$\begin{aligned}g_n&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi)\circ D^j(k)|_0\circ D^j(\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(k)\\&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi)\circ D^j(\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(k)\\&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi\circ D\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(k)\\&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi\circ\varepsilon)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(k)\\&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi)\circ D^{j-1}\varepsilon\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(k)\\&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi)\circ D^j(k)\circ\cdots\circ D^{n-2}(k)\circ D^{n-2}(\xi)\circ D^{n-1}(k)\circ D^{j-1}\varepsilon_{n-j-1}\\&=g_{n-1}\circ D^{n-1}\varepsilon_{n-j-1}\end{aligned}$$

If $t_n=1$,

$$\begin{aligned}g_n&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(k)|_1\\&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(i)\circ D^n(r)\\&=h_{n-1}\circ D^n(g_o).\end{aligned}$$

Finally, if $t_j=1$, then we have

$$\begin{aligned}g_n&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi)\circ D^j(k)|_1\circ D^j(\xi)\circ\cdots\circ D^{n-1}(\xi)\circ D^n(k)\\&=r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi)D^j(i)\circ D^j(r)\circ D^j(\xi)\circ\cdots\circ D^{n-1}(\xi)\circ D^n(k)\\&=(r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(\xi)\circ D^j(i))\circ(D^j(r)\circ D^j(\xi)\circ\cdots\circ D^{n-1}(\xi)\circ D^n(k))\\&=h_{j-1}\circ D^j g_{n-j},\end{aligned}$$

which shows that r is an s.h. D-map.

To see that $i: Y\longrightarrow X$ is an *s.h.* D-map we define

$$g_n: I^n\times D^nY\longrightarrow X$$

by $\quad g_n=k\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(i).$

To verify the compatibility conditions of Defnition 2.5, note that if $t_0=0$,

$$\begin{aligned}g_h&=k|_0\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(i)\\&=\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(i)\\&=\xi\circ Dg_{n-1}.\end{aligned}$$

If $t_0=1$, $g_h=k|_1\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(i)$

$$\begin{aligned}&=i\circ r\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{n-1}(k)\circ D^{n-1}(\xi)\circ D^n(i)\\&=i\circ h_{n-1}.\end{aligned}$$

When $t_j=0$,

$$\begin{aligned} g_n &= k\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(k)\circ D^{j-1}(\xi)\circ D^j(k)|_0\circ D^j(\xi)\circ\cdots\circ D^n(i) \\ &= k\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(k)\circ D^{j-1}(\xi)\circ D^j(\xi)\circ D^{j+1}(k)\circ\cdots\circ D^n(i) \\ &= k\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(k)\circ D^{j-1}(\xi)\circ D^{j-1}(\varepsilon)\circ D^{j+1}(k)\circ\cdots\circ D^n(i) \\ &= g_{n-1}\circ D^{j-1}\varepsilon_{n-j-1}. \end{aligned}$$

Finally, if $t_j=1$,

$$\begin{aligned} g_n &= k\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(k)\circ D^{j-1}(\xi)\circ D^j(k)|_1\circ D^j(\xi)\circ\cdots\circ D^n(i) \\ &= (k\circ\xi\circ D(k)\circ D(\xi)\circ\cdots\circ D^{j-1}(k)\circ D^{j-1}(\xi)\circ D^j(i))\circ(D^j(r)\circ D^j(\xi)\circ\cdots\circ D^n(i)) \\ &= g_j\circ D^j h_{n-j-1}. /// \end{aligned}$$

4. Summary of May's Results

The definitions and statements of May's theorem in this section will be used in the proofs of our later theorems. Most theorems and propositions will be stated here without proofs; the reader may refer to May [11] for details.

It was mentioned earlier that the functor $\Omega^n S^n$ is a monad. Beck [2] proved

Theorem 4.1 An $\Omega^n S^n$-space is of the weak homotopy type of an n-fold loop space.

