ON GENERALIZATIONS OF SELF-INJECTIVE AND STRONGLY REGULAR RINGS

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Introduction

A generalization of injectivity, called $E$-injectivity, is studied in connection with von Neumann regular rings and continuous regular ring (in the sense of Y. Utumi). Various properties of $E$-injectivity are developed. Conditions are given for modules over commutative $E$-injective rings with nilpotent singular ideal to have isomorphic injective hulls. Regular rings whose essential left ideals are two-sided ideals are also considered.

Throught, $A$ denotes an associative ring with identity and left (right) $A$-modules are unital. $J, Z, Y$ will stand respectively for the Jacobson radical, the left singular ideal and the right singular ideal of $A$. As usual, a left (right) ideal of $A$ is called reduced if it contains no non-zero nilpotent element. An ideal of $A$ will always mean a two-sided ideal. $A$ is a left $V$-ring (after Williamyor) iff every simple left $A$-module is injective. In the commutative case, $V$-rings coincide with von Neumann regular rings (I. Kaplansky). Recall that a left $A$-module $M$ is $p$-injective if, for any principal left ideal $P$ of $A$, every left $A$-homomorphism of $P$ into $M$ extends to one of $A$ into $M$. $A$ is von Neumann regular iff every left (right) $A$-module is flat iff every left (right) $A$-module is $p$-injective. In general, flat modules need not be $p$-injective and the converse is not true either. But if $I$ is a $p$-injective left ideal of $A$, then $A/I$ is a flat left $A$-module. Also, if $M$ is a maximal left ideal of $A$ which is an ideal, then $A^A/M$ is injective iff $A^A/M_A$ is $p$-injective iff $A/M_A$ is flat [17].

The concept of injectivity is among the most important fundamental concepts in the theory of rings and modules. Recall that $A$ is a left self-injective ring iff for any essential left ideal $E$ of $A$, every left $A$-homomorphism of $E$ into $A$ extends to one of $A$ into $A$.

We here introduce the following generalization of injectivity.

DEFINITION. A left $A$-module $M$ is called $E$-injective if, for any non-zero complement left submodule $C$ of $M$ and relative complement $K$ of $C$, any
essential left submodule $E$ of $M$ containing $K \oplus C$, any left $A$-monomorphism $g : E \to M$ and $A$-homomorphism $f : E \to M$, there exists an endomorphism $h$ of $A^M$ such that $hg = f$.

Continuous rings were introduced by Y. Utumi ([11], [12]). A left $A$-module $M$ is called continuous if every left submodule isomorphic to a complement left submodule of $M$ is a direct summand of $M$. Continuous modules generalize quasi-injective modules. $A$ is called left $E$-injective (resp. (1) continuous; (2) $p$-injective) if $A$ is $E$-injective (resp. (1) continuous; (2) $p$-injective). A similar definition holds on the right side.

REMARK 1. Any continuous left $A$-module $M$ is $E$-injective.

RROOF. Let $K$ be a non-zero complement submodule of $A^M$, $C$ a relative complement of $K$ in $M$. Since $M$ is continuous, then $M = K \oplus C$. If $g : M \to M$ is a left $A$-monomorphism, $f : M \to M$ a left $A$-homomorphism, then $g$ induces an isomorphism $G$ of $M$ onto $M' = g(M)$ and if $i : M' \to M$ is the inclusion map, since $M'$ is a direct summand of $M$, then there is an epimorphism $p : M \to M'$ such that $pi =$ identity map on $M'$. Then $h = fG^{-1} p : M \to M$ such that $hg = hiG = f$ which shows that $A^M$ is $E$-injective.

Although $E$-injective modules are distinct from NCI modules (considered in [19]), the proof of [19, Proposition 2.2] yields

PROPOSITION 1. If $M$ is an $E$-injective left $A$-module, then any complement left submodule is a direct summand.

COROLLARY 1.1. A non-singular left $E$-injective ring is Baer.

Applying Remark 1, we get

COROLLAY 1.2. The following conditions are equivalent:
(a) $A$ is left continuous regular;
(b) $A$ is a left $E$-injective ring whose principal left ideals are complement left ideals;
(c) $A$ is right $p$-injective left non-singular left $E$-injective.

The next corollary then follows.

COROLLARY 1.3.
(1) A right self-injective left non-singular left $E$-injective ring is left self-injective regular (cf. [6, Theorem 2.28]).
(2) A right continuous left non-singular left $E$-injective ring is left continuous regular.

