ENDOMORPHISM RINGS OF MODULES OVER DEDEKIND-LIKE RINGS*

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One of the ways of measuring the complexity of the category of modules over a ring $R$ is to ask what rings can occur as endomorphism rings of $R$-modules (always finitely generated in this paper). (See, for example, [2]).

Given an $R$-module $M$ let

$$E(M) = (\text{end } M) / (\text{nilradical of end } M)$$

To state the motivating case of the two main results of this paper, let $R = \mathbb{Z}G_p$, the integral group ring of the cyclic group $G_p$ of prime order $p$. We prove:

(0.2) If $M$ is indecomposable and non-artinian, then $E(M)$ can only be one of three different rings; but

(0.3) Every finite field of characteristic $p$ can occur as $E(M)$ for some indecomposable artinian $M$.

A strikingly similar result, in quite a different setting, has been obtained by D. Farkas and R. Snider. The considered the (noncommutative) Weyl algebra $W = K[x, y]$ where $xy - yx = 1$ and $K$ is a field of characteristic zero. If $M$ is an indecomposable non-artinian $W$-module, then it is easily seen that $\text{end } M$ can only be an integral domain Morita-equivalent to $W$. Farkas and Snider showed that every finite dimensional central $K$-division algebra can occur as $\text{end } M$ for a suitable simple $W$-module $M$. See [3].

To place our results (0.2) and (0.3) into an appropriate context, we generalize them somewhat. A Bass ring is a commutative ring without nilpotent elements, with module finite integral closure, such that every ideal is generated by two elements. In [6], L. Levy defined a subclass of Bass rings called Dedekind-like ring (This definition is repeated in §1.) and classified all finitely generated modules over the ring. Examples of Dedekind-like rings are the group ring $\mathbb{Z}G_n$ with $G_n$ cyclic of squarefree order $n$, some rings of algebraic integers that are not integrally closed in their quotient field, and subrings of $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$

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with square-free index as a subgroup.

Now let $M$ be an $R$-module, $R$ Dedekind-like. Our generalization of $(0.2)$ is:

$(0.4)$ As $M$ ranges over all indecomposable non-artinian modules, $\tilde{E}(M)$ can be only finitely many non-isomorphic rings. Each of these is the endomorphism ring of some ideal of $R$.

Before stating the generalization of $(0.3)$ we remark that if $M$ is an indecomposable module of finite length over any commutative ring $R$, then all of the composition factors of $M$ are isomorphic to each other, because $M \cong M_P$ for some maximal ideal $P$ of $R$. ([1], Chap. 4, §2.5, Prop.8).

We prove, for modules $M$ over Dedekind-like rings $R$:

$(0.5)$ Let $P$ be a maximal ideal such that $R_P$ is not a discrete valuation ring, and let $M$ range over all indecomposable artinian $R$-modules whose composition factors are $\cong R/P$. Then, the rings $\tilde{E}(M)$ are precisely the simple algebraic extension fields of $R/P$.

In proving $(0.5)$, the following theorem from linear algebra will be crucial: any matrix which commutes with a companion matrix $F$ is a polynomial in $F$. ([4], Chap. III, §2, Corollary 1 to Theorem 2). If $R_P$ is a discrete valuation ring, the problem becomes uninteresting because $M \cong R/P^e$ for some $e$, hence $\tilde{E}(M) \cong R^e$.

1. Fixed notation, Translation to $R$-diagrams

DEFINITION 1.1 (of Dedekind-like ring) To define a Dedekind-like ring $R$, let $f$ and $g$ be ring homomorphisms: $R$ onto $\tilde{R}$, where $\tilde{R}$ is a direct sum of Dedekind domains, none of which is a field, and $R$ is a direct sum of fields.

$$R = \bigoplus c R_c \quad \tilde{R} = \bigoplus k \tilde{R}_k$$

Here the subscripts $c$ and $k$ extend over fixed, but unspecified finite index sets. We suppose that $f$ and $g$ satisfy the independence condition

$$(2) \quad K_f + K_g = \tilde{R} \quad (K_f = \ker f, \ K_g = \ker g)$$

and then define our Dedekind-like ring to be the (generalized) pullback

$$(3) \quad \tilde{R} = \text{pbk}(f, g) = \{ x \in \tilde{R} | f(x) = g(x) \}$$

Notations $(1) \sim (3)$ will retain its significance throughout this paper, unless otherwise stated.

EXAMPLES 1.2 For $R = \mathbb{Z}G_p$, we have

$$(1) \quad \tilde{R} = \mathbb{Z} \oplus \mathbb{Z}[\xi] \quad R = \mathbb{Z}/p\mathbb{Z}$$

where $\xi$ is a primitive $p^{th}$ root of unity (in the complex numbers). In fact, if
we define $f$ and $g : \bar{R}$ onto $\bar{R}$ by
\[
(2) \quad f(a,b) = \bar{a}, \quad g(a, \Sigma b_i \epsilon^i) = \Sigma b_i.
\]
Then the pullback (3) of 1.1 is isomorphic to $\mathbb{Z}G_{p'}$. See [5].

For the description of $\mathbb{Z}G_n$, $n$ square-free, as a Dedekind-like ring, see [7].

**REDUCTION 1.3** The fundamental tool used to describe $R$-modules ($R$ Dedekind-like) in [6] is called an "$R$-diagram". It consists of 3 modules and 4 maps, as shown in diagram $\mathcal{D}$.

\[
(\mathcal{D}) \quad \bar{K} \xrightarrow{\gamma, \delta} \bar{S} \xrightarrow{\tilde{f}, \tilde{g}} \bar{S}
\]

Here $\bar{S}$ is an $\bar{R}$-module; and $\bar{K}$ and $\bar{S}$ are $\bar{R}$-modules. The maps $\gamma, \delta$ are $\bar{R}$-monomorphisms. The maps $\tilde{f}, \tilde{g}$ are $\bar{R}$-epimorphisms, and these maps are required to satisfy some additional conditions stated in [6], Definition 2.1. Instead of stating these conditions in general, we will state particular forms they take, as we need them.

With each $R$-diagram $\mathcal{D}$ we associate an $R$-module
\[
(1) \quad M(\mathcal{D}) = \text{pbk}(\tilde{f}, \tilde{g})/\text{im}(\gamma + \delta)
\]
where pbk$(\tilde{f}, \tilde{g})$ denotes the set of all $s$ in $\bar{S}$ such that $f(s) = g(s)$. The significance of this is that ([6], Theorem 2.4)

(2) The correspondence $\mathcal{D} \rightarrow M(\mathcal{D})$ is an additive functor that is a representation equivalence from the category of all $R$-diagrams to that of all (finitely generated) $R$-modules. In particular:
(i) Every $R$-module is $M(\mathcal{D})$ for some $\mathcal{D}$
(ii) $M(\mathcal{D}) \simeq M(\mathcal{E})$ if and only if $\mathcal{D} \simeq \mathcal{E}$
(iii) $M(\mathcal{D})$ is an indecomposable $R$-module if and only if $\mathcal{D}$ is an indecomposable $R$-diagram.

An endomorphism $\theta$ of $\mathcal{D}$ is defined to be a triple of module homomorphisms $\hat{\theta}, \tilde{\theta}$, and $\bar{\theta}$ such that the squares in (3) commute, and such that the squares in
\[
\begin{array}{ccc}
\bar{K} & \xrightarrow{\gamma} & \bar{S} & \xrightarrow{\tilde{f}} & \bar{S} \\
\downarrow{\hat{\theta}} & & \downarrow{\tilde{\theta}} & & \downarrow{\bar{\theta}} \\
\bar{K} & \xrightarrow{\gamma} & \bar{S} & \xrightarrow{\tilde{f}} & \bar{S}
\end{array}
\]
($\hat{\theta}$ and $\tilde{\theta}$ $\bar{R}$-linear, $\bar{\theta}$ $\bar{R}$-linear)

the diagram (3)' obtained from (3) by replacing $\gamma$ and $\tilde{f}$ by $\delta$ and $\tilde{g}$ also-commute.
The direct sum of two R-diagrams can be constructed by taking the direct sum of the modules and maps in each of them. This is again an R-diagram. Thus we can call an R-diagram indecomposable if it is non-zero and is not isomorphic to the direct sum of two nonzero R-diagrams.

