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A Characterization of the Semisimplicity of Commutative Banach Algebras

by

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1. Introduction

The notion of semisimplicity and the associated notions of the radical and spectrum are of fundamental importance in the study of general structure properties of Banach algebras. It is known that there are several equivalent ways of characterizing the semisimplicity of Banach algebras. Some of necessary and sufficient conditions for the semisimplicity of a commutative Banach algebra A are as follows: (i) The Gelfand transformation on A is injective ([1], [3], [4]). (ii) If $x\in A$ is such that $\sigma(x)=\{0\}$, then $x=0$ ([3], [4]). (iii) The only topologically nilpotent element in A is zero vector ([3], [4]). (iv) If $x\in A$ is such that $\|\hat{x}\|_\infty=0$, where $\hat{x}$ is the Gelfand transform of x, then $x=0$ ([4]). (v) The complex homomorphisms of A separate the points of A ([4]). The purpose of this paper is to establish another characterization for semisimplicity of finitely generated commutatative Banach algebras using the joint spectrum of the generators. In Theorem 9 we shall prove the following: Let A be a commutative Banach algebra finitely generated by $\{x_1,\dots,x_n\}\subset A$. Then A is semisimple, with $x_1^{e_1}\dots x_n^{e_n}\neq 0$ for integers $e_i\geq 0$ $(i=1,\dots,n)$ if and only if (i) $\tilde{h}_i\neq\tilde{h}_j$ for $i\neq j$ $(i,j=1,\dots,n)$ (ii) $(\tilde{h}_1)^{e_1}\dots(\tilde{h}_n)^{e_n}\neq 0$ for integers $e_i\geq 0$ $(i=1,\dots,n)$, where $\tilde{h}_k$ is the restriction of the k^{th} coordinate function h_k: $C^n\longrightarrow C$ on $\sigma(x_1,\dots,x_n)$ (the joint spectrum of $x_1,\dots,x_n$).

2. Joint spectra and semisimplicity of Banach algebras

Throughout this paper we shall use A as a commutative Banach algebra with identity e with $\|e\|=1$. It is noted that for each $x\in A$ the spectrum of x, denoted by

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$\sigma(x)$, is a non-empty compact subset of $\{\zeta\in C \mid |\zeta|\leq\|x\|\}$. We let $\triangle(A)$ be the maximal ideal space of A. Then $\triangle(A)$ is a compact space with its Gelfand topology ([1], [4] [5]). We also note that the Gelfand topology on $\triangle(A)$ is the weakest topology on $\triangle(A)$ such that the Gelfand transform $\hat{x}$ is continuous function on $\triangle(A)$ for all $x\in A$.

Definition 1. A subset E of A is said to *generate* A if, whenever $B\subset A$ is a closed subalgebra of A such that $e\in B$ and $E\subset B$, then $B=A$. The algebra A is said to be *finitely generated* if there exists a finite subset $E\subset A$ which generates A,

Proposition 2. A is generated by $\{x_1, x_2, \ldots, x_n\}\subset A$ if and only if every element of A is the norm limit of a sequence of polynomials in e, $x_1,\ldots,x_n$ with coefficients in C.

Proof. Suppose that A is generated by $\{x_1,\ldots,x_n\}$. We take B as the smallest closed subalgebra of A containing of all polynomials in e, $x_1,\ldots,x_n$ with coefficients in C. That is, if $\{p_m(e,x_1,\ldots,x_n)\}_{m=1}^{\infty}$ is a Cauchy sequence then

$$\lim_{m\to\infty}\|p_m(e_1,x_1,\ldots,x_n)-y\|=0,\quad (y\in A)$$

implies that $y\in B$. Then since $\{e,x_1,\ldots,x_n\}\subset B$, by Definition 1 $B=A$. Therefore, every element of A can be denoted by the norm limit of a sequence of polynomials in e, $x_1,\ldots,x_n$ with coefficients in C. The converse can be easily seen from the definition. ■

Definition 3. Let $x_1,x_2,\ldots,x_n$ be in A. Then the *joint spectrum* of $x_1,\ldots,x_n$ is a subset of C^n defined by

$$\begin{aligned}\sigma(x_1,\ldots,x_n)&=\{(\tau(x_1),\ldots,\tau(x_n))\in C^n \mid \tau\in\triangle(A)\}\\&=\{(\hat{x}_1(\tau),\ldots,\hat{x}_n(\tau))\in C^n \mid \tau\in\triangle(A)\}.\end{aligned}$$

We note that if $n=1$ in the definition, then it is evident that the joint spectrum of a single element reduces the notion of spectrum previously mentioned.