PROPOSITION 2. Let $A$ be left $E$-injective. Then

(1) An element of $A$ is right invertible iff its left annihilator is zero. Consequently, every left (or right) $A$-module is divisible;

(2) If every complement left ideal of $A$ is an ideal, then $A/Z$ is von Neumann regular and $Z=J$.

PROOF. (1) One implication is obvious. Let $c \in A$ such that $l(c)=0$. The map $g : A \to A$ defined by $g(a)=ac$ for all $a \in A$ is a monomorphism. If $j : A \to A$ is the inclusion map, there exists $h \in \text{End}_A$ such that $hg=j$. Then $1=j(1)=hg(1)=h(c)=ch(1)$, showing that $c$ is right invertible in $A$. It follows that any non-zero-divisor is invertible in $A$ and every left (right) $A$-module is divisible.

(2) Now suppose that every complement left ideal of $A$ is an ideal. Let $b \in A/Z$, $b=b+Z$, $b \in A$, $b \in Z$, $K$ a non-zero complement left ideal such that $E=l(b) \oplus K$ is an essential left ideal. The set of left ideals containing $l(b)$ and having zero intersection with $K$ has a maximal member $C$ which is a relative complement of $K$ in $A$. If $L=C \oplus K$, define a map $g : L \to A$ by $g(c+k)=c+kb$, $c \in C$, $k \in K$. Then $g$ is a monomorphism (in as much as $l(b) \cap K=0$ and $K$ is an ideal of $A$) and if $i : L \to A$ is the inclusion map, there exists an endomorphism $h$ of $A$ such that $hg=i$. With $h(1)=d$, for any $k \in K$, $k=i(k)=hg(k)=h(kb)=kbh(1)=kdb$, whence $K \subseteq l(b-bdb)$, yielding $E=l(b) \oplus K \subseteq l(b-bdb)$, which proves that $A/Z$ is von Neumann regular (because $b-bdb \in Z$). If $z \in Z$, $a \in A$, then $l(1-za)=0$ which implies that $(1-za)$ is right invertible in $A$ by (1), whence $z \in J$. This shows that $Z \subseteq J$. Finally, since for any ideal $T$, $(J+T)/T$ is contained in the Jacobson radical of $A/T$, then $J \subseteq Z$ ($A/Z$ being regular) which yields $J=Z$.

Since a commutative fully Goldie ring is Noetherian [3], the next corollary then follows.

COROLLARY 2.1. A commutative $E$-injective fully Goldie ring is Artinian.

$A$ is called a left $\Sigma$-ring if the set of left ideals which are left annihilators of subsets of the injective hull of $A$ satisfies the ascending chain condition (cf. [5]).

Applying [5, Theorem 11.4.1 B], we get
COROLLARY 2.2. A commutative E-injective Σ-ring is Artinian.

The next two remarks also hold.

REMARK 2. The following conditions are equivalent for a commutative ring $A$: (a) $A$ is quasi-Frobeniusean; (b) $A$ is an $E$-injective Σ-ring whose socle is principal; (c) $A$ is a $p$-injective Σ-ring. (cf. [10, Propositions 2 and 3].)

REMARK 3. The following conditions are equivalent for a commutative ring $A$: (a) Every factor ring of $A$ is quasi-Frobeniusean; (b) $A$ is an $E$-injective Σ-ring whose finitely generated ideals are principal.

We do not know whether Proposition 2(2) holds for arbitrary left E-injective rings.

Note that if $I$ is a non-singular left ideal of $A$, then $I(I)$ is an ideal of $A$ which is a complement left ideal. If $A$ is left continuous regular such that every non-zero ideal contains a non-zero nilpotent element, then $A$ is left self-injective [11, Theorem 3].

In the next remark, Utumi's decomposition [11, p.604] of left continuous regular rings is extended in (1) while (2) gives a non-trivial generalization of the self-injective regular case [7, Theorem 9.14].

REMARK 4. Let $A$ be semi-prime left $E$-injective.

(1) $S$, the sum of all reduced ideals of $A$, is the unique maximal strongly regular ideal of $A$ and we have a ring direct sum $A = S \oplus T$, where $S$ is a left and right continuous strongly regular ring, $T$ is the minimal direct summand of $A$ containing all the nilpotent elements;

(2) If $M, N$ are non-singular left $A$-modules with injective hulls $F, K$ respectively, there exists a central idempotent $e$ such that $AeM$ embeds in $AeK$ and $A(1-e)N$ embeds in $A(1-e)F$.

