Unitary Representations of G and $*$-Representation of $L^1(G)$

by

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1. Introduction

The operator theory has given rise to C^*-algebra and the research of C^*-algebra has been arrived to a great degree([3], [14]). In recent years, for a Hilbert space H, since $\mathscr{L}(H)$ is a C^*-algebra, a rapid progress has been made in the research of algebra homomorphism from each Banach algebra to $\mathscr{L}(H)$([4], [13], [17]). Moreover, since for a locally compact Hausdorff topological group G, $L^1(G)$ is a $*$-algebra and $L^2(G)$ is a C^*-algebra, the interest of $*$-homomorphism from $L^1(G)$ to $L^2(G)$ is rising. Under the above tendency, by finding some properties of unitary representation of G and $*$-representation of $L^1(G)$, we would like to prove them.

In more details, it can be descrived as follows:

§2 contains the definition of Haar measure which is used for the convolution of functions and the proof of Theorem 2.7 which disserts a left(right) translate to be continuous.

§3 presents how to form $*$-algebra $L^1(G)$ from $\mathscr{L}^1(G)$ and the proof of Theorem 3.9 that G is discrete iff $L^1(G)$ has a unity.

In §4, we prove Theorem 4.5 which consists of one of the main Theorem in this dissertation. Theorem 4.5 says:

If φ: $L^1(G)\longrightarrow\mathscr{L}(H)$ is a continuous and nondegenerate algebra homomorphism with $\|\varphi\|\leq 1$, there exists a unique continuous unitary representation of G and, at the same time, the algebra homomorphism becomes a $*$-homomorphism.

2. Haar Measures

Thoroughout this dissertation, by G we mean a locally compact Hausdorff topological group.

A subset A of G is called a *G_δ-set* if there exists a sequence $\{U_n | U : \text{open}\}$ such that $A=\bigcap_{n=1}^{\infty} U_n$.

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Then, we have the following properties([2]):

1°. Every open set and the intersection of any sequence of G_δ-sets are also G_δ-sets.

2°. The union of finitely many G_δ-sets is a G_δ-set.

3°. For a continuous real-valued function $f: G \longrightarrow \boldsymbol{R}$(reals) and a real number c, each of the following sets is a closed G_δ-set:

$$A = \{x \in G \mid f(x) \geq c\},$$
$$B = \{x \in G \mid f(x) \leq c\},$$
$$C = \{x \in G \mid f(x) = c\}.$$

In particular, if f has a compact support and $c > 0$ then A and B are compact G_δ-sets.

Definition 2.1. The σ-ring generated by the class of all compact G_δ-sets in G is called the class of *Baire sets* in G. Thus, if A, B and $A_n (n=1,2,\cdots\cdots)$ are compact G_δ-sets then A, B, $A_n (n=1,2,\cdots\cdots)$, A-B, AUB, $A \cap B$, $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ are Baire sets. A real-valued function f on G which is measurable, *i.e.*, for a real number c, $\{x \in G \mid f(x) \geq c\}$ is a Baire set, is called a *Baire function*.

For a function $f: G \longrightarrow \boldsymbol{C}$(complexes), if Ref and Imf are Baire functions, then f is called a *Baire function*.

Let us put

$$\mathscr{L}(G) = \{f: G \longrightarrow \boldsymbol{C} \mid f \text{ is continuous with compact support}\}.$$

(Note that $\mathscr{L}(G)$ is a vector space over $\boldsymbol{C}$.)

The followings have been already proved([2], [10], [12], [13], [14], [16], [17]):

4°. Every open set is the union of open Baire sets.

5°. Each $f \in \mathscr{L}(G)$ is a Baire function.

6°. The class of all Baire functions is the smallest class of complex-valued functions on G such that ⓐ it contains $\mathscr{L}(G)$ ⓑ it is closed under sequential pointwise limits.

Definition 2.2. A *Baire measure* on G is a measure μ defined on the class of all Baire sets such that $\mu(C) < \infty$ for each compact G_δ-set C.

For a Baire measure μ and for each Baire set E, it can be proved that

$$\begin{aligned}\mu(E) &= \text{l.u.b of } \{\mu(C) \mid E \supset C \text{ which is a compact } G_\delta\text{-set}\} \\ &= \text{g.l.b of } \{\mu(U) \mid E \subset U \text{ which is an open Baire set}\},\end{aligned}$$

where l.u.b = the least upper bound and g.l.b = the greatest lower bound. That is, a Baire measure is a regular measure. A Baire measure ν on G is called a (left) *Haar* *measure* iff for each Baire set E and each $s\in G$, $\nu(E)=\nu(sE)$.

Definition 2.3. A(left) *Haar integral* for G is a positive linear form $I:\mathscr{L}(G)\longrightarrow \boldsymbol{C}$,

$$i.e.,\ I(f+g)=I(f)+I(g)\ (f,g\in\mathscr{L}(G)),$$

$$I(cf)=cI(f),\ I(f)\geq 0 \text{ if } f\geq 0,$$

such that for each $s\in G$, $I(f)=I(f_s)$, where $f_s(x)=f(s^{-1}x)$ for all $x\in G$.

The following properties hold([6], [8], [11], [15], [17]):

7°. There always exists a Haar measure for G.

8°. Let I and J be Haar integrals for G. Then there exists a unique complex number c such that $J=cI$.

9°. For a Haar integral I for G, there exists a unique Haar measure ν such that for each $f\in\mathscr{L}(G)$

$$I(f)=\int_G f d\nu.$$

Hereafter, we shall write

$$\int_G f(x)dx=I(f)=\int_G f d\nu,$$

and thus dx stands for a Haar measure.

For each $f\in\mathscr{L}(G)$ and $s\in G$, let us define

$$f^s(x)=f(xs)\ (x\in G).$$

Then, for any Haar measure dx there exists a unique continuous groug homomorphism ([2], [6])

$$\triangle : G\longrightarrow \boldsymbol{R}_+ (\boldsymbol{R}_+=\{a\in\boldsymbol{R}\,|\,a>0\})$$

such that for each $f\in\mathscr{L}(G)$

$$\int_G f(xs)dx=\int_G f^s(x)dx=\triangle(s)\int_G f(x)dx.$$

In this case, the continuous group homomorphism $\triangle : G\longrightarrow \boldsymbol{R}_+$ is called the *modular function* for G.