Proposition 4. For $\{x_1,\ldots,x_n\}\subset A$, $\sigma(x_1,\ldots,x_n)$ is a nonempty compact subset of C^n which is contained in the polydisk

$$D_n=\{(z_1,\ldots,z_n)\in C^n \mid\ |z_k|\leq\|x_k\| \text{ for } k=1,\ldots,n\}.$$

Proof. We define a mapping

$$\varphi:\ \triangle(A)\longrightarrow \boldsymbol{C}^n$$

by for each $\tau\in\triangle(A)$, $\varphi(\tau)=(\hat{x}_1(\tau),\ldots,\hat{x}_n(\tau))$. Let h_k denote the k^{th} coordinate function in $\boldsymbol{C}^n$, that is, $h_k(z_1,\ldots,z_k)=z_k$, $k=1,\ldots,n$. Then for each $h_k(k=1,\ldots,n)$, it follows that

$$\begin{array}{rccc} h_k\circ\varphi: & \triangle(A) & \longrightarrow & \boldsymbol{C} \\ & \Uparrow & & \Uparrow \\ & \tau & \longmapsto & h_k\circ\varphi(\tau)=\hat{x}_k(\tau). \end{array}$$

That is, $h_k\circ\varphi=\hat{x}_k$: $\triangle(A)\longrightarrow\boldsymbol{C}$ is continuous for all $k=1,\ldots,n$. Therefore φ is contunuous. Sinece $\triangle(A)$ is compact and each continuous function preserves the compactness, $\varphi(\triangle(A))=\sigma(x_1,\ldots,x_n)$ is a compact subset of $\boldsymbol{C}^n$. We now note that the spectrum of x, $\sigma(x)$ is a non-empty subset of $\{z\in\boldsymbol{C}|\ |z|\leq\|x\|\}$. Therefore, it follows that

$$\sigma(x_1,\ldots,x_n)\subset\sigma(x_1)\times\ldots\times\sigma(x_n)\subset D_n.$$

Since $\hat{x}_k(\tau)\in\boldsymbol{C}$ for each $\tau\in\triangle(A)$ and x_k $(k=1,\ldots,n)$, it is obvious that $\sigma(x_1,\ldots,x_n)$ $=\{(\hat{x}_1(\tau),\ldots,\hat{x}_n(\tau))\in\boldsymbol{C}^n\,|\,\tau\in\triangle(A)\}$ is nonempty. ■

Lemma 5. If A is generated by $\{x_1,\ldots,x_n\}\subset A$, then $\triangle(A)$ and $\sigma(x_1,\ldots,x_n)$ are homeomorphic.

Proof. We define the mapping

$$\varphi:\ \triangle(A)\ \longrightarrow\ \sigma(x_1,\ldots,x_n)$$

by $\varphi(\tau)=(\hat{x}_1(\tau),\ldots,\hat{x}_n(\tau))$ $(\tau\in\triangle(A)$. Then as in the proof of Proposition 4, φ is continuous. φ is also surjective by Definition 3. We assume that $(\tau(x_1),\ldots,\tau(x_n))=$ $(w(x_1),\ldots,w(x_n))\in\sigma(x_1,\ldots,x_n)$ for $\tau,w\in\triangle(A)$. This means that

$$\tau(x_k)=w(x_k),\ k=1,2,\ldots,n.$$

It follows from Proposition 2 that for each $x\in A$ there is a sequence of polynomials $\{p_m(e,x_1,\ldots,x_n)\}_{m=1}^{\infty}$ such that

$$\lim_{m\to\infty}\|p_m(e, x_1, \ldots, x_n) - x\| = 0.$$

Since $\tau(e)=1=w(e)$ and $\tau(x_k)=w(x_k)$ $(k=1,\ldots,n)$, $\tau(p_m(e, x_1, \ldots, x_n))=w(p_m(e, x_1, \ldots, x_n))$, and hence $\tau(x)=w(x)$. Consequently we have $\tau=w$. That is, φ is injective and surjective. In particular, we can claim that φ is a closed mapping. F is closed in compact space $\triangle(A)$, then F is also compact and thus $\varphi(F)$ is compact in the Hausforff space $\sigma(x_1,\ldots,x_n)$. Hence $\varphi(F)$ is closed in $\sigma(x_1,\ldots,x_n)$. Therfore φ^{-1} is continuous, and thus φ is a homeomorphism. ■

Let us put

$$C(\triangle(A))=\{f:\ \triangle(A)\longrightarrow \mathbf{C} \mid f \text{ is continuous}\}.$$

Then, as is well-known, $C(\triangle(A))$ is a commutative Banach algebra with identity ([4]). The norm $\|\eta\|_\infty$ of $\eta\in C(\triangle(A))$ is defined by

$$\|\eta\|_\infty=\operatorname{Sup}_{\tau\in\triangle(A)}|\eta(\tau)|.$$

Then for each $x\in A$, we have $\|\hat{x}\|_\infty\le\|x\|$.