Definition 4.2 *An operad* $\mathscr{D}$ consists of a family of topological spaces $\mathscr{D}(j)$, $j\geq 0$, such that $\mathscr{D}(0)=*$, a point, and

1) continuous functions γ: $\mathscr{D}(k)\times\mathscr{D}(j_1)\times\cdots\times\mathscr{D}(j_k)\longrightarrow\mathscr{D}(j)$, $j=\sum_{i=1}^{k} j_i$, that satisfy an associativity relation
2) an identity element $1\in\mathscr{D}(1)$ such that $\gamma(1;d)=d$ for $d\in\mathscr{D}(j)$ and $\gamma(c;1^k)=c$ for $c\in\mathscr{D}(k)$
3) a right action of the symmetric group Σ_j on $\mathscr{D}(j)$ such that for all $c\in\mathscr{D}(k)$, $d_s\in\mathscr{D}(j_s)$, $\sigma\in\Sigma_k$ and $t_s\in\Sigma_{j_s}$, we have

$$\gamma(c\sigma;d_1,\cdots,d_k)=\gamma(c;d_{\sigma^{-1}(1)},\cdots,d_{\sigma^{-1}(k)})\sigma(j_1,\cdots,j_k)$$

and

$$\gamma(c;d_1t_1,\cdots,d_kt_k)=\gamma(c;d_1,\cdots,d_k)\ (t_1\oplus\cdots\oplus t_k),$$

where $\sigma(j_1,\cdots,j_k)$ denotes that permutation of j letters which permutes the k blocks of letters determined by the given partition of j as σ permutes k letters, and $t_1\oplus\cdots\oplus t_k$ denotes the image of $(t_1,\cdots,t_k)$ under the evident inclusion of $\Sigma_{j1}\times\cdots\times\Sigma_{jk}$ in Σ_j.

We will refer to these maps γ as "multiplication" on $\mathscr{D}$.

Definition 4.3 *A morphism of operads* $\theta: \mathscr{C} \longrightarrow \mathscr{D}$ is a collection of Σ_j equivariant maps $\theta_j: \mathscr{C}(j) \longrightarrow \mathscr{D}(j)$, $j \geq 0$ respecting the multiplication on $\mathscr{C}$ and $\mathscr{D}$.

Definition 4.4 An operad $\mathscr{D}$ is said to be Σ-*free* if Σ_j acts freely on $\mathscr{D}(j)$ for each j.

A fundamental example of an operad is the following:

Let X be any topological space with a base point. Define an operad $\mathscr{E}_X$ by letting $\mathscr{E}_X(j)$, the space of base point preserving maps $X^j \longrightarrow X$. Define $X^o = *$, a point; then $\mathscr{E}_X(o)$ is the inclusion $* \longrightarrow X$.

We can choose $1 \in \mathscr{E}_X(1)$ to be the identity map $X \longrightarrow X$.

Define $\gamma: \mathscr{E}_X(k) \times \mathscr{E}_X(j_1) \times \cdots \times \mathscr{E}_X(j_k) \longrightarrow \mathscr{E}_X(j)$ by $\gamma(g; f_1, \cdots, f_k) = g \circ (f_1 \times \cdots \times f_k)$. This operad is called the endomorphism operad of X.

Construction 4.5 Let $\mathscr{D}$ be an opered. Construct the monad associated to $\mathscr{D}$ by defining

$$DX = \coprod_j \mathscr{D}(j) \times X^j / \sim$$

where $\sim$ is determined by Σ_j actions and by degeneracies. Precisely, we have

1) $(d\sigma, y) \sim (d, \sigma y)$ where $\sigma \in \Sigma_j$, $y \in X^j$, $d \in \mathscr{D}(j)$, and $\sigma y = \sigma(x_1, \cdots, x_j)$ is defined by $(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(j)})$ and

2) $(w_i d, y) \sim (d, s_i y)$ where $d \in \mathscr{D}(j)$, $y \in X^{j-1}$ and maps $s_i: X^{j-1} \longrightarrow X^j$ are defined by $s_i(x_1, \cdots, x_{j-1}) = (x_1, \cdots, x_i, *, x_{i+1}, \cdots, x_{j-1})$ and maps $w_i: \mathscr{D}(j) \longrightarrow \mathscr{D}(j-1)$ are defined by $w_i d = \gamma(d; v_i)$ where $v_i = 1^i \times * \times 1^{j-i-1} \in \mathscr{D}(1)^i \times \mathscr{D}(o) \times \mathscr{D}(1)^{j-i-1}$.
   Here we let $0 \leq i < j$.