If G is commutative, then $\triangle(s)=1$ for all $s\in G$.

If for all $s\in G$ $\triangle(s)=1$, then G is said to be ***unimodular***.

If G is compact or discrete, then G is necessarily unimodular.

Let dx be a fixed Haar measure for G. We shall put such that

$$\mathscr{L}^1(G)=\{f:\text{ Baire function}\,|\int_G |f(x)|dx<\infty\}.$$

Lemma 2.4. For $f\in\mathscr{L}^1(G)$ and any $\varepsilon>0$, there exists a $g\in\mathscr{L}(G)$ such that

$$\int_G |f(x)-g(x)|dx \leq \varepsilon.$$

Proof. Since there exist mutually disjoint compact G_δ-sets D_j and complex numbers α_j $(j=1,2,\cdots\cdots,n)$ such that

$$\int_G |f(x)-\sum_{j=1}^{m}\alpha_j\chi_{D_j}(x)|dx \leq \frac{\varepsilon}{2} \quad ([2],[10])$$

and

$$\int_G |f(x)-g(x)|dx \leq \int_G |f(x)-\sum_{j=1}^{m}\alpha_j\chi_{D_j}(x)|dx +$$

$$\int_G |\sum_{j=1}^{m}\alpha_j\chi_{D_j}(x)-g(x)|dx,$$

we may assume

$$f=\sum_{j=1}^{m}\alpha_j\chi_{D_j}$$

where χ_{D_j} is the characteristic function of $D_j(j=1,\cdots\cdots,n)$. Mereover, it is sufficient to consider $f=\chi_D$ for a compact G_δ-set. Then since there exists a sequence $\{g_n\}$ each of which is a real-valued function in $\mathscr{L}(G)$ such that $g_n \downarrow f([2])$, *i.e.*, $g_n \geq g_{n+r}$ $(n=1,2,\cdots\cdots)$ and $f=g.l.b\ g_n$, by the monotone convergence theorem ([2]) we have

$$\int_G |g_n(x)-f(x)|dx=\int_G (g_n(x)-f(x))dx$$
$$=\int_G g_n(x)dx-\int_G f(x)dx \longrightarrow 0,$$

and so there exists a positive integer n such that for $m \geq n$

$$\int_G |f(x)-g_m(x)|dx \leq \varepsilon. \qquad ///$$

Lemma 2.5. For any $\varepsilon>0$ and $f \in \mathscr{L}(G)$, there exists a neighborhood V of e (the identity of G) such that

$$|f(x)-f(y)|<\varepsilon$$

whenever $x^{-1}y \in V$.

Proof. Assume that our assertion is true.

Then there exists an element $s \in V$ such that $y=xs$ and

$$|f(x)-f(xs)|<\varepsilon \text{ for all } x \in G.$$

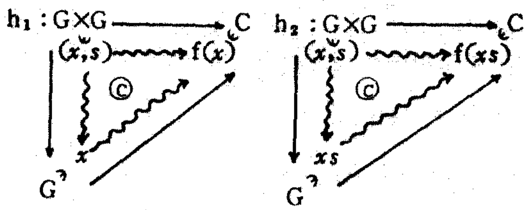
Thus, it is sufficient to prove that

$$V=\{s\in G \mid |f(x)-f(xs)|<\varepsilon \text{ for all } x\in G\}$$

is a neighborhood of e. Define a functon

$$h:G\times G\longrightarrow \mathbf{R}(h(x,s)=|f(x)-f(xs)|),$$

then h is continuous. In fact, since



are continuous, $h(x,s)=|h_1(x,s)-h_2(x,s)|$ is continuous.
In particular, for all $x\in G$ $h(x,e)=0$ and

$$V=\{s\in G \mid h(x,s)<\varepsilon \text{ for all } x\in G\}.$$

Since $f\in\mathscr{L}(G)$, there exists a compact subset C of G such that $f|_{G-C}=0$. We can choose a compact and symmetric neighborhood W of e since G is locally compact and Hausdorff ([11]).

Since $C\times W$ is compact in $G\times G$ and $G\times G\longrightarrow G((x,y)\longmapsto xy)$ is continuous,

$$D=CW=\{xy \mid x\in C \text{ and } y\in W\}$$

is compact in G.

Since $e\in W$, $C\subset D$ is clear. From the continuity of h at (y,e) and $h(y,e)=0$ for any y in D, we get a neighbborhood N_y of (y,e) such that for $(x,s)\in N_y$, $h(x,s)<\varepsilon$.

That is, $N_y=h^{-1}([0,\varepsilon))$ and thus there exist a neighborhood A_y of y and a neighborhood U_y of e such that $N_y=A_y\times U_y$.

Since D is compact, there are $y_1,\cdots\cdots$, and y_n in G such that

$$D\subset A_{y_1}\cup\cdots\cdots\cup A_{y_n}.$$

We put

$$U=U_{y_1}\cap\cdots\cdots\cap U_{y_n}\cap W,$$

then U is a neighborhood of e.

In consequence, we have to prove that $U\subset V$ (if this holds, the V is a neighbohood of e).

Given any $s \in U$ and $x \in G$, there are two cases such that $x \in D$ or $x \notin D$.

In case $x \in D$: Then there exists A_{y_i} such that $x \in A_{y_i}$ for some i.

Since $s \in U \subset U_{y_i}$ for all i, we have $h(x, s) < \varepsilon$.

$$i.e.,\ |f(x) - f(xs)| < \varepsilon.$$

In case $x \notin D$: Then $f(x) = 0$ because that $C \subset D$. Also $xs \notin C$ because that if $xs \in C$, then $x \in Cs^{-1} \subset CW = D$. Thus $f(xs) = 0$,

$$i.e.,\ |f(x) - f(xs)| = 0 < \varepsilon.$$

Accordingly, if $s \in U$, then for all $x \in G$

$$h(x, s) = |f(x) - f(xs)| < \varepsilon$$

and hence $U \subset V$. ///

Corollary 2.6. Given any $\varepsilon > 0$ and $f \in \mathscr{L}(G)$, there exists a neighborhood V of e such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $yx^{-1} \in V$.