Note that continuous regular rings (even commutative with non-zero socle) need not be self-injective [12, p.172]. Thus semiprime $E$-injective rings generalize effectively continuous regular rings and semi-prime self-injective rings.

A consequence of Remark 4(2) is that if $A$ is prime left $E$-injective, then for any non-singular left $A$-modules $M, N$, either $M$ embeds in the injective hull of $AN$ or $N$ embeds in the injective hull of $AM$.

Characteristic properties of semi-simple Artinian rings are now given in
On Generalization of Self-Injective and Strongly Regular Rings

PROPOSITION 3. The following conditions are equivalent:

1. \( A \) is semi-simple Artinian;
2. \( A \) is a semi-prime left \( E \)-injective left \( \Sigma \)-ring;
3. A left \( A \)-module is flat iff it is \( E \)-injective.

PROOF. Obviously, (1) implies (2).

Assume (2). By Proposition 2(1), \( A \) is its own classical left (and right)-quotient ring. Then (2) implies (3) by [5, Corollary 5.13].

Assume (3). If \( M \) is an \( E \)-injective left \( A \)-module, \( H \) the injective hull of \( M \), then \( M \) and \( H \) are flat which implies that \( Q=M\oplus H \) is flat. Inasmuch as \( _AQ \) is now \( E \)-injective, if \( j:M\to H, k:H\to Q \) are the inclusion maps, \( i:M\to M \) the identity map, \( p:M\oplus C\to M \) the natural projection, where \( C \) is a relative complement of \( M \) in \( _AQ \), then \( kp \) extends to an endomorphism \( h \) of \( _AQ \). Now \( i \) is the restriction of \( h \) to \( M \) and if \( q \) is the natural projection of \( Q \) onto \( M \), then \( g=ghk \) is a map of \( H \) into \( M \) such that \( gj=i \). This proves that \( M \) is a direct summand of \( _AH \), whence \( M=H \) is injective. Then every projective left \( A \)-module (being flat) is injective, whence \( A \) is quasi-Frobenian [4, Theorem 24.20]. Since every simple left \( A \)-module is \( E \)-injective, then \( A \) is a left \( \Sigma \)-ring which yields \( A \) semi-simple Artinian. Thus (3) implies (1).

Recall that \( A \) is left uniform iff every non-zero left ideal is essential. Note that \( E \)-injectivity does not imply \( p \)-injectivity and the converse is not true either. Rings whose simple right modules are either \( p \)-injective or projective (called right \( p-V' \)-rings) need not be semi-prime (cf. for example [13, p.297]).

REMARK 5. (1) If \( A \) is left uniform left \( E \)-injective, then \( A \) is a directly finite left continuous local ring such that \( Z=J \) is the unique maximal right (left) ideal of \( A \); (2) If \( A \) is right uniform right \( p-V' \)-ring, then the right singular ideal \( Y \) is the unique maximal two-sided ideal of \( A \); (3) A left \( p \)-injective ring is left continuous iff it is left \( E \)-injective; (4) If \( A \) is left Noetherian such that each prime factor ring is either left or right \( p \)-injective, then \( A \) is left Artinian.

REMARK 6. If \( M \) is an \( E \)-injective left \( A \)-module, \( Q \) a non-singular left \( A \)-module, for any \( A \)-homomorphism \( f:M\to Q \), \( \ker f \) is a direct summand of \( M \). (If \( E \) is an essential extension of \( \ker f \) in \( M \), for any \( b\in E \), \( f(b)\in Z(Q) \), the singular submodule of \( Q \), which is zero. Thus \( E=\ker f \) is a direct summand.
of $M$ by Proposition 1).

Write $Z_2(A) = \{a \in A \mid La \subseteq Z \text{ for some essential left ideal } L \text{ of } A\}$. Since $Z_2(A)$ is an ideal which is a complement left ideal of $A$, the next remark follows.

REMARK 7. If $A$ is left $E$-injective whose complement left ideals are ideals, then $A = Z_2(A) \oplus B$, where $B$ is a left and right continuous strongly regular ring.

REMARK 8. $A$ is left non-singular iff every $E$-injective left $A$-module contains its singular submodule as a direct summand.