The relation between R-diagrams and endomorphism rings of R-modules is that, for each R-diagram \( \mathcal{D} \) and the corresponding R-module \( M(\mathcal{D}) \), (4) as follows holds.

(4) There is a natural ring homomorphism of endomorphism rings:
\[
\text{end}\mathcal{D} \to \text{end} M(\mathcal{D})
\]
whose kernel is an ideal of square zero. ([6], Lemma 2.7)

Let \( \overline{E}(\mathcal{D}) = (\text{end}\mathcal{D})/(\text{nilradical of end}\mathcal{D}) \). Then it follows immediately from (4) that

(5) \( \overline{E}(\mathcal{D}) \cong \overline{E}(M(\mathcal{D})) \) (ring isomorphism)

It then follows from (2) and (4) that we can now translate the problem of computing \( \overline{E}(M) \) for an arbitrary R-module \( M \) to that of computing \( \overline{E}(\mathcal{D}) \) for an arbitrary R-diagram \( \mathcal{D} \). What has been gained is that R-diagrams consist of familiar objects: modules over \( \hat{R} \) (a direct sum of Dedekind domains) and over \( \hat{R} \) (a direct sum of fields), together with homomorphisms of these modules.

2. \( \overline{E}(M) \) when \( M \) is indecomposable of finite length

Over any commutative ring, an indecomposable R-module \( M \) of finite length is isomorphic to \( M_p \) for some maximal ideal \( P \) of \( R \). In turn, \( M_p \cong M_p(P) \) the \( P \)-adic completion. Thus, we may assume that \( R = R_p(P) \), which is either a discrete valuation ring or isomorphic to a pullback,

(1) \[
\begin{array}{c}
R_1 \xrightarrow{f} \hat{R} \xleftarrow{g} R_2 \\
\end{array}
\]

where the rings \( R_1 \) and \( R_2 \) are discrete valuation rings, the maps \( f \) and \( g \) are ring homomorphisms, and \( \hat{R} \cong R/P \) ([6], §6).

The objective of this section is to study what rings can occur as \( \overline{E}(M) \). (see (0.1)) If \( R \) is a discrete valuation ring, it is immediate that \( \overline{E}(M) \cong R/P \) since \( M \cong R/P^e \) for some \( e \). Hence, we focus on the case of \( R \) a pullback as in (1) for the rest of this section and prove that the rings \( \overline{E}(M) \) are precisely the simple algebraic extension fields of \( \hat{R} \cong R/P \).

The indecomposable R-modules of finite length are of two types called "artinian deleted cycle indecomposables" and "block cycle indecomposables". (See [5], Theorem 9.4 and the proof) We begin by describing the R-diagrams of the block cycle indecomposables and then proceed to work with the endo-
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morphism rings of these \( R \)-diagrams.

**R-DIAGRAM 2.1** (of a block cycle indecomposable) Let \( R \) be a pullback as in (1) above. The \( R \)-diagram \( D \) for a block cycle indecomposable \( M(D) \) is given as follows:

\[
\begin{array}{ccc}
\gamma = A & \xrightarrow{(T_\mu)^r} & B = C \\
\mu = 1 & & \mu = 1 \\
R^n & \xrightarrow{\delta = B} & (S_\mu)^r \\
\end{array}
\]

where

(i) The exponent \( r \) denotes a direct sum of \( r \) copies and \( n = mr \). Each \( T_\mu \) (respectively, \( S_\mu \)) is an \( R_1 \) (respectively, \( R_2 \)) module of the following form:

\[
T_\mu = R_1/(\ker f)^{d(\mu)}, \quad d(\mu) > 0 \quad (\text{respectively}, \quad S_\mu = R_2/(\ker g)^{e(\mu)}, \quad e(\mu) > 0) \quad \text{for} \quad 1 \leq \mu \leq m.
\]

In what follows, whenever \( d(\mu) \) and \( e(\mu) \) do not appear as superscripts, we write \( d_\mu \) for \( d(\mu) \) and \( e_\mu \) for \( e(\mu) \).

(ii) The maps \( \gamma \) and \( \delta \) are \( R_1 \)-linear when \( \bar{R} \) is considered as an \( R_1 \)-module via the map \( f : R_1 \to \bar{R} \) in the definition of \( R \). Similarly, the maps \( \delta \) and \( \bar{g} \) are \( R_2 \)-linear via the map \( g \).

(iii) The map \( \gamma \) is identified with left multiplication by a matrix \( A \) of maps, where the entry \((\mu, j)\) of \( A \) maps the coordinate \( j \) of \( \bar{R}^n \) to \( T_\mu \). A similar comment applies to each of \( \delta, \bar{f} \) and \( \bar{g} \). Moreover, we identify each entry of these matrices with an element of \( \bar{R} \) as follows:

For each \( \mu \), fix an arbitrary \( R \)-isomorphism called \( \bar{i} : \bar{R} \to \text{Socle } T_\mu \). Then, the \((\mu, j)\) entry \( \bar{\gamma} \) of the matrix \( A \) denotes the map \( \bar{i} \) followed by actual multiplication by \( \bar{f} \). Each entry of the matrix \( B \) denotes a similar map with \( S_\mu \) in place of \( T_\mu \). The \((i, \mu)\) entry \( \bar{\gamma} \) of the matrix \( C \) denotes a map from \( T_\mu \) to the coordinate \( i \) of \( \bar{R}^n \) given by \( x + (\ker f)^{e(\mu)} \to \bar{f}f(x) \). Similarly, the \((i, \mu)\) entry \( \bar{\gamma} \) of the matrix \( D \) denotes a map from \( S_\mu \) to the coordinate \( i \) of \( \bar{R}^n \) given by \( x + (\ker g)^{e(\mu)} \to \bar{g}g(x) \).

(iv) The matrices \( A, B, C \) and \( D \) which actually occur are as follows. \( C \) and \( D \) are the identity matrix, that is \( C = D = I_n' \).

\((A, B)\) is a block cycle pair (over \( \bar{R} \)), that is:
the companion matrix of some power $g(x)^e$ of an irreducible polynomial $g(x) \neq x$ over $\overline{R}$.

We note that each block $A_i$ (respectively, each block $I_r$ in $B$) is a map from a direct sum of $r$ copies of $\overline{R}$ into some $(T_\mu)^r$ (respectively, $(S_\mu)^r$).

(v) With each $(T_\mu)^r$ and $(S_\mu)^r$, a pair of integers $(d_\mu, e_\mu)$ is associated, where $d_\mu =$ the composition length of $T_\mu$ and $e_\mu =$ the composition length of $S_\mu$. $(d_\mu, e_\mu)$ is called a block weight. The sequence of block weights $(d_1, e_1), (d_2, e_2), \ldots, (d_m, e_m)$ is nonrepeated, that is, it cannot be obtained by merely writing a shorter sequence two or more times.

**ENDOMORPHISMS OF R-DIAGRAMS 2.2** An endomorphism of an $R$-diagram was defined in Reduction 1.3. Translated into the particular context of the $R$-diagram $\mathcal{D}$, an endomorphism of $\mathcal{D}$ consists of matrix 4 tuples $(t, s, w, v)$ such that all the squares in the following diagram commute.
For the needs of the proof that follows, we consider \( t, s, w \) and \( v \) as \( m \times m \) matrices of blocks of size \( r \), and hence \((t)_{ij}\) denotes the \((i, j)\) block of the matrix \( t \), etc. Each entry of \((t)_{ij}\) is a homomorphism from \( T_j/R_1/P_1^{d(j)} \) to \( T_i/R_1/P_1^{d(i)} \) and thus may be considered a multiplication by an element in \( R_1 \) (not necessarily unique). Similarly, each entry of \((s)_{ij}\) is in \( R_2 \).

Before investigating some relations among the entries of \( t, s, w, \) and \( v \) imposed by the commutativity of (1) above, we make a remark on a notation that will come up frequently.

**RemarK 2.3** We write \((\bar{t})_{ij}\) for the matrix obtained by taking the homomorphic image in \( \bar{R} \) of each entry of \((t)_{ij}\) under the ring homomorphism \( f \), and similarly \((\bar{s})_{ij}\) for the matrix obtained from \((s)_{ij}\) using the homomorphism \( g \) in place of \( f \).