Proof. As in the proof of Lemma 2.5, it suffices to prove that

$$V = \{s \in G \mid |f(x) - f(sx)| < \varepsilon \text{ for all } x \in G\}$$

is a neighborhood of e. Define a continuous function

$$h : G \times G \longrightarrow \boldsymbol{R}$$
$$(s, x) \longmapsto h(s, x) = |f(x) - f(sx)|.$$

Then, for all $x \in G$ $h(e, x) = 0$ and

$$V = \{s \in G \mid h(s, x) < \varepsilon \text{ for all } x \in G\}.$$

Take a compact sets C and W such that

$$f|_{G-C} = 0,\ W = W^{-1},$$

and W is a neighborhood of e. Then

$$D = WC = \{xy \mid x \in W,\ y \in C\}$$

is a compact subsct of G.

Since $e \in W$, it is clear that $C \subset D$.

If we put $N_y = h^{-1}([0, e))$ for (e, y) $(h(e, y) = 0)$, then there exist a neighborhood A_y of y and a neighborhood U_y of e such that $N_y = U_y \times A_y$. Since D is compact, there are $y_1, \cdots\cdots, y_n \in G$ such that

$$D \subset A_{y_1} \cup \cdots\cdots \cup A_{y_n}.$$

Put

$$U = U_{y_1} \cap \cdots\cdots \cap U_{y_n} \cap W,$$

then we can prove that (i) U is a neighborhood of e and (ii) $U \subset V$ by the same way as in the proof of Lemma 2.5. ///

For all $f \in \mathscr{L}^1(G)$, we define

$$\|f\|_1 = \int_G |f(x)|\, dx$$

then $f \longmapsto \|f\|_1$ defines a seminorm on $\mathscr{L}^1(G)$. Thus there exists a unique topology on $\mathscr{L}^1(G)$ such that (i) it is compatible with the vector space structure of $\mathscr{L}^1(G)$ and (ii) the sets

$$B_\varepsilon = \{f \in \mathscr{L}^1(G) \mid \|f\|_1 \leq \varepsilon\} \quad (\varepsilon > 0)$$

are a fundamental system of neighborhoods of 0 $(0(G) = 0)$. This topology is called the *seminorm topology.*

Theorem 2.7. Let f be a fixed element in $\mathscr{L}^1(G)$. Then

$$\begin{array}{ccccccc} \phi : G & \longrightarrow & \mathscr{L}^1(G) & \text{and} & \chi : G & \longrightarrow & \mathscr{L}^1(G) \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ s & \longmapsto & f_s & & s & \longmapsto & f^s \end{array}$$

are continuous with the seminorm topology on $\mathscr{L}^1(G)$.

Proof. It is clear that f_s and f^s are in $\mathscr{L}^1(G)$. Thus, it suffices to prove that for any $\varepsilon > 0$, there exists a neighborhood V of e such that for all $s \in V$ $\|f - f_s\|_1 < \varepsilon$ and

$$\|f - f^s\|_1 < \varepsilon$$

We only prove the contiouity of ϕ because that the continuity of χ can be prove by the same way. By Lemma 2.4, there exists an element $h \in \mathscr{L}(G)$ such that

$$\int_G |f(x) - h(x)|\, dx < \frac{\varepsilon}{3}.$$

Put $F(x) = |f(x) - h(x)|$, which is in $\mathscr{L}^1(G)$. Since $F(sx) = |f(sx) - h(sx)|$, we have the following(Definition 2.3):

$$\int_G |f(x)-h(x)|\,dx=\int_G F(x)\,dx$$
$$=\int_G F(sx)\,dx=\int_G |f(sx)-h(sx)|\,dx<\frac{\varepsilon}{3}.$$

By Corollary 2.6, there exists a neighborhood V of e such that for all $x\in G$ and $s\in V$

$$|h(x)-h(sx)|<\frac{\varepsilon}{3M}$$

where for a compact G_δ-set Ω containing the support of $|h(x)-h(sx)|$

$$M=\int_\Omega dx<\infty.$$

(Note that since the support of $|h(x)-h(sx)|\in\mathscr{L}(G)$ (by 3°) is compact, there always exists a compact G_δ-set Ω with its measure $<\infty$ (Definition 2.2) as above ([2]).)

Since

$$|f(x)-f(sx)|$$
$$=|f(x)-h(x)+h(x)-h(sx)+h(sx)-f(sx)|$$
$$\leq|f(x)-h(x)|+|f(sx)-h(sx)|+|h(x)-h(sx)|$$

for all $s\in V$, we have the following:

$$\int_G |f(x)-f(sx)|\,dx$$
$$\leq\int_G |f(x)-h(x)|\,dx+\int_G |f(sx)-h(sx)|\,dx+\int_G |h(x)-h(sx)|\,dx$$
$$<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

That is,

$$\|f-f_s\|_1<\varepsilon \text{ for all } s\in V. \qquad ///$$

3. The Banach *-algebra $L^1(G)$

Let dx be a fixed Haar measure on G, and let us recall the seminormed space $\mathscr{L}^1(G)$ defined in §2. For each $f,\ g\in\mathscr{L}^1(G)$, we shall put

$$D_{(f,g)}=\{s\in G \mid \int_G f(sx)g(x^{-1})\,dx \text{ exists and} <\infty\},$$

then there exists a Baire set $\Omega\subset G$ such that

$$\int_\Omega dx=0 \text{ and } G-\Omega\subset D_{(f,g)} \text{ ([2]).}$$

This means that for each $s \in D_{(f,g)}$ the function $x \longmapsto f(sx)\, g(x^{-1})$ is an element of $\mathscr{L}^1(G)$.

Proposition 3.1. With the above notations, we have

$$\int_G f(sx)\, g(x^{-1})dx = \int_G f(x)g(x^{-1}s)dx$$

for all $s \in D_{(f,g)}$.

Proof. Put $F(x)=f(x)g(x^{-1}s)$, then $F(sx)=f(sx)g(x^{-1})$. Thus,

$$\int_G f(x)\, g(x^{-1}s)dx = \int_G F(x)dx$$
$$= \int_G F(sx)\, dx = \int_G f(sx)g(x^{-1})dx$$

by Definition 2.2. ///

Definition 3.2. Under the above situation, the *convolution* $f*g$ of f and g is defined by

$$f*g(s) = \int_G f(sx)g(x^{-1})dx \text{ for all } s \in D_{(f,g)} = D_{f*g}.$$

That is, the convolution $f*g$ is defined on D_{f*g}. For this situation, we say that $f*g$ is *defined a.e.*. We have the following property with respect to convolutions([2]):

10°. (i) For $f, g \in \mathscr{L}^1(G)$, there exists a function $h \in \mathscr{L}^1(G)$ such that $h=f*g$ *a.e.* and $\|h\|_1 \leq \|f\|_1 \|g\|_1$.