A left $A$-module $M$ is called semi-simple if the intersection of all the maximal left submodules of $M$ is zero. Then $A$ is a left $V$-ring iff every left $A$-module is semi-simple [9, Theorem 2.1].

REMARK 9. The following conditions are equivalent: (a) $A$ is a regular left $V$-ring; (b) A cyclic left $A$-module is semi-simple iff it is $p$-injective.

For any left $A$-module $M$, $r_M(J) = \{y \in M \mid Jy = 0\}$ is a left submodule of $M$. If $f$ is a left $A$-homomorphism of $AP$ into $AM$, then $f(r_P(J)) \subseteq r_M(J)$. We now turn to sufficient conditions for two modules to have isomorphic injective hulls.

THEOREM 4. Let $A$ be a commutative $E$-injective ring with nilpotent singular ideal. If $M, N$ are $A$-modules such that there exist isomorphic $A$-modules $P, R$ with monomorphism $f : P \rightarrow M$, $g : R \rightarrow N$ having the properties that $f(r_P(J))$ (resp. $g(r_R(J))$) is essential in $r_M(J)$ (resp. $r_N(J)$), then $M$ and $N$ have isomorphic injective hulls.

PROOF. Let $E, Q$ denote the injective hulls of $M, N$ respectively and $i : M \rightarrow E$, $i : N \rightarrow Q$ the inclusion maps, Then $F = jf$, $G = ig$ are monomorphisms and since $Q$ is an injective $A$-module, there exists an $A$-homomorphism $H : E \rightarrow Q$ such that $HF = Gh$. For any $0 \neq u \in r_E(J)$, since $M$ is essential in $E$, there exists $b \in A$ such that $0 \neq bu \in M$ and then $Jbu = bJu = 0$ implies that $bu \in r_M(J)$, showing that $r_M(J)$ is an essential submodule of $r_E(J)$. Therefore $F(r_P(J)) = f(r_P(J))$ is an essential submodule of $r_E(J)$. Similarly, $G(r_R(J))$ is an essential submodule of $r_Q(J)$. Now $ker H \cap Im F = 0$ implies $ker H \cap F(r_P(J)) = 0$ and since $F(r_P(J))$ is essential in $r_E(J)$, we have $ker H \cap r_E(J) = 0$. At this point, we note that, by
On Generalization of Self-Injective and Strongly Regular Rings

Proposition 2(2), $A/J$ is von Neumann regular and $J=Z$ is nilpotent. If $J=0$, then $f(P)\cong g(R)$ and $f(P)$ (resp. $g(R)$) essential in $E$ (resp. $Q$) imply $E\cong Q$. Now assume that $J\neq 0$. Let $n$ be the least positive integer such that $J^n=0$. Then $J^{n-1} \cap \ker H=0$ and similarly, if $n-1>1$, we have $J^{n-2} \cap \ker H=0$ and so on. Thus we reach $\ker H=0$, yielding an isomorphism $K: H(E)\rightarrow E$ (where $KH(u)=u$ for all $u\in E$) Since $Q=H(E)\oplus S$ for some submodule $S$ of $Q$, define $t: Q\rightarrow E$ as follows: for any $q=v+s$, $q\in Q$, $v\in H(E)$, $s\in S$, $t(q)=K(v)$. Then $t$ is a well-defined $A$-homomorphism and since $HF=Gk$, we get $G(r_R(J)) \cap \ker t=0$ which implies $r_Q(J)\cap \ker t=0$. The preceding proof then yields $\ker t=0$. Now $\ker t=S$ and we finally have $Q=H(E)$ and $K: Q\rightarrow E$ is an isomorphism, which completes the proof.

A necessary and sufficient condition for two injective modules to be isomorphic now follows.

COROLLARY 4.1. If $A$ is commutative $E$-injective with nilpotent singular ideal, $M, N$ injective $A$-modules, then $M\cong N$ iff $r_M(J)\cong r_N(J)$.