**Lemma 2.4** Let \((t, s, w, v)\) be as in 2.2. Then, \((\bar{t})_{ij} = w_{ij} = (\bar{s})_{ij}\) for \( 1 \leq i, j \leq m \).

**Proof.** We obtain from (1) of 2.2 for fixed \( i \) and \( j \),

\[
\begin{array}{c}
(T)_i^r \xrightarrow{f_r} \bar{R}^r \xleftarrow{f_r} (S)_i^r \\
\downarrow \quad \quad \downarrow \\
(T)_j^r \xrightarrow{f_r} \bar{R}^r \xleftarrow{f_r} (S)_j^r
\end{array}
\]

From the commutativity of this diagram it immediately follows that \((\bar{t})_{ij} = (w)_{ij} = (\bar{s})_{ij}\).

**Lemma 2.5** (i) Each matrix \((\bar{t})_{ii}\) for \( 1 \leq i \leq m \) satisfies \((\bar{t})_{ii}A_m = A_m(\bar{t})_{ii}\). In particular, the matrix \( F \) is a polynomial in \( A_m \).

(ii) \((\bar{t})_{ii} = (\bar{t})_{jj}\) for all \( i, j \) such that \( 1 \leq i, j \leq m \).
PROOF. We first define a block diagram \((D)_{ij}\) to be

\[
\begin{array}{ccc}
(T_i)_{r'} & \leftarrow & R' & \rightarrow & (S_{ii})_{r'} \\
\downarrow & & \downarrow & & \downarrow \\
(T_i)_{r'} & \leftarrow & R' & \rightarrow & (S_{ii})_{r'}
\end{array}
\]

where we consider the indices \(i\) and \(j\) to be in the group \(\mathbb{Z}/m\mathbb{Z}\). Suppose \(i = j\) and \(1 \leq i \leq m\). Then, \(A_{i-1} = I_r\), and from the commutativity of (2) above it follows that \((\tilde{I})_{ii} = (v)_{i-1i-1} = (s)_{i-1i-1}\). Combined with Lemma 2.4 we obtain

\[
(s)_{mm} = (\tilde{I})_{mm} = (s)_{m-1m-1} = (\tilde{I})_{m-1m-1} = \ldots = (s)_{22} = (\tilde{I})_{22} = (3)_{11} = (\tilde{I})_{11}.
\]

Hence, (ii) is proved. Now suppose \(i = j = 1\) in (2). We obtain \((\tilde{I})_{11} A_m = A_m (v)_{mm}\) and \((v)_{mm} = (3)_{mm}\) by the commutativity of the diagram. Consequently, \((\tilde{I})_{11} A_m = A_m (s)_{mm} = A_m (\tilde{I})_{mm} = A_m (\tilde{I})_{11}\), that is, \((\tilde{I})_{11}\) commutes with \(A_m\). Any matrix which commutes with \(A_m\), a companion matrix, is a polynomial in \(A_m\). ([4], Chap III, §2, Corollary 1 to Theorem 2). Thus, (i) is proved.

Our first goal is to show that there exists a permutation of the matrizing choices that occur in the diagram \(D\) such that every \(\tilde{I}\) (hence \(s\)) is \(r \times r\) block upper triangulizable. That is, there exists a permutation matrix \(Q\) such that every \(Q\tilde{I}Q^{-1}\), when viewed in \(r \times r\) blocks, is upper triangular. This fact will facilitate the analysis of the structure of the endomorphism ring of the diagram \(D\). Our triangularization is based upon the following easily proven statement.

REMARK 2.6 Let \(N = (n_{ij})\) be a square matrix over \(\mathbb{R}\). \(N\) can be put into an upper triangular form by a simultaneous permutation of its rows and columns if and only if there exists a total ordering \(\geq\) on the set of subscripts such that \(n_{ij} = 0\) whenever \(i \geq j\).

We begin by defining an ordering on pairs of integers as follows.

DEFINITION 2.7 For integers \(a, b, c\) and \(d\), we denote \((a, b) \geq (c, d)\) if

(i) \(a \geq c\) or
(ii) \(a = c\) and \(b < d\)

DEFINITION 2.8 We inductively define an ordering \(\geq\) on the set of integers:
modulo $m$: Let $i \nerves j$ mean that

(i) $(d_i, e_{i-1}) \nerves (d_j, e_{j-1})$ or

(ii) $(d_i, e_{i-1}) = (d_j, e_{j-1})$ and $i-1 \nerves j-1$

**Lemma 2.9** If $i \nerves j$, then $(\bar{e})_{ij} = 0$

**Proof.** We consider the block diagram $(\mathcal{Q})_{ij}$ as defined in the proof of Lemma 2.5. If $d_i > d_j$, it is immediate that $(\bar{e})_{ij} = 0$ since $(t)_{ij}$ is a homomorphism from $(T_j)^T$ to $(T_j)^{T'}$, and the length of $T_j = d_j < d_i$ is the length of $T_i$. If $d_i = d_j$ and $e_{i-1} < e_{j-1}$, we obtain $(t)_{ij}A_j = A_{i-1}(v)_{i-1j-1}$ from the commutativity of the left-hand square of $(\mathcal{Q})_{ij}$. Also, the map $(s)_{i-1j-1}$ sends the socle of $(S_{j-1})^r$ to zero since the length of $S_{j-1} = e_{j-1} > e_{i-1}$ is the length of $S_{i-1}$. This combined with the commutativity of the right-hand square and the invertibility of the matrix $I_r$ forces $(v)_{i-1j-1} = 0$. Hence it follows that $(\bar{e})_{ij} = 0$. If $d_i = d_j$ and $e_{i-1} = e_{j-1}$, then $i-1 \nerves j-1$ and thus $(\bar{e})_{i-1j-1} = 0$ by induction. This together with the commutativity of $(\mathcal{Q})_{ij}$ results in

$$(\bar{e})_{ij}A_j = A_{i-1}(v)_{i-1j-1} = A_{i-1}(3)_{i-1j-1} = A_i(\bar{e})_{i-1j-1} = 0$$

Therefore, $(\bar{e})_{ij} = 0$.

Next lemma shows that $\nerves$ is a total ordering.

**Lemma 2.10** Suppose $i \nerves j$ and $j \nerves k$ modulo $m$. Then,

(i) $i \nerves j$ or $i \lhd j$.

(ii) $i \nerves j$ and $j \nerves k$ imply $i \nerves k$.

**Proof.** For (i) we only have to show there exists an integer $n \geq 0$ such that $(d_{i-n}, e_{i-n}) \nerves (d_{j-n}, e_{j-n})$. Suppose not. Then, we have

$$(3) \quad (d_i, e_{i-1}) = (d_j, e_{j-1}), \ (d_{i-1}, e_{i-2}) = (d_{j-1}, e_{j-2}), \ldots$$

$$(d_{i-m+1}, e_{i-m}) = (d_{j-m+1}, e_{j-m}).$$

We will derive a contradiction to the non-repeatedness of the block weights as mentioned in 2.1 (v). Let $p = i-j$ and $(p, m) = q$. There exist integers $a$ and $b$ such that $ap + bm = q$. It is easily seen from (3) that

$$(d_i, e_i) = (d_{i+ap}, e_{i+ap}) = (d_{i+ap+bm}, e_{i+ap+bm}) = (d_{i+q}, e_{i+q}).$$

Therefore, the sequence of block weights $(d_1, e_1), (d_2, e_2), \ldots, (d_m, e_m)$ is obtained by writing the sequence $(d_1, e_1), (d_2, e_2), \ldots, (d_q, e_q)$ $m/q$ times. A contradiction.
For (ii) we note that there exists a smallest integer \( u \geq 0 \) such that \((d_{i-u}, e_{j-u-1}) \gg (d_{j-u}, e_{j-u-1})\) or \((d_{j-u}, e_{j-u-1}) \gg (d_{k-u}, e_{k-u-1})\). It is easy to see that this implies \((d_{i-\mu}, e_{i-\mu-1}) = (d_{k-\mu}, e_{k-\mu-1})\) for \(0 \leq \mu < u\) and \((d_{i-u}, e_{i-u-1}) \gg (d_{k-u}, e_{k-u-1})\). Thus, \(i \gg k\).