(ii) If $f=f'$ *a.e.*, $g=g'$ *a.e.* and $h'=f'*g'$ *a.e.*, then $h=h'$ *a.e.*.

Definition 3.3. If $f \in \mathscr{L}^1(G)$, the *adjoint* of f is the function $\tilde{f}$ defined by the formula

$$\tilde{f}(x) = \triangle(x) f(x^{-1})^*, \quad (x \in G)$$

where $\triangle$ is the modular function of G and $f(x^{-1})^*$ is the conjugate of $f(x^{-1})$.

Under Definition 3.3, the following holds ([2]):

11°. (i) $\tilde{f} \in \mathscr{L}^1(G)$ and

$$\int_G \tilde{f}(x)\, dx = \left(\int_G f(x)dx\right)^*.$$

where ( )* is the conjugate of ( ). In particular, $\|f\|_1 = \|\tilde{f}\|_1$.

(ii) For $f, f' \in \mathscr{L}^1(G)$, $f=f'$ *a.e.* implies that $\tilde{f}=\tilde{f}'$ *a.e.*.

(iii) For $f, g, h \in \mathscr{L}^1(G)$, $h=f*g$ implies $\tilde{h}=\tilde{f}*\tilde{g}$ *a.e.*.

Definition 3.4. Let $\mathscr{F}$ be the set of all functions $g \in \mathscr{L}^1(G)$ such that ⓐ $g \geq 0$ ⓑ g is essentially bounded(*i.e.*, there exists a positive number $M < \infty$ such that $g \leq M$ *a.e.*) and ⓒ

$\int_G g(x)dx=1$. For a subset $A\subset G$, we define

$$\mathscr{F}_A=\{f\in\mathscr{F} \mid f|_{G-A}=0\}.$$

Proposition 3.5. (i) For $f\in\mathscr{L}^1(G), \tilde{\tilde{f}}=f$.

(ii) For a neighborhood V of e, $\mathscr{F}_V$ contains a continuous function $g\in\mathscr{L}(G)$ such that $g=\tilde{g}$.

(iii) For any $f\in\mathscr{L}^1(G)$ and $g\in\mathscr{F}$, $f*g$ is bounded and continuous.

Proof. (i): For each $x\in G$

$$\begin{aligned}\tilde{\tilde{f}}(x)&=\triangle(x)\,(\tilde{f}(x^{-1}))^*\\&=\triangle(x)(\triangle(x^{-1})(f(x))^*)^*\\&=\triangle(xx^{-1})f(x)\\&=f(x).\end{aligned}$$

(ii): Since G is a locally compact Hausdorff space, there exists a symmetric neighborhood W such that $W\subset V$. Moreover, there is a nonzero continuous function $h:G\longrightarrow C$ such that ⓐ $h\geq 0$ ⓑ $h|_{G-W}=0$ and ⓒ the support of h is compact and contained in W. Then $\tilde{h}|_{G-W}=0$ which is a nonzero continuous function. Thus, $h+\tilde{h}$ is a nonzero continuous function with compact support such that $h+\tilde{h}|_{G-W}=0$.
By (i), we can suppose that $h=\tilde{h}$.
Put $g=\alpha^{-1}h$, where $\alpha=\int_G h(x)dx$. Then $g=\tilde{g}$, and $g\geq 0$ and $\int_G g(x)dx=1$.

(iii): Since $f\in\mathscr{L}^1(G)$ and $g\in\mathscr{F}$, it is abvious that $\|f\|_1<\infty$ and since there exist null set E, number $M(<\infty)$ such that for all $x\notin E$, $g(x)\leq M$,

$$\sup_{t\in G-E}|g(t)|=\sup_{t\in G-E} g(t)=\|g\|_\infty<\infty.$$

For each $s\in G$,

$$\begin{aligned}|f*g(s)|&=|\int_G f(sx)g(x^{-1})dx|\leq\int_G|f(sx)||g(x^{-1})|dx\\&\leq\|g\|_\infty\int_G|f(sx)|dx\\&=\|g\|_\infty\|f\|_1<\infty\end{aligned}$$

Therefore, $f*g$ is bounded on G. Accordingly, $f*g$ is everywhere defined on G. On the other hand, for $s,t\in G$

$$\begin{aligned}|f*g(s)-f*g(t)| &= \left|\int_G f(sx)g(x^{-1})dx - \int_G f(tx)g(x^{-1})dx\right| \\ &= \left|\int_G (f(sx)-f(tx))g(x^{-1})dx\right| \\ &\leq \|g\|_\infty \int_G |f(sx)-f(tx)|dx \\ &= \|g\|_\infty \int_G |f_{ts^{-1}}(x)-f(x)|dx \\ &= \|g\|_\infty \|f_{ts^{-1}}-f\|_1.\end{aligned}$$

(Note that: $F(x)=|f(sx)-f(tx)| \Longrightarrow F(t^{-1}x)=|f(st^{-1}x)-f(x)|=|f_{ts^{-1}}(x)-f(x)|$ and $\int_G F(x)dx=\int_G F(t^{-1}x)dx$.)

Given $\varepsilon>0$, there exists a neighborhood V of e such that

$$\|f_{ts^{-1}}-f\|_1<\varepsilon$$

whenever $ts^{-1}\subset V$ (*i.e.*, $t\in Vs$) by Theorem 2.7. That is, $f*g$ is continuous. ///

The following was proved ([1]):

12°. Given any $\varepsilon>0$ and $f\in\mathscr{L}^1(G)$, there exists a neighborhood V of e such that for all $g\in\mathscr{F}_V$ $\|f*g-f\|_1\leq\varepsilon$ and $\|g*f-f\|_1\leq\varepsilon$.

Definition 3.6. We introduce an equivalence relation "$\sim$" on $\mathscr{L}^1(G)$ such that for $f,g\in\mathscr{L}^1(G)$

$$f\sim g \Longleftrightarrow f=g \ a.e.$$

Define $L^1(G)=\mathscr{L}^1(G)/\sim$, and denote the class of $f\in\mathscr{L}^1(G)$ by $[f]=u$.