Recall that a ring is fully idempotent (resp. fully left idempotent) if every ideal (resp. left ideal) is idempotent. Note that if $A$ is fully idempotent, then any ideal $T$ of $A$ is a fully idempotent ring (indeed, for any $t\in T$, $t\in (AtA)^6\subseteq (TtT)^2$). Similarly, if $A$ is fully right idempotent, then any ideal of $A$ is a fully right idempotent ring. It may also be noted that if $A$ is semi-prime, then any essential left ideal of $A$ which is an ideal must be right essential. Various generalizations of strongly regular rings have been considered in [14] and [15]. $A$ is called ERT (resp. ELT) if every essential right (resp. left) ideal is an ideal while $A$ is called MERT if any maximal essential right ideal (if it exists) is an ideal. We know that ELT fully right idempotent rings are von Neumann regular [8, Corollary 6]. ELT regular rings generalize effectively strongly regular rings and semi-prime rings whose left ideals are quasi-injective.

The next result improves [15, Corollary 1.4] and [16, Remark 6].

THEOREM 5. The following conditions are equivalent for an ELT ring $A$:

1. $A$ is von Neumann regular;
2. $A$ has a maximal ideal which is a fully idempotent ring;
3. $A$ is a left $p-V'$-ring containing a prime ideal which is a fully idempotent ring;
(4) Every essential left ideal is idempotent and there exists a prime ideal which is a fully idempotent ring;

(5) Every cyclic semi-simple right \( A \)-module is flat;

(6) \( A \) is right non-singular whose simple right modules are flat such that every essential left ideal is right essential;

(7) For each factor ring of \( A \), every essential left ideal is right essential while the intersection of the right singular ideal and the Jacobson radical is zero;

(8) \( A \) is right \( p\)-\( V' \)-ring such that every essential left ideal is right essential;

(9) There exists an ideal \( T \) such that \( A/T \) and \( T \) are fully idempotent rings.

PROOF. Obviously, (1) implies (2) through (8).

If \( A \) has a maximal ideal \( M \), then \( A/M \) is a simple ring which is therefore fully left (and right) idempotent. Thus (2) implies (9).

(3) implies (4) by [14, Proposition 3].

Assume (4). Let \( P \) be a prime ideal which is a fully idempotent ring. Since \( A/P \) is a prime ELT fully idempotent ring, then (4) implies (9).

If \( A/J_A \) is flat, then \( J = 0 \). Therefore (5) implies (6).

Assume (6). It is sufficient to show that \( A \) is semi-prime for then, (6) will imply (9) by [1, Theorem 3.1]. Suppose there exists a non-zero ideal \( T \) such that \( T^2 = 0 \). Let \( K \) be a complement left ideal such that \( L = r(T) \oplus K \) is an essential left ideal of \( A \). Then \( TK \subseteq K \cap T \subseteq K \cap r(T) = 0 \) implies that \( K \subseteq r(T) \), whence \( K = 0 \). Thus \( r(T) \) is an essential left ideal and since \( Tr(T) = 0 \), the hypothesis that \( r(T) \) is right essential leads to \( T \subseteq Y = 0 \), a contradiction! This proves that \( A \) is semi-prime.

Assume (7). The proof of "(6) implies (9)" shows that every factor ring of \( A \) is semi-prime. Therefore \( A \) is fully idempotent and (7) implies (9).

Assume (8). Suppose that \( A \) is not fully right idempotent. Then there exists \( b \in A \) such that \( AbA + r(b) \neq A \). If \( K \) is a complement left ideal such that \( L = AbA \oplus K \) is an essential left ideal of \( A \), then \( L_A \) is essential and since \( bK \subseteq K \cap AbA = 0 \), then \( K \subseteq r(b) \) which implies that \( AbA + r(b) \) is proper right essential in \( A_A \). Let \( M \) be a maximal right ideal containing \( AbA + r(b) \). Then \( A/M_A \) cannot be projective (in as much as \( M_A \) is essential in \( A_A \)) which implies that \( A/M_A \) is \( p' \)-injective. Now the map \( f : bA \to A/M \) given by \( f(ba) = a + M \) for all \( a \in A \) yields \( 1 + M = f(b) = db + M \) for some \( d \in A \), whence \( 1 - db \in M \), leading to \( 1 \in M \), which contradicts \( M \neq A \). This proves that \( A \) is fully right idempotent and then (8) implies (9).
If \( B \) is a prime ring whose essential left ideals are idempotent, for any non-zero ideal \( T \) of \( B \), there exists a complement left subideal \( K \) of \( T \) such that \( L = Bt \oplus K \) is an essential left subideal of \( T \), and therefore an essential left ideal of \( B \) (since \( _BT \) is essential in \( _BB \)). Then \( t \in L = L^2 \) which leads to \( t \in T \). It follows that for any \( b \in B \), \( b \in (BbB)b \) which shows that \( B \) is fully left idempotent.