Before stating our "triangularization theorem" as was mentioned immediately preceding Remark 2.6, we make a pertinent definition.

**DEFINITION 2.11** (of a permuted \(R\)-diagram) Let \(D\) be an \(R\)-diagram for a block cycle indecomposable, and \(Q\) an \(m \times m\) permutation matrix of \(r \times r\) blocks over \(R\), that is, an \(m \times m\) matrix each of whose entries is \(I_r\) or \(0_r\). Then, the corresponding permutation of \((T_\mu)^r\)s and \((S_\mu)^r\)s gives the permuted diagram \(D^Q\):

\[
\begin{array}{c}
\text{\(QA\)} \\
\text{\(QB\)} \\
\text{\(CQ^{-1}\)} \\
\text{\(DQ^{-1}\)}
\end{array}
\]

\[
\begin{array}{c}
\mu = 1 \quad \{T_\mu\}^r \\
\mu = 1 \quad \{S_\mu\}^r
\end{array}
\]

where \(\sum_{\mu = 1}^{m} (T_\mu)^r_Q\) and \(\sum_{\mu = 1}^{m} (S_\mu)^r_Q\) denote the permuted direct sums.

**THEOREM 2.12** Let \(D\) be an \(R\)-diagram for a block cycle indecomposable. Then, there exists a permuted diagram \(D^Q\) such that the matrices \(\bar{t}\) and \(\bar{s}\) obtained from every endomorphism \((t, s, w, v)\), when viewed in \(r \times r\) blocks, are upper triangular. Furthermore, \(T_\mu\)s after being permuted are arranged according to increasing composition lengths.

**PROOF.** The fact that the matrices \(\bar{t}\) are \(r \times r\) block upper triangular is immediate from Lemma 2.9, Lemma 2.10 and Remark 2.6 with \(r \times r\) blocks in place of \(n_{ij}\)'s. Then, it follows from Lemma 2.4 that the matrices \(\bar{s}\) are \(r \times r\) block upper triangular. The second statement follows immediately from the way the ordering \(\gg\) is defined in Definition 2.7 and Definition 2.8.

We now easily obtain some interesting properties of end \(D\).
COROLLARY 2.13 Let $\mathcal{D}$ be an $R$-diagram of a block cycle indecomposable and $E=\text{End} \mathcal{D}$. Let

$$N=\{(t, s, w, v) \in E | (\bar{t})_{ii}=0 \text{ for } 1 \leq i \leq m\}.$$ 

Then, $N$ is a nilpotent ideal of $E$, and $E/N \cong R[x]/\langle g(x) \rangle^e$, where the polynomial $g(x)$ is as in (iv) of 2.1.

PROOF. We may assume that $\bar{t}$ is $r \times r$ block upper triangular by Theorem 2.12. Therefore, it is immediate that $N$ is a nilpotent ideal of $E$. It follows from Lemma 2.5 that

$$E/N \cong \text{the ring of } r \times r \text{ block diagonal matrices of the form}$$

$$\begin{pmatrix}
(\bar{t})_{11} & & & 0 \\
& (\bar{t})_{22} & & \\
& & & \\
0 & \cdots & & (\bar{t})_{mm}
\end{pmatrix}
$$

Moreover, $(\bar{t})_{ii}$ is of the form $f(A_m)$ for some $f(x) \in R[x]$.

Therefore, $E/N \cong R[x]/\langle g(x) \rangle^e$ because $A_m$ is the companion matrix of the polynomial $g(x)^e$.

COROLLARY 2.14 Keep notations as in Corollary 2.13. Then, $\bar{E}(\mathcal{D}) \cong \text{the field } R[\alpha]$ where $g(\alpha)=0$.

PROOF. The ideal $\langle g(x) \rangle/\langle g(x) \rangle^e$ is the nilradical of $R[x]/\langle g(x) \rangle^e$, and $R[x]/\langle g(x) \rangle^e$ modulo this ideal is $\cong R[x]/\langle g(x) \rangle \cong R[\alpha]$ where $g(\alpha)=0$.

We now look at the artinian deleted cycle indecomposables briefly.

$R$-DIAGRAM 2.15 (of an artinian deleted cycle indecomposable) The $R$-diagram $\mathcal{D}$ for an artinian deleted cycle indecomposable is given as follows:
where

(i) \( T_\mu = R_1 / (\ker f)^{d(\mu)}, \quad d(\mu) > 0 \) and \( S_\mu = R_2 / (\ker g)^{e(\mu)}, \quad e(\mu) > 0 \) for \( 1 \leq \mu \leq n \).

(ii) The maps \( r \) and \( \bar{r} \) are \( R_1 \)-linear, and the maps \( \delta \) and \( \bar{g} \) are \( R_2 \)-linear as mentioned in (ii) of 2.1. Each of the maps \( r, \delta, \bar{f} \) and \( \bar{g} \) is viewed as left multiplication by the matrices \( A, B, C \) and \( D \) of the maps respectively as in (iii) of 2.1.

(iii) The matrices \( A, B, C \) and \( D \) that actually occur are as follows: \( C = D = I_n \). (\( A, B \)) is a deleted cycle pair (over \( \bar{R} \)), that is,

\[
A = \begin{bmatrix}
... & 0 & ... \\
1 & 0 & 1 \\
I_{n-1} & 0 & I_{n-1}
\end{bmatrix}
\]

An endomorphism of \( \mathcal{D} \) consists of matrix 4 tuples \( (t, s, w, v) = ((t_{ij}), (s_{ij}), (w_{ij}), (v_{ij})) \) similar to 2.2 except that each “block” is \( 1 \times 1 \), that is, each entry \( t_{ij} \) is a map from \( T_j \) to \( T_i \), etc. Concerning the endomorphism ring of \( \mathcal{D} \) we have the following.

**Lemma 2.16** If \( \mathcal{D} \) is an \( \mathcal{R} \)-diagram of an artinian deleted cycle indecomposable, then \( \bar{E}(\mathcal{D}) \cong \bar{R} \).

**Proof.** The proof carries over with some modification from that of the non-artinian case in §3. We may think of \( \mathcal{D} \) as a \( \mathcal{D}_{\text{del}} \) in 3.3 in which \( (A, B) \) is a single deleted cycle pair and \( T_\mu \)'s and \( S_\mu \)'s have finite lengths. Then, the results in Lemma 3.7 through Theorem 3.12 still remain true. We now define a ring homomorphism

\[
\tau : \text{end} (\mathcal{D}) \longrightarrow \bar{R} \text{ by } \theta = (t, s, w, v) \longrightarrow \bar{t}_{11}.
\]

It is clear that \( \tau \) is a ring homomorphism since we may assume that \( \bar{t} \) and \( \bar{s} \) are upper triangular by Theorem 3.13. The surjectivity of \( \tau \) is easily established using Lemma 3.7 and Lemma 3.8.

Suppose \( \tau(\theta) = 0 \). Then, by Lemma 3.8, the diagonal entries of \( \bar{t} \) are all zero, and hence \( \bar{t} \) is strictly upper triangular. We also conclude from Lemma 3.7 that \( \bar{s} \) is strictly upper triangular. Conversely, it follows from the definition of \( \tau \) that an endomorphism which satisfies these properties is in \( \ker \tau \). Hence \( \bar{t}^n = \bar{s}^n = 0 \). This says that the entries of \( t^n \) are contained in \( \ker f \) and those of \( s^n \) in \( \ker g \). If \( e = \max \{d_i, e_i\} \leq n \), \( t^{ne} \) and \( s^{ne} \) are the zero map. Thus, \( \ker \tau \) is nilpotent, and in fact is the nilradical of \( \text{end} (\mathcal{D}) \) since \( \bar{R} \) is semiprime.
Therefore, $E(D) \simeq \mathbb{R}$.

Putting together the results which are obtained so far, we obtain our main theorem.

**THEOREM 2.18** Let $R$ be a Dedekind-like ring, and let $P$ be a maximal ideal such that $R_P$ is not a discrete valuation ring. Then, as the modules $M$ range over all indecomposable artinian $R$-modules whose composition factors are isomorphic to $R/P$, the rings $E(M)$ are precisely the simple algebraic extension fields of $R/P$.