For $u=[f]$, $v=[g]$, $w=[h]$ and $c\in C$ ($f,g,h\in\mathscr{L}^1(G)$), we define

$$\begin{aligned}&u+v=[f+g], \quad cu=[cf]\\ &uv=w, \qquad\quad \tilde{u}=[\tilde{f}],\end{aligned}$$

where $h=f*g$ *a.e.*. Then these are well-defined by 10° and 11°. Moreover, for $[f]=u\in L^1(G)$ we define

$$\|u\|_1=\|f\|_1,$$

then $L^1(G)$ is a normed space.

We can easily prove that

$$\begin{aligned}&\|uv\|_1\leq\|u\|_1\|v\|_1 && (\tilde{u})^\sim=u \\ &u(v_1+v_2)=uv_1+uv_2 && (u_1+u_2)^\sim=\tilde{u}_1+\tilde{u}_2 \\ &(u_1+u_2)v=u_1v+u_2v && (\lambda u)^\sim=\lambda^*\tilde{u} \qquad (**) \\ &u(\lambda v)=(\lambda u)v=\lambda(uv) && (u_1u_2)^\sim=\tilde{u}_2\tilde{u}_1 \\ &(u_1u_2)v=u_1(u_2v) && \|\tilde{u}\|_1=\|u\|_1\end{aligned}$$

where $u, v, u_1, u_2, v_1, v_2 \in L^1(G)$ and $\lambda \in C$(see 10°, 11° and [2]). Since $\mathscr{L}^1(G)$ is a complete seminormed space([2]) by the left side of (**), $L^1(G)$ is a Banach algebra.

By the right side of (**), the following is obvious:

"$L^1(G)$ is a Banach *-algebra, with involution $u \longmapsto \tilde{u}$ defined as above: furthermore, for all $u \in L^1(G)$ $\|u\|_1 = \|\tilde{u}\|_1$."

Definition 3.7. Let A be a Banach algebra and let J be an increasingly directed set. An *approximate identity* in A is a net $\{a_j | j \in J\}$ in A such that ⓐ for all $j \in J$, $\|a_j\| = 1$ and ⓑ given any $\varepsilon > 0$ and $b \in A$, there exists $j_0 \in J$ such that for all $j \geq j_0$

$$\|a_j b - b\| \leq \varepsilon, \quad \|b a_j - b\| \leq \varepsilon.$$

Example 3.8. Let $\mathscr{V}$ be the set of all neighborhoods of e in G. For $W, V \in \mathscr{V}$, we define $W \geq V$ if and only if $W \subset V$. Then $\mathscr{V}$ is an increasing directed set. Thus a family $\{g_V | V \in \mathscr{V}\ g_V \in \mathscr{F}\}$ may be regarded as a net. Given any $\varepsilon > 0$, if there exist $g \in \mathscr{L}^1(G)$ and $V \in \mathscr{V}$ such that

$$\forall\, W \geq V, \ \|g - g_W\|_1 \leq \varepsilon$$

then we say that $g_V \longrightarrow g$ as $V \longrightarrow e$. If we put $u_V = [g_V] \in L^1(G)$ for $V \in \mathscr{V}$, then for each $u \in L^1(G)$ we have

$$\|uu_v - u\|_1 \longrightarrow 0 \text{ and}$$
$$\|u_v u - u\|_1 \longrightarrow 0 \text{ by Theorem 2.7.}$$

Theorem 3.9. G is a discrete space if and only if $L^1(G)$ has a unity element.

Proof. Let G be a discrete space. Then $\{e\}$ (e is the identity of G) is an open neighborhood of itself. By (ii) of Proposition 3.5, there exists a function $g_{\{e\}} \in \mathscr{L}^1(G)$ such that $\int_G g_{\{e\}}(x)dx = 1$, $g_{\{e\}} \geq 0$, $g_{\{e\}} = g|_{G-\{e\}} = 0$ and $g_{\{e\}} = \tilde{g}_{\{e\}}$. Since for every $V \in \mathscr{V}$ $\{e\} \subset V$, by 12° for all $f \in \mathscr{L}^1(G)$ and all $\varepsilon > 0$

$$\|f * g_{\{e\}} - f\|_1 \leq \varepsilon, \quad \|g_{\{e\}} * f - f\|_1 \leq \varepsilon.$$

As in Example 3.8, if we put $[g_{\{e\}}] = u_{\{e\}}$ then for all $v \in L^1(G)$

$$\|vu_{\{e\}} - v\|_1 = 0, \quad \|u_{\{e\}} v - v\|_1 = 0.$$

Thus $vu_{\{e\}} = v$ and $u_{\{e\}} v = v$, and thus $u_{\{e\}}$ is a unity of $L^1(G)$.

Conversely, we assume that $u_{\{e\}}$ is a unity of $L^1(G)$ and put $[g_{\{e\}}] = u_{\{e\}}$. We show that there is a positive number a such that the measure of every nonempty open Baire set is at least a. Suppose that this does not hold. Then, given any positive number ε

there exists an open neighborhood V of the identity e of G whose measure is less then ε, and hence one such that

$$\int_V |g_{(e)}(x)|\,dx<\varepsilon.$$

We can find a symmetric neighborhood W of e such that $W^2 \subset V$ and let χ_W be its characteristic function. Then

$$\begin{aligned}\chi_W(x)&=(g_{(e)}*\chi_W)(x)=\int g_{(e)}(y)\chi_W(y^{-1}x)\,dy\\&=\int_{xW} g_{(e)}(y)\,dy\leq\int_V |g_{(e)}(y)|\,dy<\varepsilon\end{aligned}$$

for almost all x in W. This is a contradiction for $\chi_W(x)\equiv 1$ in W. Therefore, there exists a number $a>0$ such that the measure of every nonempty open Baire set is at least a. If A is any infinite set and open set of G whose closure is compact, then its measure is infinite. Therefore if A is an open set whose closure is compact and which has finite measure, then A is a finite set.

Hence, every point is an open set, and the topology is discrete. ///

For the next section, we shall note the following([1]):

13°. (i) For all $u, v \in L^1(G)$, $s, t \in G$ and $\lambda \in C$,

$$\begin{array}{l|l}(u+v)_s=u_s+v_s & (u+v)^s=u^s+v^s\\(\lambda u)_s=\lambda u_s & (\lambda u)^s=\lambda u^s\\u_{st}=(u_t)_s & u^{st}=(u^s)^t\\(uv)_s=u_s v & (uv)^s=uv^s\\\|u_s\|_1=\|u\|_1 & \|u^s\|_1=\triangle(s)\|u\|_1\end{array}$$

$$(\tilde{u})_s=\triangle(s^{-1})(u^s)^\sim,$$

where for $[f]=u(f\in\mathscr{L}^1(G))[f_s]=u_s$ and $[f^s]=u^s$.