The point is that the multiplication γ in the operad structure induces the natural transformation $D^2 \longrightarrow D$ and the inclusion $X \longrightarrow \mathscr{D}(1) \times X$ given by $x \longrightarrow (1, x)$ induces the natural transformation $1 \longrightarrow D$.

Another important example of an operad is the little cube operad $\mathscr{C}_n$. Let I^n denote the unit n-cube and J^n its interior. A little n-cube is a linear embedding $f: J^n \longrightarrow J^n$ with parallel axes. Define $\mathscr{C}_n(j)$ to be the set of j-tuples $\langle c_1, \cdots, c_j \rangle$ of little n-cubes where the images of the c_r are pairwise disjoint. Topologize $\mathscr{C}_n(j)$ as a subspace of

$J^n(\amalg J^n)$.

Let $\mathscr{C}_n(0)$: $\phi \longrightarrow J^n$ be the inclusion. Define γ by

$$\gamma(c: d_1, \cdots, d_k) = c \circ (d_1 \amalg \cdots \amalg d_k): (\coprod_{j_1} J^n) \amalg \cdots \amalg (\coprod_{j_k} J^n) \longrightarrow J^n$$

and let $1 \in \mathscr{C}_n(1)$ be the identity map.

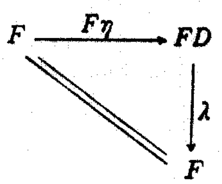
There is also a natural morphism of operads σ_n: $\mathscr{C}_n \longrightarrow \mathscr{C}_{n+1}$ defined by

$$\sigma_{n,j}: \langle c_1, \cdots, c_j \rangle \longrightarrow \langle c_1 \times 1, \cdots, c_j \times 1 \rangle$$

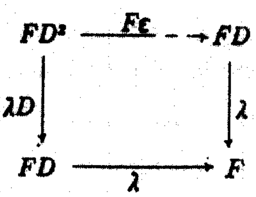
Where $1: J \longrightarrow J$.

Let us denote the monad that arise from $\mathscr{C}_n$ by C_n.

Definition 4.6 Let (D, ε, η) be a monad in $\mathscr{T}$.
A *D-functor* (F, λ) in a category ν is a functor
F: $\mathscr{T} \longrightarrow \nu$ together with a natural transformafion
λ: $FD \longrightarrow F$ such that the diagrams

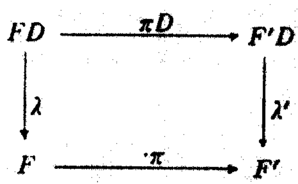


and



are commutative.

A morphism of D-functors π: $(F, \lambda) \longrightarrow (F', \lambda')$ is a natural transformation π: $F \longrightarrow F'$ such that



commutes.

Lemma 4.7 ([11], p.87)

(1) If α: $(D, \varepsilon)\longrightarrow(D', \varepsilon')$ is a morphism of monads and $(X, h_{\circ})$ is a D′-space, then $(X, h_{\circ}\circ\alpha)$ is a D-space.

(2) If (F, λ) is a D′-functor, then $(F, \lambda\circ F\alpha)$ is a D-functor.

Construction 4.8 Let (D, ε, η) be a monad, $(X, h_{\circ})$ a D-space, and (F, λ) a D-functor. Define a simplicial space $B_*(F,D,X)$ by

$$B_q(F,D,X)=FD^qX$$

with face and degeneracy operators given by

$$\begin{array}{lll} \partial_0=\lambda & \lambda:\ FD^qX\longrightarrow FD^{q-1}X & \\ \partial_i=FD^{i-1}\varepsilon & \varepsilon:\ D^{q-i+1}X\longrightarrow D^{q-i}X & 0<i<q \\ \partial_q=FD^{q-1}h_0 & h_0:\ DX\longrightarrow X & \\ s_i=FD^i\eta & \eta:\ D^{q-i}X\longrightarrow D^{q-i+1}X & 0\leq i\leq q. \end{array}$$

A morphism $B_*(\pi,\alpha,f)$: $B_*(F,D,X)\longrightarrow B_*(F',D',X')$ is induced by a morphism of monads α: $D\longrightarrow D'$, a morphism of D-functors π: $(F,\lambda)\longrightarrow(F', \lambda'\circ F'\alpha)$ and a morphism of D-spaces f: $(X,h_0)\longrightarrow(X',h_0'\circ\alpha)$.

