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A Note on the Cousin 1 Problem

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As is well known, a Stein manifold X is a Cousin 1 Domain ([1], [3]) because of $H^1(X,\mathscr{A})=0$. From this idea we want to prove, in this paper, that a complex manifold X is a Cousin 1 Domain if and only if $H^1(X,\mathscr{A})\to H^1(X,\mathscr{M})$ is injective(Theorem 3).

Let X be a topological space. X is said to be an *n-dimensional complex manifold* if there exists an atlas $A=\{(U_i,\varphi_i),\ i\in I\}$, ($I$ is an index set) of charts of X such that

(i) φ_i is a homeomorphism of U_i, ($X=\bigcup_{i\in I} U_i$, U_i open) onto the open subset $\varphi_i(U_i)$ of $\boldsymbol{C}^n$($\boldsymbol{C}$: complex numbers) for all $i\in I$,

(ii) For all $i, j\in I$, $\varphi_i\varphi_j^{-1}$ is a biholomorphic map $\varphi_j(U_{ij})$ onto $\varphi_i(U_{ij})$ where $U_{ij}=U_i\cap U_j$. ([1])

Let X be a complex manifold. We denote by $\mathscr{A}(U)$ the set of all analytic functions from U to $\boldsymbol{C}$, where U is an open subset of X. Then $\mathscr{A}(U)$ is a commutative algebra with identity over $\boldsymbol{C}$. We put

$$\mathscr{A}_x=\varinjlim_{x\in U}\mathscr{A}(U) \quad \text{and} \quad \mathscr{A}=\dot{\bigcup_{x\in X}}\mathscr{A}_x \quad \text{(disjoint union)}.$$

Then there is the ring homomorphism

$$\mathscr{A}(U)\longrightarrow\mathscr{A}_x(f\longmapsto f_x).$$

We next introduce the topology to $\mathscr{A}$ taking as an open base the family of sets $\{f_x|x\in U$ which is open in X, $f\in\mathscr{A}(U)\}$.

Thus the projection

$$\begin{array}{rccc}\pi: & \mathscr{A} & \longrightarrow & X\\ & \bigcup & \longmapsto & \Downarrow\\ & \mathscr{A}_x & & x\end{array}$$

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is a sheaf which is called *the Oka Sheaf of* $\boldsymbol{X}$ or *the sheaf of germs of analytic functions of* $\boldsymbol{X}$.

Let $\boldsymbol{C}\{z_1-a_1, \cdots, z_n-a_n\}$ or simply $\boldsymbol{C}\{z-a\}$ $(z=(z_1, \cdots, z_n),\ a=(a_1, \cdots, a_n)\in \boldsymbol{C}^n)$ denote the ring of convergent power series. Under the above notations we have the following.

Proposition 1. For all $x\in X$, $\mathscr{A}_x\cong \boldsymbol{C}\{z_1, \cdots, z_n\}$, where X is an n-dimensional complex manifold.

Proof. Recall the atlas $\{(U_i, \varphi_i),\ i\in I\}$ of X, and assume that $x\in U_i$. Since $f\in\mathscr{A}(U_i)$ implies that $f\varphi_i^{-1}: \varphi(U_i)\to \boldsymbol{C}$ is analytic, it is clear that $\mathscr{A}(U_i)\cong\{f: \varphi(U_i)(\subset \boldsymbol{C}^n)\to \boldsymbol{C}\,|\, f$ is analytic$\}$.

Put $\varphi_i(x)=(a_1, \cdots, a_n)=a\in \boldsymbol{C}^n$, then $f\in\{f: \varphi(U_i)\to \boldsymbol{C}\,|\,f$ is analytic$\}$ can be represented by a convergent power series

$$f(z)=\sum_{m=0}^{\infty} a_m(z-a)^m \quad ([1]),$$

where $m=(m_1, \cdots, m_n)$ with $m_1, \cdots, m_n\in \boldsymbol{Z}^+\cup\{0\}$ and $(z-a)^m=(z_1-a_1)^{m_1}\cdots(z_n-a_n)^{m_n}$. Since $\varphi_i(U_i)$ is an open subset of $\boldsymbol{C}^n$, we can put as follows

$\varphi_i(x)=0$ is the origin
$\varphi_i(U_i)$ is an open neighborhood of the origin.

Therefore, we have

$$\begin{aligned}\mathscr{A}(U_i)&\cong\{f(z)=\sum_{m=0}^{\infty} a_m z^m \,|\, \sum_{m=0}^{\infty} a_m z^m \text{ is convergent}\}\\ &=\boldsymbol{C}\{z_1, \cdots, z_n\}.\end{aligned}$$

Hence, we get that $\mathscr{A}_x\cong \boldsymbol{C}\{z_1, \cdots, z_n\}$.

Since $\mathscr{A}_x\cong \boldsymbol{C}\{z_1, \cdots, z_n\}$ and $\boldsymbol{C}\{z_1, \cdots, z_n\}$ is an integral domain, we can put as follows

the quotient field of $\mathscr{A}_x\cong \mathscr{M}_x (x\in X)$.