(ii) For a fixed element $u\in L^1(G)$, the functions

$$\begin{array}{ccc}G\longrightarrow L^1(G), & & G\longrightarrow L^1(G)\\ \cup\!\!\!\mid \qquad \cup\!\!\!\mid & & \cup\!\!\!\mid \qquad \cup\!\!\!\mid\\ (s\longmapsto u^s) & & (s\longmapsto u_s)\end{array}$$

are continuous with the norm topology on G(see Theorem 2.7).

§4. Unitary and $*$-Representations

For a Hilbert space H, we put such that

$$\mathscr{L}(H)=\{T:H\longrightarrow H\,|\,T \text{ is linear and continuous}\}.$$

Then, as is welll-known $\mathscr{L}(H)$ is a C^*-algebra with unity([4], [5]). For each $x \in H$, the function

$$\rho_x : \mathscr{L}(H) \longrightarrow \boldsymbol{C} \ (T \longmapsto \rho_x(T) = \|Tx\|)$$

is a seminorm ($\|Tx\| = (Tx|Tx)^{\frac{1}{2}}$, (1) is the inner product in H). The topology τ on $\mathscr{L}(H)$ generated by $\mathscr{F} = \{\rho_x | x \in H\}$ is the coarsest topology, *i.e.*, this topology is the supremum topology of $\{\tau_{\rho_x} | x \in H\}$, where the topology τ_{ρ_x} is defined such that $\{T | \rho_x(T_0 - T) = \|T_0 x - Tx\| < \varepsilon\}$ is an open neighborhood of $T_0 \in \mathscr{L}(H)$. This topology τ is called the *strong operator topology*. We put

$$\mathscr{U}(H) = \{T \in \mathscr{U}(H) \mid TT^* = T^*T = 1_H\}$$

which is the set of all unitary operators in $\mathscr{L}(H)$.

Proposition 4.1. $\mathscr{U}(H)$ is a topological group with the strong operator topology.

Proof. It is clear that $\mathscr{U}(H)$ is a multiplicative group. Thus it is sufficient to prove that the mappings

$$\begin{array}{ccccccc} \mathscr{U}(H) \times \mathscr{U}(H) & \longrightarrow & \mathscr{U}(H), & & \mathscr{U}(H) & \longrightarrow & \mathscr{U}(H) \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ (S, T) & \longmapsto & ST & & S & \longmapsto & S^{-1} \end{array}$$

are continuous with the strong operator topology. Since

$$\begin{aligned} \|(ST - S_0T_0)x\| &= \|S(T - T_0)x + (S - S_0)T_0x\| \\ &\leq \|S\| \, \|(T - T_0)x\| + \|(S - S_0)T_0x\| \\ &\leq \|(T - T_0)x\| + \|(S - S_0)T_0x\| \end{aligned}$$

for any S, S_0, T, $T_0 \in \mathscr{U}(H)$, it follows that the mapping

$$(S, T) \longmapsto ST$$

is strongly continuous. In order to prove the strog-operator continuity of the latter mapping, it suffices to note that for any $T \in \mathscr{U}(H)$

$$\begin{aligned} \|T^*x - T_0^*x\|^2 &= \|T^*x\|^2 + \|T_0^*x\|^2 - 2Re(T^*x | T_0^*x) \\ &= \|Tx\|^2 + \|T_0x\|^2 - 2Re(T_0^*x | T_0^*x) - 2Re(T^*x - T_0^*x | T_0^*x) \\ &= \|Tx\|^2 - \|T_0x\|^2 - 2Re(x | (T - T_0)T_0^*x) \\ &\leq (\|Tx\| - \|T_0x\|)(\|Tx\| + \|T_0x\|) + 2\|x\| \, \|(T - T_0)T_0^*x\| \\ &\leq \|(T - T_0)x\| (\|(T - T_0)x\| + 2\|T_0x\|) + 2\|x\| \, \|(T - T_0)T_0^*x\|. \end{aligned}$$

///

Definition 4.2. For a $*$-algebra A and a Hilbert space H, a $*$-homomorphism $\varphi : A \longrightarrow \mathscr{L}(H)$ is called a *$*$-representation* of A on H.

Example 4.3. As in §3, $L^1(G)$ is $*$-algebra. Put $\mathscr{L}^2(G) = \{f : G \longrightarrow \boldsymbol{C} | f$ is a Baire function such that $\int_G |f(x)|^2 dx < \infty\}$

where dx is a fixed Haar measure defined on G. We introduce an equivalence relation "$\sim$" on $\mathscr{L}^2(G)$ such that

$$f \sim g \Longleftrightarrow f = g \ a.e..$$

We put

$$L^2(G) = \mathscr{L}^2(G)/\sim$$

then it is well-known that $L^2(G)$ is a Hilbert space with inner product defined by

$$(u|v) = \int_G f(x) g(x)^* dx$$

for $u=[f]$ and $v=[g]$ in $L^2(G)$ ([1]), where $g(x)^*$ is the conjugate of $g(x)$. The *left regular representation*

$$\varphi(u) : L^2(G) \longrightarrow L^2(G)$$

is defined by $\varphi(u)v = uv$ for each $u \in L^1(G)$ and $v \in L^2(G)$, where $[f^* g] = uv$ if $u=[f] \in L^1(G)$ and $v=[g] \in L^2(G)$. We shall put $\varphi(u) = T_u$, then it was already proved that

$$\begin{array}{ccc} L^1(G) & \longrightarrow & \mathscr{L}(L^2(G)) \\ \Cup & & \Cup \\ u & \longmapsto & T_u \end{array}$$

is a faithful $*$-representation([1]). *i.e.*,

ⓐ $u \longmapsto T_u$ is a $*$-representation($T_{u^*} = (T_u)^*$ is the adjoint operator of T_u).

ⓑ if $u \neq v$ in $L^1(G)$, then $T_u \neq T_v$.