Using the above notation, let us define $B(F,D,X)$ to be the geometric realization of $B_*(F,D,X)$; i.e.,

$$B(F,D,X)=\coprod_q \triangle_q\times FD^qX/\sim$$

where $\triangle_q$ is the standard q-simplex and $\sim$ is the usual realization relation.

Theorem 4.9 (May) Let π: $D\longrightarrow C_n$ be a morphism of monads that results from a morphism π: $\mathscr{D}\longrightarrow\mathscr{C}_n$ of Σ-free operads having the property that π_j: $\mathscr{D}(j)\longrightarrow\mathscr{C}_n(j)$ is a homotopy equivalence for each j. Let X be connected and let (X,h_0) be a D-space. Then we have morphisms of D-spaces

$$X\xleftarrow{H}B(D,D,X)\xrightarrow{B(\alpha_n\circ\pi,1,1)}B(\Omega^nS^n,D,X)\xrightarrow{\gamma^n}\Omega^nB(S^n,D,X)$$

such that

1) H is a strong deformation retraction (H is induced by the map h_0: $DX\longrightarrow X$)

2) B $(\alpha_n \circ \pi,\ 1,1)$ and γ^n are weak homotopy equivalences. In other words, X is of the same weak homotopy type as $\Omega^n B(S^n, D, X)$ as a D-space via D-maps.

5. Main Theorems

Let (D, ε, η) be a monad and (F, λ) a D-functor. Let us assume that F is a continuous functor; this means that given any two objects X, Y in $\mathcal{T}$, the set of morphisms $X \longrightarrow Y$, denoted $\text{Mor}(X, Y)$ forms a topological space and that F: $\text{Mor}(X,Y) \longrightarrow \text{Mor}(FX,\ FY)$ is a continuous map. The requirement that $\text{Mor}(X,Y)$ be a topological space is satisfied in our category of compactly generated spaces. In category theory there is a more general concept of a topological category; such categories also enjoy the property that Mor (X,Y) is a topological space.

The continuity of F will be crucial in the construction to be considered. Given a homotopy h: $I \times X \longrightarrow Y$ and a functor F, we have the natural map Fh: $F(I \times X) \rightarrow FY$. We would like to have a similarly induced map $I \times FX \longrightarrow FY$; i.e., we would like to apply F to a homotopy and get back another homotopy. Our original homotopy gives us a collection of maps h_t: $X \longrightarrow Y$ for all $0 \leq t \leq 1$. Applying F, we obtain another family of maps Fh_t: $FX \longrightarrow FY$ for all t. For F to be continuous means that these Fh_t "fit together" continuously to yield a map $I \times FX \longrightarrow FY$.

From an adjoint point of view we can think of our original homotopy h: $I \times X \longrightarrow Y$ as a continuous map $I \longrightarrow Y^X$. The continuity of F by defintion means that F: $Y^X \longrightarrow FY^{FX}$ is continuous. Thus the composition of the homotopy with the continuous functor gives us the desired map $I \longrightarrow Y^X \longrightarrow FY^{FX}$.

We will use the symbol Fh for the map $I \times FX \longrightarrow FY$ as well as for the usual map $F(I \times X) \longrightarrow FY$ and hope that the context will be clear enough to avoid confusion.

In the category of CW complexes, a pair (Y, A) is said to be retractile if the homology exact sequence reduces to $O \longrightarrow H(A) \longrightarrow H(Y) \longrightarrow H(Y, A) \longrightarrow O$ [8]. It is not required that this sequence split. Retractile pairs have not only the homotopy extension property, but also in some sence a relative homotopy extension property. Stasheff [12. p. 291] showed

Lemma 5.1 Let (X, m) be an H-space. If (Y, A) is a retractile pair, given homotopic maps f_0, f_1: $Y \longrightarrow X$ and a homotopy g_t: $A \longrightarrow X$ such that $g_i = f_i | A$ for $i = 0, 1$, then g_t extends to a homotopy f_t: $Y \longrightarrow X$.