Definition 6. The *radical* of A, denoted by Rad(A), is defined as the intersection of all the maximal ideals in A; that is,

$$\operatorname{Rad}(A)=\bigcap_{M\in\triangle(A)} M.$$

Definition 7. If Rad$(A)=\{0\}$, then A is called semisimple.

One of the most useful characterization for the semisimplicity of A is the following ([1], [3], [4]).

Lemma 8. A is semisimple if and only if the Gelfand transformation on A ($x\longmapsto \hat{x}$, $x\in A$) is injective.

We now recall that $\sigma(x_1,\ldots,x_n)$ is non-empty and compact in $\mathbf{C}^n$, and let h_k denote the k^{th} coordinate function in $\mathbf{C}^n$, that is,

$$h_k:\ \mathbf{C}^n\longrightarrow \mathbf{C}((z_1,\ldots,z_n)\longmapsto z_k) \text{ for } k=1,\ldots,n.$$

By Lemma 5, we have the identification of $\triangle(A)$ with $\sigma(x_1,\ldots,x_n)$ such that

$$\begin{array}{ccc} \tau & \longleftrightarrow & (\tau(x_1),\ldots,\tau(x_n))=(\hat{x}_1(\tau),\ldots,\hat{x}_n(\tau)). \\ \cap\!\!\!| & & \cap\!\!\!| \\ \triangle(A) & & \sigma(x_1,\ldots,x_n) \end{array}$$

Therefore, for each $x \in A$

$$\begin{array}{rcl} \hat{x}:\ \sigma(x_1,\ldots,x_n) & \longrightarrow & C \\ \Cup & & \Cup \\ (\tau(x_1),\ldots,\tau(x_n)) & \longmapsto & \tau(x) \end{array}$$

is defined. For example, $\hat{o}(\sigma(x_1,\ldots,x_n)=\{0\}$ and $\hat{e}(\sigma(x_1,\ldots,x_n))=\{1\}$ since for each $\tau\in\triangle(A)$ $\hat{o}(\tau)=\tau(0)=0$ and $\hat{e}(\tau)=\tau e=1$. We shall put

$$\underset{\equiv 0}{\hat{0}|\triangle(A)=\hat{o}|\sigma(x_1,\ldots,x_n)} \text{ and } \underset{\equiv 1}{\hat{e}|\triangle(A)=\hat{e}|\sigma(x_1,\ldots,x_n)}$$

Mereover, if we put

$$\tilde{h}_k=h_k|\sigma(x_1,\ldots,x_n),\ k=1,\ldots,n$$

then we have for each $\tau\in\triangle(A)$

$$\hat{x}_k((\hat{x}_1(\tau),\ldots,\hat{x}_n(\tau))=\hat{x}_k(\tau)=\tilde{h}_k(\hat{x}_1(\tau),\ldots,\hat{x}_n(\tau)).$$

That is, $\hat{x}_k=\tilde{h}_k(k=1,\ldots,n)$ on $\triangle(A)=\sigma(x_1,\ldots,x_n)$

Theorem 9. Let A be generated by $\{x_1,\ldots,x_n\}\subset A$. Then A is semisimple with $x_1^{e_1}\ldots x_n^{e_n}\neq 0$ for integers $e_i\geq 0$ $(i=1,\ldots,n)$ if and only if $\tilde{h}_i\neq\tilde{h}_j$ for $i\neq j$ $(i,j=1,\ldots,n)$ and $(\tilde{h}_1)^{e_1}\ldots(\tilde{h}_n)^{e_n}\neq 0$ for integers $e_i\geq 0$ $(i=1,\ldots,n)$.

Proof. We first note that A is semisimple if and only if $\|\hat{x}\|_\infty=0$ implies $x=o$([4]). Thus if A is semisimple, then $\|(\hat{x}_1)^{e_1}\ldots(\hat{x}_n)^{e_n}\|_\infty\neq 0$ since $x_1^{e_1}\ldots x_n^{e_n}\neq 0$, and thus we have $(\hat{x}_1)^{e_1}\ldots(\hat{x}_n)^{e_n}\neq 0$. Moreover, since $x_i\neq x_j$ for $i\neq j$ and $\hat{x}_k=\tilde{h}_k$ $(k=1,\ldots,n)$, we have all conditions for the necessity.