**PROOF.** Given any such module $M$, let $D$ be the corresponding $R$-diagram. Then, $\tilde{E}(D)$ is a simple algebraic extension field of $\bar{R}=R/P$ by Corollary 2.14. But, $\tilde{E}(D) \simeq \tilde{E}(M)$ by (5) of 1.3. Conversely, consider a simple algebraic extension field $F$ over $\bar{R}$. Then, $F=\bar{R}[\alpha]$ for some $\alpha \in F$. Let $g(x) \in \bar{R}[x]$ be the minimal polynomial of $\alpha$. It is easy to construct an $R$-diagram $D$ of a block cycle indecomposable $M$ such that $A_m$ is the companion matrix of $g(x)$. (See 2.1) Then, $\tilde{E}(M) \simeq \tilde{E}(D) \simeq F$ by Corollary 2.14.

**EXAMPLES 2.19** (i) The integral group ring $\mathbb{Z}G_p$, $G_p$ cyclic of order $p$, is a Dedekind-like ring as shown in Examples 1.2. Let $R=\mathbb{Z}G_p$, and let $g,g^2,\ldots,g^p$ denote the elements of $G_p$. Also denote an arbitrary element in $R$ by $\sum_{i=1}^p a_i g^i$, $a_i \in \mathbb{Z}$. Then,

$$P=\{\sum_{i=1}^p a_i g^i \in R | \sum_{i=1}^p a_i \equiv (p)\}$$

is the kernel of the map $f \oplus g : R \to \bar{R}$ in Examples 1.2. Hence, $P$ is a maximal ideal of $R$ such that $R_P$ is not a discrete valuation ring ([6], Proposition 6.2). Therefore, as the modules $M$ range over all indecomposable artinian $R$-modules whose composition factors are $\simeq R/P \simeq \mathbb{Z}/p\mathbb{Z}$, the rings $E(M)$ are precisely the finite fields of characteristic $p$.

(ii) Let $m$ be a positive integer such that $m \equiv 1 \pmod{4}$. Then, $R=\mathbb{Z}[\sqrt{m}]$ is a Dedekind-like ring: The integral closure of this ring in $\mathbb{Q}(\sqrt{m})$ is $\mathbb{Z}[\omega]$ where $\omega=(-1+m)/2$. Define two ring homomorphisms: $R=\mathbb{Z}[\omega]$ onto $R=\mathbb{Z}/2\mathbb{Z}$ by $f(a+b\omega)=\overline{a}$, $g(a+b\omega)=\overline{a+b}$. Then,

$$\{a+b\omega \in \mathbb{Z}[\omega] | f(a+b\omega)=g(a+b\omega)\} = \mathbb{Z}[2\omega] = \mathbb{Z}[\sqrt{m}] .$$

Let $P$ be either of the two maximal ideals (ker $f$ and ker $g$) of $R=\mathbb{Z}[\sqrt{m}]$ such that $R/P \simeq \mathbb{Z}/2\mathbb{Z}$. Then, as the modules $M$ range over all indecomposable artinian $R$-modules whose composition factors are $\simeq R/P \simeq \mathbb{Z}/2\mathbb{Z}$, the rings
\[ \bar{E}(M) \text{ are precisely the finite fields of characteristic 2.} \]

3. **\( \bar{E}(M) \) when \( M \) is Non-Artinian Indecomposable**

In this section we show that as \( M \) ranges over all indecomposable non-artinian modules, \( \bar{E}(M) \) can be only finitely many non-isomorphic rings, unlike the artinian case. It also turns out that each of these rings is isomorphic to the endomorphism ring of some ideal of \( R \).

The outline of the development of the proofs is similar to that of the artinian case although some details are quite different. We first look at the structure of the \( R \)-diagrams of indecomposable non-artinian \( R \)-modules and then work with the endomorphism rings of these \( R \)-diagrams. In order to study the \( R \)-diagrams, we note that a Dedekind-like ring \( R \) as in Definition 1.1 can be redefined as follows.

**REMARK 3.1** A Dedekind-like ring \( R \) as in Definition 1.1 can be redefined in terms of notation which focuses on the coordinate rings from which \( \bar{R} \) and \( \bar{R} \) are built. Let \( \varphi_k \) be the composition of \( f \) with the coordinate projection: \( \bar{R} \rightarrow \bar{R}_k \). There is a unique index \( i(k) \) such that the projection of ker \( \varphi_k \) in \( R_{i(k)} \) is a maximal ideal of \( R_{i(k)} \). Let \( f_k \) be the restriction of \( \varphi_k \) to \( R_{i(k)} \). Define \( g_k \) similarly. Hence for each \( k \) we obtain a pair of ring homomorphisms:

\[ R_{i(k)} \xrightarrow{f_k} \bar{R}_k \]

\[ R_{j(k)} \xrightarrow{g_k} \bar{R}_k \]

[possibly \( i(k) = j(k) \)]

It is easy to check that ([6], §3) whenever two terms of the sequence \( f_1, g_1, f_2, g_2, \ldots \) are defined on the same coordinate \( R_c \) of \( \bar{R} \), they have distinct kernels. (See (2) of Definition 1.1), and \( R = \{ (r_1, r_2, \ldots) \in \bar{R} \mid (\text{for all } k) f_k(r_{i(k)}) = g_k(r_{j(k)}) \} \).

We make some definitions that are necessary for stating the structure of \( R \)-diagrams.

**DEFINITION 3.2** From each isomorphism class of nonzero ideals of each \( R_c \), choose one ideal \( H_c \) such that \( H_c \) is prime to ker \( f_k \) whenever \( i(k) = c \), and to ker \( g_k \) whenever \( j(k) = c \). If \( H_c \) is principal, take \( H_c = R_c \). \( H_c \) is called the-
standard $R_c$-ideal in its class. An $(f, k)$-matrizing choice means either a standard $R_i(k)$-ideal or a module of the form $\frac{R_i(k)}{(\ker f_j)^d}$ ($d > 0$). A $(g, k)$-matrizing choice means either a standard $R_j(k)$-ideal or a module of the form $\frac{R_j(k)}{(\ker g_j)^e}$ ($e > 0$).

**R-DIAGRAMS 3.3** (of indecomposable non-artinian $R$-modules) The $R$-diagram $D$ of an $R$-module $M$ without artinian direct summands is of the form (See 1.3):

\[
\begin{align*}
\bar{K} &= \oplus (\text{various } \bar{R}_k) \\
\bar{S} &= \oplus U_\mu \\
\end{align*}
\]

where each $U_\mu$ is an $(f, k)$-matrizing choice or a $(g, k)$-matrizing choice for some $k$, and at least one $U_\mu$ is a standard ideal. Also, an $\bar{R}_k$ can occur more than once in $\bar{K}$ and $\bar{S}$. The maps $\gamma$ and $\delta$ (respectively, $\delta$ and $\bar{\delta}$) are $R$-linear if $\bar{K}$ and $\bar{S}$ are considered to be $R$-modules via $f : R \to R$ (respectively, $g : R \to R$). Also, $\text{im } \gamma$ (respectively, $\text{im } \delta$) is contained in the direct sum of all $(f, k)$-matrizing choices (respectively, $(g, k)$-matrizing choices) which occur in $\bar{S}$ for all $k$. Any matrizing choice other than the $(f, k)$-matrizing choices (respectively, $(g, k)$-matrizing choices) for all $k$ is contained in $\ker \bar{f}$ (respectively, $\ker \bar{g}$).

The homomorphisms $\gamma$, $\delta$, $\bar{\delta}$, and $\bar{\gamma}$ must satisfy some additional technical conditions, whose general statement we omit for now (See [6], Definition 2.1). Instead we give these complete details only in specific situations in which they are needed.