Construction 5.2 Define a topological space that depends upon a monad (D,ε,η), a strong homotopy D-space $(X, \{h_q\})$, and a continuous D-functor (F,λ) by

$$\tilde{B}\ (F,D,X)=\coprod_q I^q\times FD^qX/\sim$$

where the equivalence relation $\sim$ is defined by

$$(t_1,\cdots,t_q,x)\sim\begin{cases}(t_2,\cdots,t_q,\lambda(x))\in I^{q-1}\times FD^{q-1}X \text{ if } t_1=0\\ (t_1,\cdots,\hat{t}_j,\cdots,t_q,FD^{j-2}\varepsilon_{q-j+1}(x))\in I^{q-1}\times FD^{q-1}X \text{ if } t_j=0\\ (t_1,\cdots,t_{j-1},FD^{j-1}h_{q-j},(t_{j+1},\cdots,t_q,x))\in I^{j-1}\times FD^{j-1}X \text{ if } t_j=1\end{cases}$$

where $x\in FD^qX$.

The primary example of such a space is given by taking the D-functor to be $(D^n,D^{n-1}\varepsilon)$.

Theorem 5.3 Let (DY,Y) be retractile, D come from an operad, and consider the D-functor (D,ε). Then $\tilde{B}(D,D,X)$ is a strict D-space.

Proof: We claim that $D\tilde{B}(D,D,X)$ is homeomorphic to $\tilde{B}(D^2,D,X)$. To verify this, we need the existence of a Σ_j-equivarient, 1-1, onto map

$$\mathscr{D}(j)\times(\coprod_q I^q\times D^{q+1}X/\sim)^j\longrightarrow\coprod_n I^n\times(\mathscr{D}(j)\times(D^{n+1}X)^j)/\sim$$

for all j. May proved such a theorem and exhibited such a map for the strict D case. This map is essentially defined by using the concept of simplicial subdivision. The only difference between our required map and May's map is that his is defined on simplices whereas ours must be defined on cubes. However, if one looks closely at the definition of $\tilde{B}(D,D,X)$ and thinks of cubes as "thickened" simplices, the identifications in $\tilde{B}(D,D,X)$ collapse these extra faces. Although these faces are not collapsed to points in general, they have lost their parameters from the cube.

We thus describe the required map as follows: to map

$$\mathscr{D}(j)\times I^{q_1}\times D^{q_1+1}X\times\cdots\times I^{q_j}\times D^{q_j+1}X\longrightarrow I^{\Sigma q_i}\times\mathscr{D}(j)\times(D^{\Sigma q_i+1}X)^j,$$

first calculate the map for simplices. Here, it simplifies calculations if we define

$$\triangle_n\subset R^n \text{ to be } \{(t_1,\cdots,t_n)\in R^n \text{ such that } 0\leq t_i\leq 1 \text{ and } t_1\leq\cdots\leq t_n\}.$$

It is necessary to subdivide the product $\triangle_{q_1}\times\cdots\times\triangle_{q_j}$ to define the map. Then "thicken" the appropriate faces of each $\triangle_{q_i}$ and obtain the cube $I^{\Sigma q_i}$ which contains $\triangle_{q_1}\times\cdots\times\triangle_{q_j}$. Now subdivide the cube $I^{\Sigma q_i}$ in exactly the same manner that $\triangle_{q_1}\times\cdots\times\triangle_{q_j}$ was subdivided and use exactly the same degeneracy maps on

$$D^{q_1+1}X\times\cdots\times D^{q_j+1}X\longrightarrow(D^{\Sigma q_i+1}X)^j$$

that are used in the simplicial case.

The equivalence relation in $\tilde{B}(D^2,D,X)$ will guarantee that our map is well-defined and continuous if we require our higher homotopies be relative homotopies with respect to the subspaces of D^qX given by the various $\eta's$: $D^{q-1}X\longrightarrow D^qX$.