For the converse, we note that since

$$(\tilde{h}_1)^{e_1}\ldots(\tilde{h}_n)^{e_n}=(\hat{x}_1)^{e_1}\ldots(\hat{x}_n)^{e_n}\neq 0,$$

we have

$$\|(\hat{x}_1)^{e_1}\ldots(\hat{x}_n)^{e_n}\|_\infty\neq 0.$$

And thus

$$0<\|(\hat{x}_1)^{e_1}\ \dots\ (\hat{x}_n)^{e_n}\|_\infty<\|x_1^{e_1}\ \dots\ x_n^{e_n}\|,$$

which implies that $x_1^{e_1}\dots x_u^{e_n}\neq 0$. Now to show A is semisimple, it suffices to show that the Gelfand transformation on $A(x\mapsto\hat{x},\ x\in A)$ is injective (Lemma 8). We take $x\neq y$ in A, and put $\delta=\|x-y\|>0$. Take a positive real number $\varepsilon>0$ with $0<\varepsilon<\delta/3$. Then, by Proposition 2, there are polynomials $p_\varepsilon\ (e,x_1,\dots,x_n)$ and $p_\varepsilon'(e,x_1,\dots,x_n)$ with coefficients in C such that

$$\|x-p_\varepsilon(e,x_1,\dots,x_n)\|<\varepsilon,\ \ \|y-p_\varepsilon(e,x_1,\dots,x_n)\|<\varepsilon$$

and

$$\|p_\varepsilon(e,x_1,\dots,x_n)-p_\varepsilon'(e,x_1,\dots,x_n)\|>\delta/3>\varepsilon\cdots\cdots\cdots(*)$$

Since $\|\hat{\lambda}\|_\infty\leq\|x\|$ for each $x\in A$,

$$\begin{aligned}\|\hat{x}-p_\varepsilon(e,x_1,\dots,x_n)^\wedge\|_\infty&=\|\hat{x}-p_\varepsilon(\hat{e},\hat{x}_1,\dots,\hat{x}_n)\|_\infty\\&=\|\hat{x}-p_\varepsilon(l,\tilde{h}_1,\dots,\tilde{h}_n)\|_\infty\\&<\varepsilon\end{aligned}$$

and

$$\begin{aligned}\|\hat{y}-p_\varepsilon'(e,x_1,\dots,x_n)^\wedge\|_\infty&=\|\hat{y}-p_\varepsilon'(\hat{e},\hat{x}_1,\dots,\hat{x}_n)\|_\infty\\&=\|y-p_\varepsilon'(l,h_1,\dots,\tilde{h}_n)\|_\infty\\&<\varepsilon\end{aligned}$$

Moreover, from (*) and our assumption we have

$$\begin{aligned}0&<\|p_\varepsilon(e,x_1,\dots,x_n)^\wedge-p_\varepsilon'(e,x_1,\dots,x_n)^\wedge\|_\infty\\&=\|p_\varepsilon(l,\tilde{h}_1,\dots,\tilde{h}_n)-p_\varepsilon'(l,\tilde{h}_1,\dots,\tilde{h}_n)\|_\infty\\&\leq\|\hat{x}-p_\varepsilon(1,\hat{h}_1,\dots,\hat{h}_n)\|_\infty+\|x-y\|_\infty+\|y-p_\varepsilon'(1,h_1,\dots,h_n)\|_\infty\end{aligned}$$

Therefore, thaking the limits,

$$0<\lim_{\varepsilon\to 0}\|\hat{x}-p_\varepsilon(l,\hat{h}_1,\dots,\hat{h}_n)\|_\infty+\lim_{\varepsilon\to 0}\|x-y\|_\infty+\lim_{\varepsilon\to 0}\|y-p_\varepsilon'(l,\hat{h}_1,\dots,\hat{h}_n)\|_\infty=\|x-y\|_\infty.$$

Thus, $x\neq y$ in A implies $\hat{x}\neq\hat{y}$. That is, the Gelfand transformation on A $(x\mapsto\hat{x})$ is injective. Therefore, by Lemma 8, A is semisimple. ■

References

1. G. Bachman and L. Narici, Funcional Analysis, Academic Press, 1966.
2. A.T. Dash, Joint Spectra, Studia Math., pp.225~236, 1973.
3. R.G. Douglas, Banach Algebra Techniques in Operator Theorey, Academic Press, New York, 1972.
4. R. Larsen, Banach Algebra an introduction, Marcel Dekker, Inc., 1973.
5. W. Rudin, Functional Analysis, McGraw-Hill, Inc., 1973.