Next, we put

$$\mathscr{M}=\dot{\bigcup_{x\in X}}\mathscr{M}_x,$$

which is called *the sheaf of germs of meromorphic functions* on X. Of course, $\mathscr{M}$ has the topology which is defined by the same as in $\mathscr{A}$.

For a complex manifold X, a map σ: $X \to \mathscr{M}$ is said to define a *meromorphic function on* X if

(i) For all $x \in X$ $\sigma(x) \in \mathscr{M}_x$

(ii) there exists an open neighborhood U of every point $x \in X$ and analytic functions $f, g \in \mathscr{A}(U)$ such that

$$\sigma(y) = f_y / g_y, \quad y \in U.$$

In this case, σ is a section of the sheaf $\mathscr{M} \to X (\mathscr{M}_x \mapsto x)$.

We shall put as

$$\mathscr{M}(X) = \text{the set of all sections of } \mathscr{M} \to X.$$

Similarly, we shall put

$$\mathscr{A}(X) = \text{the set of all sections of the sheaf } \mathscr{A} \to X.$$

Then $\mathscr{A}(X)$ is a commutative algebra over $\boldsymbol{C}$ and $\mathscr{M}(X)$ is a commutative algebra over $\boldsymbol{C}$ as an $\mathscr{A}(X)$-module.

Let X be a complex manifold, and let $\mathscr{B}$ be a sheaf of $\mathscr{A}(X)$-modules over X. Then $\mathscr{B}$ has a fine resolution(the canonical resolution) such that

$$0 \longrightarrow \mathscr{B} \xrightarrow{\varepsilon} \mathscr{B}_0 \xrightarrow{d_0} \mathscr{B}_1 \xrightarrow{d_1} \cdots\cdots$$

where ker $d_i = Im\ d_{j-1} (j \geq 1)$ and each $\mathscr{B}_j (j \geq 0)$ is a *fine sheaf*([4]).

Then

$$0 \longrightarrow \mathscr{B}(X) \xrightarrow{\varepsilon^*} \mathscr{B}_0(X) \xrightarrow{d_0^*} \mathscr{B}_1(X) \xrightarrow{d_1^*} \cdots\cdots$$

is a complex. We define

$$H^0(X, \mathscr{B}) = \text{Ker } d_0^*, \ H^k(X, \mathscr{B}) = \text{Ker } d_k^* / Im\ d_{k-1}^* \quad (k \geq 1).$$

We say that $H^p(X, \mathscr{B})$ is the ***p-th sheaf cohomology group of X with coefficients in the sheaf*** $\mathscr{B}$.

Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be a short exact sequence of sheaves over X(*i.e*, for all $x \in X$

$$0 \longrightarrow \mathscr{F}_x \longrightarrow \mathscr{G}_x \longrightarrow \mathscr{H}_x \longrightarrow 0$$

is a short exact sequence of rings if each sheaf is a sheaf of rings). Then

$$0 \longrightarrow \mathscr{F}(X) \longrightarrow \mathscr{G}(X) \longrightarrow \mathscr{H}(X) \quad \cdots\cdots\cdots\cdots (1)$$

is an exact sequence ([1],[2],[3]). If, moreover, $\mathscr{F}$ is a fine sheaf then

$$0 \longrightarrow \mathscr{F}(X) \longrightarrow \mathscr{G}(X) \longrightarrow \mathscr{H}(X) \longrightarrow 0 \cdots\cdots\cdots\cdots (2)$$

is a short exact sequence.

By(1) above it follows that $H^0(X,\mathscr{B}) \cong \mathscr{B}(X)$ for each sheaf $\mathscr{B}$ over X. By(2) above we have the following commutative diagram:

$$\begin{array}{ccccccccl} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{F}(X) & \longrightarrow & \mathscr{G}(X) & \longrightarrow & \mathscr{B}(X) & \longrightarrow 0 & \text{(exact)} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{F}_0(X) & \longrightarrow & \mathscr{G}_0(X) & \longrightarrow & \mathscr{B}_0(X) & \longrightarrow 0 & \text{(exact)} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{F}_1(X) & \longrightarrow & \mathscr{G}_1(X) & \longrightarrow & \mathscr{B}_1(X) & \longrightarrow 0 & \text{(exact)} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

where $0 \to \mathscr{F} \to \mathscr{F}_0 \to \mathscr{F}_1 \to \cdots$ is the canonical resolution of $\mathscr{F}$. (Note that $\mathscr{F}_i (i \geq 0)$ is a fine sheaf.). By a straightforward diagram chase of the above diagram, we have the long exact sequence

$$0 \longrightarrow H^0(X,\mathscr{F}) \longrightarrow H^0(X,\mathscr{G}) \longrightarrow H^0(X,\mathscr{H}) \xrightarrow{\delta} H^1(X,\mathscr{F}) \longrightarrow \cdots.$$

Definition 2. Let X be a complex manifold and $\{U_i,\ i \in I\}$ be an open cover of X. X is said to be a Cousin 1 domain if $\sigma_i \in \mathscr{M}(U_i)$ with $\sigma_i - \sigma_j \in \mathscr{A}(U_{ij})$ for all $i,j \in I$ are given, then there exists $\sigma \in \mathscr{M}(X)$ such that $\sigma - \sigma_i \in \mathscr{A}(U_i)$ for all $i \in I$.