Noting that each D^qX is an H-space and recalling the earlier comments about our category, we utilize Lemma 5.1 to insure that our higher homotopies behave properly on subspaces. ///

Example 5.4 Let the map

$$\mathscr{D}(2)\times I\times D^2X\times I\times D^2X\longrightarrow I^2\times\mathscr{D}(2)\times D^3X\times D^3X$$

be given by

$$(d,s,x,t,y)\longrightarrow\begin{cases}(s,\ \frac{t-s}{1-s},\ d,D^2\eta(x),D\eta D(y)) & \text{if } s\leq t\\(t,\ \frac{s-t}{1-t},\ d,D\eta D(x),D^2\eta(y)) & \text{if } s\geq t.\end{cases}$$

It is clear that the coordinates of the cube in the range are just those of the thickened simplex; but for this thickening, the map is the same as in the strict D case.

The image of the point $(d,1,x,1,y)$ is $(1,\ \frac{0}{0},\ d,\ D^2\eta(x),\ D\eta D(y))$. Since this point is equivalent to $(d,\ Dh_t(\frac{0}{0},\ D^2\eta(x)),\ Dh_t(\frac{0}{0},\ D\eta D(y)))$, $\frac{0}{0}$ may be taken to be any $0\leq t\leq 1$ if

$$Dh_t|D^2\eta(D^2X)=Dh_t|D\eta D(D^2X)=D^2h_0$$

for all t; this last equality always holds for $t=0,1$ and the assumption that (DY,Y) is retractile allows us to alter Dh_t for $0<t<1$ so that the equality holds for all t.

This guarantees that the map is well-defined and continuous.

Proposition 5.5 Let $(X,\ \{h_t\})$ be a strong homotopy D-space. Then X is a deformation retract of $\tilde{B}(D,D,X)$.

Proof: Define a map i: $X \longrightarrow \tilde{B}(D,D,X)$ by $i(x)=\eta(x) \in I^{\circ} \times DX \subset \tilde{B}(D,D,X)$. Now define a map r: $\tilde{B}(D,D,X) \longrightarrow X$ by

$$r(t_1,\cdots,t_q,y)=h_q(t_1,\cdots,t_q,y).$$

To see that r is well-defined, suppose that $t_j=0$.
Then

$$\begin{aligned} r(t_1,\cdots,t_q,y) &= h_q(t_1,\cdots,t_j,\cdots,t_q,y) \\ &= h_{q-1}(t_1,\cdots,\hat{t}_j,\cdots,t_q,D^{j-1}\varepsilon_{q-j}(y)) \end{aligned}$$

by the properties of h_q.

But

$$(t_1,\cdots,t_q,y) \sim (t_1,\cdots,\hat{t}_j,\cdots,t_q,D^{j-1}\varepsilon_{q-j}(y))$$

and

$$r(t_1,\cdots,\hat{t}_j,\cdots,t_q,D^{j-1}\varepsilon_{q-j}(y))=h_{q-1}(t_1,\cdots,\hat{t}_j,\cdots,t_q,D^{j-1}\varepsilon_{q-j}(y))$$

again.

Now suppose that $t_j=1$. Then

$$\begin{aligned} r(t_1,\cdots,t_q,y) &= h_q(t_1,\cdots,t_q,y) \\ &= h_{j-1}(t_1,\cdots,t_{j-1},D^j h_{q-j}(t_{j+1},\cdots,t_q,y)). \end{aligned}$$

On the other hand, since

$$\begin{aligned} (t_1,\cdots,t_q,y) &\sim (t_1,\cdots,t_{j-1},D^j h_{q-j}(t_{j+1},\cdots,t_q,y)) \\ &\in I^{j-1} \times D^j X, \end{aligned}$$

we have

$$\begin{aligned} & r(t_1,\cdots,t_{j-1},D^j h_{q-j}(t_{j+1},\cdots,t_q,y)) \\ &= h_{j-1}(t_1,\cdots,t_{j-1},D^j h_{q-j}(t_{j+1},\cdots,t_q,y)) \end{aligned}$$

again. Thus r is indeed well-defined. We also have that $r \circ i = id_X$. Since

$$r \circ i(x)=r(\eta(x))=h_o(\eta(x))=x.$$

To show that $i \circ r$ is homotopic to the identity of $\tilde{B}(D,D,X)$, define a homotopy

$$F: I\times\tilde{B}(D,D,X)\longrightarrow\tilde{B}(D,D,X)$$

by

$$F(s,t_1,\cdots,t_q,y)=(s,t_1,\cdots,t_q,\eta_{q+1}(y))\in I^{q+1}\times D^{q+2}X$$

where $(s,t_1,\cdots,t_q,y)\in I\times I^q\times D^{q+1}X$ and $\eta_{q+1}=\eta D^{q+1}$.