Definition 4.4. Let A be an algebra and let H be a Hilbert space. An algebra homomorphism $\varphi : A \longrightarrow \mathscr{L}(H)$ is said to be ***nondegenerate*** if the set $\{\varphi(a)x \mid a \in A, x \in H\}$ is *total* in H. *i.e.*, the closed linear span of the set is H.

Theorem 4.5. For a Hilbert space H, let

$$\varphi : L^1(G) \longrightarrow \mathscr{L}(H)$$

be a continuous and nondegeneate algebra homomorphism with $\|\varphi\| \leq 1$. Then,

(i) there is a continuous unitary representation $G \longrightarrow \mathscr{U}(H)$ (see Proposition 4.1 for the notation $\mathscr{U}(H)$) $(t \longmapsto U_t)$ such that

ⓐ for all $t \in G$ and $u \in L^1(G)$, $\varphi(u_t) = U_t \varphi(u)$ (see 13°),

ⓑ for all $x, y \in H$ and $u=[f] \in L^1(G)$

$$(\varphi(u)x|y) = \int_G f(t)\ (U_t x|y) dt,$$

(ii) φ is a $*$-representation of $L^1(G)$.

Proof. (i): Let $\{U_j\}_{j\in J}$ be any approximate identity in $L^1(G)$ (see Definition 3.7 and Example 3.8).

For a fixed element $t\in G$, we put $T_j=\varphi((u_j)_t)$ $(j\in J)$. Then, for all $j\in J$

$$\begin{aligned}\|T_j\|&=\|\varphi((u_j)_t)\|\leq\|\varphi\|\,\|(u_j)_t\|_1\\&\leq\|(u_j)_t\|_1=\|u_j\|_1 \text{ by } 13°.\end{aligned}$$

Since $\{u_j|j\in J\}$ is an approximate identity for each $u\in L^1(G)$, $u=lim_j(u_ju)$, and thus

$$u_t=lim_j(u_ju)_t=lim_j((u_j)_tu) \text{ by } 13°.$$

Therefore we have

$$\begin{aligned}\varphi(u_t)&=lim_j\ \varphi((u_j)_tu)\\&=lim_j\ \varphi((u_j)_t)\varphi(u)=lim_j(T_j\varphi(u))\end{aligned}$$

that is, for all $u\in L^1(G)$ and all $x\in H$

$$\|T_j\varphi(u)-\varphi(u_t)\|\longrightarrow 0,\ \|T_j\varphi(u)x-\varphi(u_t)x\|\longrightarrow 0 \quad \cdots\cdots\cdots\cdots(1)$$

By our hypothesis, the set

$$\{\varphi(u)x\,|\,u\in L^1(G),\ x\in H\}$$

is dense in H, and hence the net $\{T_j|\ j\in J\}$ is strongly convergent. Thus there is a unique element U_t in $\mathscr{L}(H)$ such that $T_j\longrightarrow U_t$

$$i.e.,\ \varphi((u_j)_t)\longrightarrow U_t \text{ (strongly)} \cdots\cdots\cdots\cdots(2)$$

We want to prove that $U_t\in\mathscr{U}(H)$. But, by (1) and (2) we see that

$$U_t\varphi(u)x=\varphi(u_t)x$$

for all $u\in L^1(G)$ and $x\in H$. Thus, for all $u\in L^1(G)$ we have

$$U_t\varphi(u)=\varphi(u_t) \cdots\cdots\cdots\cdots(3)$$

We have to note that (3) implies that U_t is dependent only on t and independent of the particular approximate identity $\{u_j|\ j\in J\}$. By (3), it is clear that

$$U_e=1_H,\ U_{st}=U_sU_t\ (s,\ t\in G)\cdots\cdots\cdots\cdots(4)$$

Therefore $U_{t^{-1}}=U_t^{-1}$ for all $t\in G$. Moreover, since

$$\|U_t-T_j\|\longrightarrow 0 \text{ and } \|T_j\|\leq 1$$

it follows that for every $t\in G$ $\|U_t\|\leq 1$ and thus $\|U_t^{-1}\|=\|U_{t^{-1}}\|\leq 1$. Therefore, for all

$x \in H$

$$\begin{aligned}\|U_t x\| &\le \|U_t\|\,\|x\| \le \|x\| \\ &= \|U_{t^{-1}}(U_t x)\| \le \|U_{t^{-1}}\|\,\|U_t x\| \le \|U_t x\|\end{aligned}$$

and thus $\|U_t x\| = \|x\|$. It follows that

$$\begin{aligned}(x\,|\,x) &= \|x\|^2 = \|U_t x\|^2 \\ &= (U_t x\,|\,U_t x) = (U_t^* U_t x\,|\,x).\end{aligned}$$

Thus $1_H = U_t^* U_t = U_{t^{-1}} U_t$ implies that $U_t^* = U_{t^{-1}} = U_t^{-1}$. In consequence, U_t is a unitary operator for all $t \in G$.

Next, we shall prove that the mapping

$$t \longmapsto U_t \text{ is continuous.}$$

If $t \longmapsto e$ in G, then for each $u \in L^1(G)$ $u_t \longmapsto u$ by of (ii) of 13°. Thus, for all $u \in L^1(G)$ and $x \in H$

$$U_t \varphi(u) x \longrightarrow \varphi(u) x \text{ as } t \longrightarrow e.$$

Since $1 = \|U_{t^{-1}} U_t\| = \|1_H\| \le \|U_t\|$ implies $\|U_t\| = 1$ and the set $\{\varphi(u)x \mid u \in L^1(G),\ x \in H\}$ is total in H, we can assert that $U_t \longrightarrow 1_H$ strongly as $t \longrightarrow e$. Thus the mapping $t \longmapsto U_t$ is continuous.