We wish to describe $\gamma, \delta, \bar{\delta}, \bar{\gamma}$ by matrices over the fields $\bar{R}_k$. In order to deal with one field at a time, we define the "$k$-deletion" $D_{k \text{ del}}$ as follows.

\[
\begin{align*}
(D_{k \text{ del}}) & \quad \bar{R}_k^{n(k)} \\
\gamma_k & \quad S(f, k) = \oplus (\tau; \tau, k, \delta U_\mu) \\
\bar{\delta}_k & \quad S(g, k) = \oplus (\tau; \tau, k, \delta U_\mu) \\
\end{align*}
\]

where $\bar{R}_k^{m(k)}$ and $\bar{R}_k^{n(k)}$ denote direct sums of $m(k)$ and $n(k)$ copies of $\bar{R}_k$: those coordinates that remain after deleting all coordinates other than $\bar{R}_k$ from the modules $\bar{K}$ and $\bar{S}$ in $D$. The symbol $S(f, k)$ denotes the direct sum of all $(f, k)$-matrizing choices that appear in $D$, and a similar definition applies to $S(g, k)$. We also note that if $i(k) = j(k)$, every $U_\mu$ which is a standard $R_i(k)$-ideal occurs in both $S(f, k)$ and $S(g, k)$. More generally, each standard $R_c$-ideal can be an $(f, k)$ and/or a $(g, k)$-matrizing choice for several $k$'s, and hence appear in several $D_{k \text{ del}}$'s.
We now complete the detailed description of $D$ when $M = M(D)$ is indecomposable and non-artinian, in terms of the diagrams $D_{k \text{ del}}$. For notational simplicity, we call the coordinate modules of $S(f, k)$ $T_{\mu}$ and those of $S(g, k)$ $S_{\mu}$, as in (2) below. But note that, if $i(k) = j(k)$, every standard ideal that is a $T_{\mu}$ is also an $S_{\nu}$, usually with $\mu \neq \nu$ (See (4) and (7) below). Also note that the number of matrizing choices which occur in $S(f, k)$ is same as that of matrizing choices which occur in $S(g, k)$, that is, $n(k)$. We remark here that the ordering of the matrizing choices $T_{\mu}$ and $S_{\mu}$ will be important in what follows.

We represent the maps $\gamma_k$, $\delta_k$, $f_k$, and $g_k$ by matrices $A_k, B_k, C_k,$ and $D_k$ respectively over the field $R_k$ as in (iii) of 2.1 with the following modifications due to the fact that some of the modules $T_{\mu}$ and $S_{\mu}$ can now be standard ideals.

We recall that the $(i, \mu)$ entry of $C_k$ is a map: $T_{\mu} \rightarrow R_k$, and hence when $T_{\mu}$ is a standard ideal, it equals the maps $f_k$ in the coordinates definition of $R$ followed by actual multiplication by some element $\bar{r}$ in $R$. We denote this entry of $C_k$ by $\bar{r}$. We similarly define the entries of $D_k$ using the map $g_k$ when $S_{\mu}$ is a standard ideal. We also recall that the $(\mu, j)$ entry of $A_k$ is a map: $R_k \rightarrow \text{Socle } T_{\mu}$, and hence it equals the zero map when $T_{\mu}$ is a standard ideal. Similarly the corresponding entry of $B_k$ is the zero map when $S_{\mu}$ is a standard ideal.

Next we describe the matrices $(A, B, C, D) = (A_k, B_k, C_k, D_k)$ that actually occur in $D_{k \text{ del}}$ when $M = M(D)$ is non-artinian indecomposable. We omit the subscript $k$ for the matrices for notational convenience.

The matrices $C$ and $D$ are diagonal matrices given as follows: $C = \text{diag}(x_k, 1, 1, \ldots, 1)$ where $x_k \neq 0$, and $D = I_{n(k)}$.

Each $(A, B)$ is either a deleted cycle pair as defined in 2.15 (iii) or is a direct sum of two deleted cycle pairs as shown below.

\[
A = \begin{bmatrix}
0 & \cdots & 0 \\
I_{r-1} & \cdots & 0 \\
0 & \cdots & I_{s-1}
\end{bmatrix} \quad B = \begin{bmatrix}
I_{r-1} & 0 \\
0 & I_{s-1}
\end{bmatrix}
\]
We note that a deleted cycle pair has one less column that row. We make use of this fact in our notation by writing

\[ n(k) = p + q \quad \text{and} \quad m(k) = (p-1) + (q-1) \]

if two deleted cycle pairs occur.

\[ n(k) = p \quad \text{and} \quad m(k) = p-1 \]

if one deleted cycle pair occurs.

We now describe precisely which summands \( T_\mu \) and \( S_\mu \) can be standard ideals.

If \((A,B)\) is a single deleted cycle pair, the first row of \( A \) and the \( p \)th row of \( B \) are zero. From the way the entries of \( A \) and \( B \) are defined following (2), this implies that \( T_1 \) is the only summand which can be a standard \( R_{i(k)} \)-ideal among the \( T_\mu \)'s, and \( S_p \) is the only summand which can be a standard \( R_{j(k)} \)-ideal among the \( S_\mu \)'s. More precisely there exist three possibilities as follows.

1. \( T_1 \) is a standard \( R_{i(k)} \)-ideal, and \( S_p \) is a standard \( R_{j(k)} \)-ideal. Furthermore, if \( i(k) = j(k) \), \( T_1 = S_p \) is a single standard \( R_{i(k)} \)-ideal which appears in \( D_{k,\text{del}} \).
2. \( T_1 \) is a standard \( R_{i(k)} \)-ideal, and \( S_p \) is of finite length.
3. \( T_1 \) is of finite length, and \( S_p \) is a standard \( R_{j(k)} \)-ideal.

If \((A,B)\) is a direct sum of two deleted cycle pairs as in (3) above, then \( T_1 \) and \( T_{p+1} \) are the only summands which can be standard \( R_{i(k)} \)-ideals among the \( T_\mu \)'s because the rows 1 and \( p \) of \( A \) are the only zero rows of \( A \). Similarly, \( S_p \) and \( S_{p+q} \) are the summands which can be standard \( R_{j(k)} \)-ideals among the \( S_\mu \)'s. What actually occurs in this case is that

4. \( T_1 \) and \( T_{p+1} \) are standard ideals, and \( T_1 \) and \( S_{p+q} \) are of finite lengths. Furthermore, if \( i(k) = j(k) \), \( T_{p+1} = S_p \) is a single standard \( R_{i(k)} \)-ideal which appears in \( D_{k,\text{del}} \).

This completes the detailed description of each individual \( D_{k,\text{del}} \). Finally, we describe a global condition, called "connectedness" that relates the various "deleted" diagrams.

**Remark 3.4** Two deleted diagrams \( D_{a,\text{del}} \) and \( D_{b,\text{del}} \) are immediately connected if some \( T_\mu \) or \( S_\mu \) that is a standard ideal appearing in \( D_{a,\text{del}} \) also occurs as a coordinate module \( T_\nu \) and/or \( S_\rho \) in \( D_{b,\text{del}} \). Then we say that two deleted diagrams \( D_{k,\text{del}} \) and \( D_{k,\text{del}} \) are connected if there exists a sequence of diagrams \( D_{a,\text{del}} \), \( D_{b,\text{del}} \), \( \ldots \), \( D_{a,\text{del}} \) \( D_{k(1),\text{del}} \), \( D_{k(2),\text{del}} \), \( \ldots \), \( D_{k(m),\text{del}} \) beginning with \( D_{k,\text{del}} \) and ending with \( D_{k,\text{del}} \) such that any two consecutive diagrams \( D_{k(i),\text{del}} \) and \( D_{k(i+1),\text{del}} \) are immediately connected.

A non-artinian \( R \)-module \( M = M(D) \) is indecomposable if and only if the diagrams \( D_{k,\text{del}} \) as in 3.3 above are connected, that is, any two \( D_{k,\text{del}} \)'s are
ENDOMORPHISMS OF R-DIAGRAMS 3.5 (of non-artinian R-modules) An endomorphism \( \theta \) of an R-diagram \( D \) was defined in 1.3. For each \( k \), the endomorphism \( \theta \) induces an endomorphism \( \theta_k : D_{k \text{ del}} \to D_{k \text{ del}}' \). (See Remark 3.6 on how \( \theta_k \)'s are related) Thus, an endomorphism of \( D \), the R-diagram of an indecomposable non-artinian R-module, consists of a collection of matrix 4-tuples \((t, s, w, v)_k\) such that each of the following diagram commutes.

\[
\begin{array}{ccc}
R_k^{\pi_{\mu}} & \xrightarrow{A_k} & n(k) \oplus T_j \xrightarrow{C_k} R_k^{\pi_{\nu}} \\
B_k & \xrightarrow{B_k} & S_j \xrightarrow{D_k} R_k^{\pi_{\nu}} \\
\end{array}
\]

Each entry \( t_{ij} \) of \( t \) is a homomorphism from the coordinate module \( T_j \) to the coordinate module \( T_i \). Hence, \( t_{ij} \) can be considered a multiplication by an element (not necessarily unique) of \( R_{i(k)} \). Similarly, each entry \( s_{ij} \) of \( s \) can be considered a multiplication by an element (not necessarily unique) of \( R_{i(k)} \). Similarly, each entry \( s_{ij} \) of \( s \) can be considered a multiplication by an element of \( R_{j(k)} \).