To see that F is well-defined, first let $t_j=0$. Then

$$(s,t_1,\cdots,t_q,y)\sim(s,t_1,\cdots,\hat{t}_j,\cdots,t_q,D^{j-1}\varepsilon_{q-j}(y))$$

and

*) $$F(s,t_1,\cdots,t_q,y)=(s,t_1,\cdots,t_q,\eta_{q+1}(y))$$
$$\sim(s,t_1,\cdots,\hat{t}_j,\cdots,t_q,D^j\varepsilon_{q-j+1}\circ\eta_{q+1}(y)).$$

Also,

**) $$F(s,t_1,\cdots,\hat{t}_j,\cdots,t_q,D^{j-1}\varepsilon_{q-j}(y))$$
$$=(s,t_1,\cdots,\hat{t}_j,\cdots,t_q,\eta_q D^{j-1}\varepsilon_{q-j}(y)).$$

The equality of the two points on the right hand side of *) and **) follows from the equality $D^j\varepsilon_{q-j+1}\circ\eta_{q+1}=\eta_q D^{j-1}\varepsilon_{q-j}$ which is a consequence of the naturality of η.
On the other hand, if $t_j=1$,

$$(s,t_1,\cdots,t_q,y)\sim(s,t_1,\cdots,t_{j-1},D^j h_{q-j}(t_{j+1},\cdots,t_q,y))$$

and we have

*) $$F(s,t_1,\cdots,t_q,y)=(s,t_1,\cdots,t_q,\eta_{q+1}(y))$$
$$\sim(s,t_1,\cdots,t_{j-1},D^{j+1}h_{q-j}(t_{j+1},\cdots,t_q,\eta_{q+1}(y))$$

and

**) $$F(s,t_1,\cdots,t_{j-1},D^j h_{q-j}(t_{j+1},\cdots,t_q,y))$$
$$=(s,t_1,\cdots,t_{j-1},\eta_{j+1}\circ D^j h_{q-j}(t_{j+1},\cdots,t_q,y)).$$

Again, equality of the right hand sides of *) and **) follows from the naturality of η.
Thus F is well-defined.

When $s=0$, we have

$$F(o,t_1,\cdots,t_q,y)=(o,t_1,\cdots,t_q,\eta_{q+1}(y))$$
$$\sim(t_1,\cdots,t_q,\varepsilon_{q+1}\eta_{q+1}(y))$$
$$=(t_1,\cdots,t_q,y).$$

Thus $F|\{o\}\times\tilde{B}(D,D,X)=$ identity on $\tilde{B}(D,D,X)$.

When $s=1$, we have

$$\begin{aligned}F(1,t_1,\cdots,t_q,y)&=(1,t_1,\cdots,t_q,\eta_{q+1}(y))\\&\sim Dh_q(t_1,\cdots,t_q,\eta_{q+1}(y))\\&=\eta_o\circ h_q(t_1,\cdots,t_q,y)\\&=i\circ r(t_1,\cdots,t_q,y).\end{aligned}$$

Thus $F|\{1\}\times\tilde{B}(D,D,X)=i\circ r$ and we are done. ///

Proposition 5.6 r: $(\tilde{B}(D,D,X),\varepsilon)\longrightarrow(X,\ \{h_q\})$ is a strong homotopy D-map.

Proof: According to Definition 2.5 we have to find a collection of homotopies g_n: $I^n\times D^n\tilde{B}(D,D,X)\longrightarrow X$ that satisfy the requisite compatibility relations.

Let $g_o=r$. Define

$$g_n:\ I^n\times D^n\tilde{B}(D,D,X)=I^n\times\tilde{B}(D^n,D,X)\longrightarrow X$$

by

$$g_n(t_1,\cdots,t_n,z)=h_{q+n}(t_1,\cdots,t_n,s_1,\cdots,s_q,y)$$

where $z=(s_1,\cdots,s_q,y)\in I^q\times D^{q+n+1}X$.