Next, we shall prove ⓑ of (i), *i.e.*, we have to prove that for fixed elements $x, y \in H$

$$(\varphi(u)x\,|\,y) = \int_G f(x)\ (U_t x\,|\,y)dt \cdots\cdots\cdots\cdots\cdots\cdots (5)$$

for all $u = [f] \in L^1(G)$. We can suppose $\|y\| < 1$. The function $t \longmapsto (U_t x\,|\,y)$ is a locally Baire measurable function ([2]) and

$$\|(U_t x\,|\,y)\| = \|U_t\|\,\|x\|\,\|y\| = \|x\|\,\|y\|.$$

i.e., $t \longmapsto (U_t x\,|\,y)$ is a bounded function. Thus $t \longmapsto f(t)\ (U_t x\,|\,y)$ is in $\mathscr{L}^1(G)$. Therefore

$$\int_G f(t)\ (U_t x\,|\,y)dt$$

makes sense. Moreover, since

$$\begin{aligned}&|(\varphi(u)x\,|\,y)| \le \|\varphi\|\,\|u\|_1\|x\|\,\|y\| \le \|u\|_1\|x\|\,\|y\|, \\ &\left|\int_G f(t)\ (U_t x\,|\,y)dt\right| \le \left(\int_G |f(t)|\,dt\right)\|x\|\|y\| \\ &\qquad = \|u\|_1\|x\|\,\|y\| \text{ for all } u = [f] \in L^1(G),\end{aligned}$$

$u \longmapsto (\varphi(u)x\,|\,y)$ and $u \longmapsto \int_G f(t)(U_t x\,|\,y)dt$ are linear and contintous.

To prove (5), we have to note that it is sufficient to prove (5) in case $u=\chi_E$ for a Baire set E of finite Haar measure([2]). But, in this case we can prove that for each $\varepsilon>0$

$$(\varphi(u)x|y)-\int_G f(t)\,(U_t x|y)dt\leq\varepsilon\int_G f(t)dt \quad ([1]).$$

(ii): We have to prove that for all $x,y\in H$ and $u=[f]\in L^1(G)$

$$(\varphi(u)x|y)=(x|\varphi(\tilde{u})y),$$

i.e., $\varphi(u)^*=\varphi(\tilde{u})$. By (b) of (i),

$$\begin{aligned}(x|\varphi(\tilde{u})y)^*=(\varphi(\tilde{u})y|x)&=\int_G \tilde{f}(t)\,(U_t y|x)dt\\ &=\int_G \triangle(t)f(t^{-1})^*(U_t y|x)dt\\ &=(\int_G f(t)\,(U_t x|y)dt)^*\end{aligned}$$

because that

$$\begin{aligned}F(t)=\triangle(t)f(t^{-1})^*(U_t y|x)\Longrightarrow\tilde{F}(t)&=\triangle(t)F(t^{-1})^*\\ &=\triangle(t)\triangle(t^{-1})f(t)(U_{t^{-1}}y|x)^*\\ &=f(t)(U_t x|y)\end{aligned}$$

and

$$\int_G\tilde{F}(t)dt=(\int_G F(t)dt)^*.$$

Thus, $(x|\varphi(\tilde{u})y)^*=(\varphi(u)x|y)^*\Longrightarrow(\varphi(u)x|y)=(x|\varphi(\tilde{u})y)$ and thus $\varphi(u)^*=\varphi(\tilde{u})$. ///

Recall Example 4.3, *i.e.*, the left regular representation

$$\begin{array}{ccc}L^1(G)&\longrightarrow&\mathscr{L}(L^2(G))\\ \Psi&&\Psi\\ u&\longmapsto&T_u\end{array}$$

is a faithful $*$-representation. Note that every $*$-representation has its norm≤1([1]).

Proposition 4.6. Let $t\longmapsto U_t$ be the unitary representation of G corresponding to the left regular representation as above. Then, for every $v\in L^2(G)$ and $t\in G$

$$U_t(v)=v_t.$$

Proof. For $u=[f]\in L^1(G), v=[g]\in L^2(G)$ and $w=[k]\in L^2(G)$, we have

$$(T_u v|w)=\int_G f(t)(U_t v|w)dt$$

by (b) of (i) in Theorem 4.5. Let us put $U_t v=[k]\in L^2(G)$. Since

$$\begin{aligned}(T_u v|w) &= \int_{D_{f*g}} \left(\int_G f(st)g(t^{-1})dt\right)h(s)^* ds \\ &= \int_{D_{f*g}} \left(\int_G f(t)g(t^{-1}s)dt\right)h(s)^* ds \\ &= \int_G f(t)\left(\int_{D_{f*g}} g(t^{-1}s)h(s)^* ds\right)dt\end{aligned}$$

(see the first part of §3 for the notation D_{f*g}).

$$\int_G f(t)(U_t v|w)dt = \int_G f(t)\left(\int_{D_{f*g}} k(s)h(s)^* ds\right)dt.$$

Therefore, we have $g_t = k$ *a.e.* over D_{f*g} for all $f \in \mathscr{L}^1(G)$. This implies that $g_t = k$ *a.e.* over G and thus

$$v_t = [g_t] = U_t v. \qquad ///$$

References

1. S.K. Berberian: Lectures in Functional Analysis and Operator Theory, Springer-Verlag (1974).
2. ______: Measure and Integration, Chelsea (1970).
3. ______ and G.H. Orland: On the closure of the numerical range of an operator, *Proc. Amer. Math. Soc.* ***18***, pp. 499∼503 (1967).
4. J. Dixmier: Les C^*-algèbres et leurs représentations, Gauthier-Villars (1969).
5. R.G. Douglas and C. Pearcy: On the spectral theorem for normal operators, *Proc. Cambridge Philos. Soc.* ***68***, pp. 393∼400 (1970).
6. R. Godement: Sur la théorie des caractères dens les groupes localement compact unimodulaires, *J. Math. Pures Appl.* ***30***, pp. 1∼110 (1951).
7. ______: Sur la théorie des représentations unitaires, *Ann. of Math.* (2) ***53***, pp. 68∼124 (1951).
8. P.R. Halmos: A Hilbert space problem book. Van Nostrand (1967).
9. ______: Continuous functions of Hermitian Operators, *Proc. Amer. Math. Soc.* ***31***, pp. 130∼132 (1972).
10. E. Hewitt and K.A. Ross: Abstract Harmonic Analysis, Vol. I. Springer-Verlag (1963).
11. T. Husain: Introduction to Topological Groups, W.B. Saundres Company (1966).
12. C.E. Rickart: General Theory of Banach Algebras, D. Van Nostrand Company (1960).
13. W. Rudin: Functional Analysis, McGraw-Hill, Inc. (1973).
14. S. Sakai: C^*-algebras and W^*-algebras, Springer-Verlag (1971).
15. I.E. Segal: The Group algebra of a locally compact group, *Trans. Amer. Math. Soc.* ***61***, pp. 69∼105 (1947).
16. J. Wermer: Banach algebras and analytic functions, *Adv. Math.* ***1***, pp. 52∼102 (1961).
17. K. Yosida: Functional analysis, Springer-Verlag (1965).