REMARK 3.6 For an endomorphism \( \theta \) of an R-diagram \( D \) as in 3.5 above, the maps \((t, s, w, v)_k\) are connected in the following sense. Any two diagrams \( D_{h \text{ del}} \) and \( D_{k \text{ del}} \) are connected as in Remark 3.4. We assume that they are immediately connected and hence a standard ideal \( H_c \) appears in both \( D_{h \text{ del}} \) and \( D_{k \text{ del}}' \). Then, both \((t, s, w, v)_h\) and \((t, s, w, v)_k\) contain an entry which is an endomorphism of \( H_c \), and in fact these two entries are the same, that is, the endomorphism of \( H_c \) induced by \( \theta \).

In what follows, we write \( \bar{t} = (\bar{t}_{ij}) \) for the matrix obtained by taking the-
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homomorphic image in \( R_k \) of each entry \( t_{ij} \) under the ring homomorphism \( f_k \), and similarly \( \bar{s} = (\bar{s}_{ij}) \) for the matrix obtained from \( s \) using the homomorphism \( g_k \) instead of \( f_k \).

**Lemma 3.7** Let \((t, s, w, v)_k\) be as in 3.5. Then \( r_{ij} = w_{ij} = \bar{s}_{ij} \) where

(a) \( r = \bar{x}_k \) if \( 1 = i \neq j \).

(b) \( r = \bar{x}_k^{-1} \) if \( i \neq j = 1 \).

(c) \( r = 1 \) otherwise.

**Proof.** Similar to Lemma 2.4 with \( 1 \times 1 \) blocks in place of \( r \times r \) blocks and \( C_k = \text{diag}(\bar{x}_k, 1, 1, \ldots, 1) \) in place of the matrix \( C \).

**Lemma 3.8** Let \((t, s, w, v)_k\) be as in 3.5. Then, the first \( p \) diagonal entries of \( \bar{t} \) are equal to each other, and the remaining \( q \) diagonal entries are equal to each other. (That is, \( \bar{t}_{11} = \bar{t}_{22} = \cdots = \bar{t}_{pp} \) and \( \bar{t}_{p+1p+1} = \bar{t}_{p+2p+2} = \cdots = \bar{t}_{p+q+q} \))

**Proof.** Consider the following subdiagram of diagram (1) in 3.5 for \( 2 \leq i \leq p \).

\[
\begin{array}{ccl}
T_i & \leftarrow & [1] \\
& \leftarrow & \bar{R} \\
& \rightarrow & [1] \\
& \rightarrow & S_{i-1} \\
T_i & \leftarrow & [1] \\
& \leftarrow & \bar{R} \\
& \rightarrow & [1] \\
& \rightarrow & S_{i-1} \\
s_{i-1} & \rightarrow & \bar{R} \\
& \rightarrow & s_{i-1} \\
\end{array}
\]

From the commutativity of (1) it follows that \( \bar{t}_{ii} = v_{i-1i-1} = \bar{s}_{i-1i-1} \). Now by Lemma 3.7 we obtain \( \bar{t}_{ii} = \bar{s}_{i-1i-1} = \bar{t}_{i-1i-1} \).

For \( p + 2 \leq i \leq p + q = n(k) \), the proof is similar except that \( v_{i-2i-2} \) occurs in place of \( v_{i-1i-1} \) because of the fact that one more row other than the first row of \( A_k \) is zero, that is, the \( p + 1 \)st row of \( A_k \).

As in §2, our first goal is to show that there exists a permutation of the matrizing choices that occur in each diagram \( D_k \) such that every \( \bar{t} \) (hence \( \bar{s} \)) is upper triangulizable. (See Remark 2.6 concerning triangularization of a matrix)

As in Definition 2.7 we define an ordering \( \succ \) on the set of ordered pairs of integers. That is, \((a, b) \succ (c, d)\) means \((a > c)\) or \((a = c \text{ and } b < d)\). Also as in 2.1, \( d_i \) denotes the composition length of the matrizing choice \( T_i \), and \( e_i \) denotes the composition length of \( S_i \). Note that \( d_i \) and \( e_i \) can be infinite.
DEFINITION 3.9 Fix a $D_{k \, \text{del}}$. We inductively define a total ordering $\geq$ on the set \{1, 2, ..., $n(k)$\).

Let $i \geq j$ mean that $i \neq j$ and:

(a) $(d_i, e_{i-1}) \geq (d_j, e_{j-1})$ with $i \neq 1$ and $j \neq 1$

or (b) $i = 1$ and $d_1 \geq d_j$

or (c) $j = 1$ and $d_i > d_1$

or (d) $(d_i, e_{i-1}) = (d_j, e_{j-1})$ with $i \neq 1$, $j \neq 1$, and $i-1 \geq j-1$.

LEMMA 3.10 If $i \geq j$, then $\tilde{e}_{ij} = 0$.

PROOF. The proof is similar to that of Lemma 2.9 with $1 \times 1$ blocks in place of $r \times r$ blocks except that we have to consider what happens for the case, $i = 1$ and $d_1 = d_j$ in (b). So suppose this holds true. Then, from (1) of 3.5, we obtain a commutative diagram

\[
\begin{array}{ccc}
\mathbb{R}^{n(k)} & \xrightarrow{\varphi} & T_j' \\
\downarrow{V} & & \downarrow{\tilde{e}_{ij}} \\
\mathbb{R}^{n(k)} & \xrightarrow{\psi} & T_1'
\end{array}
\]

where $\varphi$ is the map $A_k$ followed by the projection to $T_j'$, and $\psi$ is the map $A_k$ followed by the projection to $T_1'$. However, we note that $\psi = 0$ (This is a result of the first row of $A_k$ being zero). From the fact that (1) is commutative, $\varphi$ maps onto the socle of $T_j'$, and $d_1 = d_j$, we can conclude that $\tilde{e}_{ij} = 0$.

Before stating our "triangularization theorem", we define the notion of a permuted diagram for an indecomposable, non-artinian $R$-module.

DEFINITION 3.11 (of a permuted $R$-diagram) Let $D$ be an $R$-diagram for an indecomposable non-artinian $R$-module. Fix a $k$-deletion $D_{k \, \text{del}}$, and let $Q$ be an $n(k) \times n(k)$ permutation matrix over $R$. Then, the corresponding permutation of $T_{\mu}$'s and $S_{\mu}$'s gives the permuted diagram $D^Q_{k \, \text{del}}$:

\[
\begin{array}{ccc}
(D^Q_{k \, \text{del}}) & \xrightarrow{QA} & (\bigoplus_{\mu=1}^{n(k)} T_{\mu})_q \\
\downarrow{QB} & & \downarrow{CQ^{-1}} \\
(R^{n(k)}) & \xrightarrow{DQ^{-1}} & (\bigoplus_{\mu=1}^{n(k)} S_{\mu})_q
\end{array}
\]
where \((\bigoplus_{\mu=1}^{n(k)} T_{\mu}) Q\) and \((\bigoplus_{\mu=1}^{n(k)} S_{\mu}) Q\) denote the permuted direct sums.

**THEOREM 3.12** For each \(D_{k,\text{del}}\) of \(D\), an \(R\)-diagram for an indecomposable non-artinian \(R\)-module, there exists a permuted diagram \(D^Q_{k,\text{del}}\) such that the matrices \(\bar{t}\) and \(\bar{s}\) obtained from every endomorphism \((t, s, w, v)_k\) are upper triangular. Moreover, \(T_{\mu}\)'s after being permuted are arranged according to increasing composition lengths.