In particular, it is important to note that $i: \mathscr{A} \to \mathscr{M}$.

Theorem 3. Let X be a complex manifold. X is a Cousin 1 Domain if and only if $i^*: H^1(X,\mathscr{A}) \to H^1(X,\mathscr{M})$ is injective, where i^* is the induced homomorphism from $i: \mathscr{A} \to \mathscr{M}$.

Proof. Suppose the short exact sequence of sheaves over X:

$$0 \longrightarrow \mathscr{A} \xrightarrow{i} \mathscr{M} \xrightarrow{p} \mathscr{M}/\mathscr{A} \longrightarrow 0$$

where $\mathscr{M}/\mathscr{A}$ is the quotient sheaf ([4]). Therefore by the above statements we have the long exact sequence

$$0 \longrightarrow H^0(X,\mathscr{A}) \xrightarrow{i^*} H^0(X,\mathscr{M}) \xrightarrow{p^*} H^0(X,\mathscr{M}/\mathscr{A}) \xrightarrow{\delta} H^1(X,\mathscr{A})$$
$$\xrightarrow{i^*} H^1(X,\mathscr{M}) \longrightarrow H^1(X,\mathscr{M}/\mathscr{A}) \xrightarrow{\delta} \cdots.$$

When we note that

$$H^0(X,\mathscr{A}) \cong \mathscr{A}(X),\ H^0(X,\mathscr{M}) \cong \mathscr{M}(X),\ H^0(X,\mathscr{M}/\mathscr{A}) \cong (\mathscr{M}/\mathscr{A})(X)$$

we have the exact sequence

$$0 \longrightarrow \mathscr{A}(X) \xrightarrow{i} \mathscr{M}(X) \xrightarrow{p} (\mathscr{M}/\mathscr{A})(X) \xrightarrow{\delta} H^1(X,\mathscr{A})$$
$$\xrightarrow{i^*} H^1(X,\mathscr{M}) \longrightarrow \cdots.$$

We first assume that X is a Cousin 1 Domain. Then for $\sigma_i \in \mathscr{M}(U_i)$, $\sigma_j \in \mathscr{M}(U_j)$ if $\sigma_i - \sigma_j \in \mathscr{A}(U_{ij})$ then

$$\sigma_i | U_{ij} \equiv \sigma_j | U_{ij}$$

in the sheaf $\mathscr{M}/\mathscr{A}$. Therefore there exists an element $\mathscr{X} \in (\mathscr{M}/\mathscr{A})(X)$ such that $\mathscr{X} | U_i = \sigma_i$ for $i \in I$. Since X is a Cousin 1 Domain there exists an element $\sigma \in \mathscr{M}(X)$ such that $\sigma - \sigma_i \in \mathscr{A}(U_i)$. Thus

$$p(\sigma) = \mathscr{X}.$$

Take an element $\mathscr{X} \in (\mathscr{M}/\mathscr{A})(X)$ and put

$$\mathscr{X} | U_i = \sigma_i \in \mathscr{M}(U_i)$$

for all $i \in I$. Then $\sigma_i - \sigma_j \in \mathscr{A}(U_{ij})$ because of

$$\sigma_i - \sigma_j = 0 \text{ in } (\mathscr{M}/\mathscr{A})(U_{ij}).$$

Therefore, each element of $(\mathscr{M}/\mathscr{A})(X)$ is a data of the Cousin 1 Problem. As before. for each $\mathscr{X} \in (\mathscr{M}/\mathscr{A})(X)$ there exists an element $\sigma \in \mathscr{M}(X)$ such that $p(\sigma) = \mathscr{X}$ under the condition that X is a Cousin 1 Domain. Thus p is surjective and hence i^*: $H^1(X,\mathscr{A}) \to H^1(X,\mathscr{M})$ is injective.

Next, we assume that i^*: $H^1(X,\mathscr{A}) \to H^1(X,\mathscr{M})$ is injective. This implies that for every $\mathscr{X}\in(\mathscr{M}/\mathscr{A})(X)$ $\delta(\mathscr{X})=0$ *i.e*, p: $\mathscr{M}(X)\to(\mathscr{M}/\mathscr{A})(X)$ is surjective. Since each data of the Cousin 1 Problem can be represented by an element $\mathscr{X}\in(\mathscr{M}/\mathscr{A})(X)$, and there exists an element $\sigma\in\mathscr{M}(X)$ such that $p(\sigma)=\mathscr{X}$ if $\mathscr{X}$ is a data $\{\sigma_i\in\mathscr{M}(U_i),\ i\in I\}$ of the Cousin 1 Problem then σ is a solution of this Problem, *i.e*, $\sigma-\sigma_i\in\mathscr{A}(U_i)$. Therefore X is a Cousin 1 Domain.

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