To verify the compatibility requirements let us first suppose that $t_n=0$. Then

$$\begin{aligned}g_n(t_1,\cdots,0,z)&=h_{q+n}(t_1,\cdots,t_{n-1},0,s_1,\cdots,s_q,y)\\&=h_{q+n-1}(t_1,\cdots,t_{n-1},s_1,\cdots,s_q,D^{n-1}\varepsilon_q(z))\\&=g_{n-1}(t_1,\cdots,t_{n-1},\varepsilon_q(z)).\end{aligned}$$

If $t_j=0$, we have

$$\begin{aligned}g_n(t_1,\cdots,0,\cdots,t_n,z)&=h_{q+n}(t_1,\cdots,0,\cdots,t_n,s_1,\cdots,s_q,y)\\&=h_{q+n-1}(t_1,\cdots,\hat{t}_j,\cdots,t_n,s_1,\cdots,s_q,D^{j-1}\varepsilon_{q+n-j}(y))\\&=h_{q+n-1}(t_1,\cdots,\hat{t}_j,\cdots,t_n,s_1,\cdots,s_q,D^{j-1}\varepsilon_{n-j-1}D^{q+1}(y))\\&=g_{n-1}(t_1,\cdots,\hat{t}_j,\cdots,t_n,s_1,\cdots,s_q,D^{j-1}\varepsilon_{n-j-1}(z)).\end{aligned}$$

On the other hand, if $t_n=1$ we have

$$\begin{aligned}g_n(t_1,\cdots,1,z)&=h_{q+n}(t_1,\cdots,1,s_1,\cdots,s_q,y)\\&=h_{n-1}(t_1,\cdots,t_{n-1},D^nh_q(s_1,\cdots,s_q,y))\\&=h_{n-1}(t_1,\cdots,t_{n-1},D^ng_o(z)).\end{aligned}$$

Finally, if $t_j=1$ we have

$$g_n(t_1,\cdots,t_n,z)=h_{q+n}(t_1,\cdots,1,\cdots,t_n,s_1,\cdots,s_q,y)$$
$$=h_{j-1}(t_1,\cdots,t_{j-1},D^j h_{q-j+n}(t_{j+1},\cdots,t_n,s_1,\cdots,s_q,y))$$
$$=h_{j-1}(t_1,\cdots,t_{j-1},D^j g_{n-j}(t_{j+1},\cdots,t_n,z)).$$

One must check that g_n is well-defined; however, this is a straightforward consequence of the fact that $\tilde{B}(D^n,D,X)$ is constructed by means of the identities in the definition of s.h. D-space and the proof is similar to the one given in the previous proposition. ///

Theorem 5.7 Let π: $D\longrightarrow C_n$ be a morphism of monads that results from a morphism π: $\mathscr{D}\longrightarrow\mathscr{C}_n$ of Σ-free operads having the property that π_j; $\mathscr{D}(j)\longrightarrow\mathscr{C}_n(j)$ is a homotopy equivalence for each j. Let $(X, \{h_q\})$be an $s.h.$ D-space where X is connected. Then X has the weak homotopy type of an n-fold loop space via $s.h.$ D-maps.

Proof: By Theorem 5.3, $\tilde{B}(D,D,X)$ is a strict D-space. Thus by Theorem 4.9 (May) we have the weak homotopy equivalences

$$\begin{array}{ccccc}\tilde{B}(D,D,X) & \xleftarrow{E} & B(D,D,\tilde{B}(D,D,X)) & \xrightarrow{B(\alpha_n\pi,1,1,)} & B(\Omega^nS^n,D,\tilde{B}(D,D,X)) \\ & & & & \downarrow r^n \\ & & & & \Omega^nB(S^n,D,\tilde{B}(D,D,X))\end{array}$$

which are all $s.h.$ D-maps. By Proposition 5.5 we have the retraction r: $\tilde{B}(D,D,X)\longrightarrow X$ which is an $s.h.$ D-map by Proposition 5.6. Consequently X has the weak homotopy type of $\Omega^nB(S^n,D,\tilde{B}(D,D,X))$ via strong homotopy D-maps. ///

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