**PROOF.** The fact that the matrices \(\bar{t}\) are upper triangular is immediate from Lemma 3.10, Lemma 3.11, and Remark 2.6. Then, it also follows from Lemma 3.7 that the matrices \(\bar{s}\) are upper triangular. The second statement is also immediate from the way the ordering \(\succ\) is defined in Definition 2.7 and Definition 3.9.

**COROLLARY 3.13** As \(\bar{D}\) ranges over all diagrams for indecomposable non-artinian \(R\)-modules, \(\bar{E}(\bar{D})\) can be only finitely many non-isomorphic rings. Each of these is the endomorphism ring of some ideal of \(R\).

**PROOF.** Fix such an \(R\)-diagram \(D\). By 3.3, each nonempty \(D_{k,\text{del}}\) contains a standard \(R_{i(k)}\)-ideal and/or a standard \(R_{j(k)}\)-ideal. On the other hand, \(D\) contains at most one standard ideal \(H_c\) of each \(R_c\), a coordinate ring of \(R\); Suppose two different ideals \(H_c\) and \(I_c\) of \(R_c\) appear in \(D\). Then, \(c=i(k)\) or \(j(k)\) for some \(k\). Say \(c=i(k)\). Then, \(H_c\) and \(I_c\) would appear as \(R_{i(k)}\)-ideals in \(D_{k,\text{del}}\). (See (1) of 3.3). However, a \(D_{k,\text{del}}\) has to satisfy one of (4), (5), (6) and (7) in 3.3, all of which allow for at most one standard \(R_{i(k)}\)-ideal. This is a contradiction.

An endomorphism \(\theta\) of \(D\), as mentioned in 3.5, consists of a matrix 4 tuple \((t, s, w, v)_k\) for each nonempty \(D_{k,\text{del}}\). Suppose a standard ideal \(H_c\) of \(R_c\) occurs in \(D\). Then, for each \(D_{k,\text{del}}\) in which \(H_c\) occurs, \((t, s, w, v)_k\) contains an entry which is an endomorphism of \(H_c\) and all these entries are the same, that is, a unique element \(r_c\) of \(R_c\) (See Remark 3.6).

Let \(r_c\) be the element of \(\bar{R}\) whose \(R_c\)-coordinate is \(r_c\) if a standard ideal \(H_c\) occurs, and zero otherwise. We define a map

\[\tau: \text{end } D \rightarrow \bar{R}\text{ by } \theta \rightarrow \bar{r}_c.\]

It follows from the preceding two paragraphs that \(\tau\) is a well-defined map. We first show that \(\tau\) is a ring homomorphism. It is obviously additive. Now, let \(\theta, \phi \in \text{end } D\). Let the entry of \(\theta\) which is an endomorphism of \(H_c\) be \(r_c\), and the entry of \(\phi\) which is an endomorphism of \(H_c\) be \(s_c\). Then, the entry of
which is an endomorphism of $H_c$ must be $r_c s_c$ because a homomorphism from an $R_c$-matrizing choice of finite length into $H_c$ is always the zero map.

Now, we obtain a description of $\ker \tau$. Suppose $\tau(\theta) = 0$ for $\theta \in \text{end } D$. Then, each entry of $\theta$ which is an endomorphism of a standard ideal is zero. Note that each of these entries is a diagonal entry of $t$ and $s$ in some $(t, s, w, v)_k$ and furthermore $(t, s, w, v)_k$ for each nonempty $D_{k\text{ del}}$ contains such an entry. In fact, if $D_{k\text{ del}}$ is as in (4), (5), or (6) of 3.3, then $t_{11} = 0$ and/or $s_{pp} = 0$. If $D_{k\text{ del}}$ is as in (7) of 3.3, $t_{p+1p+1} = 0$ and $s_{pp} = 0$. Moreover, $\bar{t}_{11} = \bar{s}_{11} = 0$ and/or $\bar{t}_{pp} = \bar{s}_{pp} = 0$ for the former, and $\bar{t}_{pp} = \bar{s}_{pp} = \bar{t}_{p+1p+1} = \bar{s}_{p+1p+1} = 0$ for the latter. Thus, it follows from Lemma 3.8 that the diagonal entries of $\bar{t}$ and $\bar{s}$ are zero. Conversely, it is easy to see from the definition of $\tau$ that an endomorphism $\theta$ which satisfies the above properties is in $\ker \tau$.

We claim that $\ker \tau$ is the nilradical of end $D$. We may assume, by Theorem 13.2, that $\bar{t}$ and $\bar{s}$ are upper triangular for $(t, s, w, v)_k \in \text{end } D_{k\text{ del}}$. Thus, by the preceding paragraph $\bar{t}$ and $\bar{s}$ are strictly upper triangular for each $(t, s, w, v)_k$ of $\theta$ in $\ker \tau$. Hence $\bar{t}^{n(k)} = \bar{s}^{n(k)} = 0$. This implies that the entries of $t^{n(k)}$ are contained in $\ker f_k$ and those of $s^{n(k)}$ in $\ker g_k$. Let $e = \max \{d_i, e_i\}$ for $1 \leq i \leq n(k)$. Then, $t^{n(k)}e$ and $s^{n(k)}e$ are the zero map. Thus, ker $\tau$ is nilpotent. Since the homomorphism $\tau$ is a map onto a subring of $\mathcal{R}$, which is semiprime, ker $\tau$ must be the nilradical of end $D$.

Let $C$ be the set of subscripts such that $D$ has an ideal $H_c$ of $R_c$. Let $K$ be the collection of subscripts $k$ for which $D_{k\text{ del}}$ satisfies (4) of 3.3. Then, let

$$R' = \{r \in \bigoplus_{c \in C} R_c | f_k(r_{i(k)}) = g_k(r_{j(k)}) \text{ whenever } k \in K\}.$$

We claim that $\text{im } \tau = R'$. It is clear that $\text{im } \tau \subseteq R'$ from the definition of $\tau$ and Lemma 3.8. The opposite inclusion is easily obtained from Lemma 3.7 and Lemma 3.8 using diagonal matrices. It now follows from the finiteness of $C$ and $K$ in (1) that there are only finitely many possibilities for the ring $\text{im } \tau = R'$.

In order to show that $R'$ is the endomorphism ring of an ideal of $R$, we first show that $R'$ is isomorphic as an $R$-module to an ideal of $R$. For this, it suffices to see that $\bigoplus R_i$ is isomorphic to an ideal of $R$ since $R' \subseteq \bigoplus_{i=1}^u R_i$. By the coordinates definition of $R$, there is an element $r_i \neq 0$ of each $R_i \cap R$ such that $r_i \in \ker f_k$ whenever $i = i(k)$ and $r_i \in \ker g_k$ whenever $i = j(k)$. Then, the multiplication by $\sum_{i=1}^u r_i$ is an isomorphism of $\bigoplus R_i$ onto an ideal of $R$. Now,
Endomorphism Rings of Modules over Dedekind-like Rings

We now state our main theorem.

**THEOREM 3.15** Let $R$ be a Dedekind-like ring. As the modules $M$ range over all indecomposable non-artinian $R$-modules, the rings $\overline{E}(M)$ can be only finitely many non-isomorphic rings. Each of these is the endomorphism ring of some ideal of $R$.

**PROOF.** Given such a module $M$, the corresponding $R$-diagram $D$ satisfies $\overline{E}(M) \simeq \overline{E}(D)$ as in (6) of 1.3. We are done by Corollary 3.14.

**EXAMPLES 3.16** (i) As the modules $M$ range over all indecomposable non-artinian $ZG_p$-modules, the rings $\overline{E}(M)$ can be only three different rings: $Z, Z[\xi]$ and $ZG_p$ ($\xi =$ primitive $p^{th}$ root of unity).

(ii) As the modules $M$ range over all indecomposable non-artinian $Z[\sqrt{m}]$-modules ($m \equiv 1 (\mod 4)$), the rings $\overline{E}(M)$ can be only two different rings: $Z[\sqrt{m}]$ and $Z[\omega]$ ($\omega = (-1 + m)/2$).

(iii) Let $G_n = \langle x_n \rangle$ denote a cyclic group of square-free order $n$, generated by an element $x_n$. In [7], it is shown that the integral group ring $ZG_n$ is Dedekind-like. So Theorem 3.15 applies.

**REFERENCES**


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