Now assume (9). For any \( b \in A \), since \( A/T \) is fully idempotent, then \( AbA + T = (AbA)^2 + T \) which implies that \( b - c \in T \) for some \( c \in (AbA)^2 \), whence \( b - c \in T(b - c)T(b - c)T \subseteq (AbA)^2 \), which yields \( b \in (AbA)^2 \), showing that \( A \) is fully idempotent. If \( P \) is a prime ideal of \( A \), then \( A/P \) is an ELT fully idempotent ring and we have just seen that in this case, \( B = A/P \) is a fully left idempotent ring. If \( S \) is the socle of \( B \), then \( B/S \) is a fully left idempotent ring whose maximal left ideals are ideals which implies that \( B/S \) is strongly regular. Since \( S \) and \( B/S \) are fully right idempotent rings, then \( B \) is a fully right idempotent ring and by [8, Corollary 6], \( B \) is von Neumann regular. Therefore (9) implies (1) by [7, Corollary 1.18].

Theorem 5(8) yields immediately

**COROLLARY 5.1.** If every essential left ideal of \( A \) is an essential right ideal and every singular right \( A \)-module is injective, then \( A \) is regular, right hereditary.

**COROLLARY 5.2.** The following conditions are equivalent for an ELT ring \( A \):
(a) \( A \) is a regular right \( V \)-ring; (b) A cyclic right \( A \)-module is flat iff it is semi-simple.

Following Birkenmeier [2], \( A \) is called (1) a right GFC ring if every faithful cyclic right \( A \)-module generates the category of right \( A \)-modules; (2) a strongly right bounded ring if every non-zero right ideal contains a non-zero ideal. We note the following results: (a) A strongly right bounded ring is right GFC [2, Lemma 1.2]; (b) A right non-singular right GFC ring is semi-prime [2, Proposition 1.4]. Right GFC rings generalize effectively right duo rings, strongly regular rings, right FPF and right pseudo-Frobeniusean rings.

**PROPOSITION 6.** Let \( A \) be a MERT right GFC ring whose simple left modules are flat. Then \( A \) is regular.

**PROOF.** Suppose that \( Y \neq 0 \). Then there exists \( 0 \neq y \in Y \) such that \( y^2 = 0 \) [18,
Lemma 7]. If $M$ is a maximal right ideal containing $r(y)$, then $M$ is an ideal of $A$ which implies that $M$ is an maximal left ideal of $A$. Now $A/A/M$ is flat and since $y \in M$, then $y = yu$ for some $u \in M$. Therefore $1 - u \in r(y) \subseteq M$ yields $1 \in M$, which contradicts $M \neq A$. This proves that $Y = 0$ and by [2, Proposition 1.4], $A$ is semi-prime. Now suppose that there exists $c \in A$ such that $AcA + r(c) \neq A$. If $R$ is a maximal right ideal containing $AcA + r(c)$, then $R$ must be right essential (otherwise, $R = r(e)$, $e = e^2 \subseteq A$, and $eAcA = 0$ implies $(aAeA)^2 = 0$, whence $cAeA = 0$, yielding $e \in r(c)$ and then $e = e^2 = 0$, which contradicts $e \neq 0$). Therefore $R$ is a maximal left ideal and $A/A/R$ being flat leads, as above, to a contradiction. This proves that $A$ is fully right idempotent. Since we know that a MERT fully right idempotent ring is ERT, then $A$ is regular by Theorem 5(6).

Since a prime right GFC ring is right Goldie [2, Corollary 3.6], we get

**COROLLARY 6.1.** Let $A$ be a MERT ring whose simple left modules are flat. If every factor ring is GFC, then $A$ is unit-regular.

**REMARK 10.** The following conditions are equivalent for a MERT ring $A$:
(a) $A$ is regular and biregular; (b) every simple right $A$-module is flat and for any $a \in A$, $AaA$ is the left annihilator of an element; (c) if $a, b \in A$ such that $AaA + AbA \neq A$, then $AaA + AbA$ is the left annihilator of a non-nilpotent central element of $A$. In that case, $A$ is a unit-regular left and right $V$-ring.

**THEOREM 7.** The following conditions are equivalent:
(1) $A$ is strongly regular;
(2) $A$ is a MERT strongly right bounded ring whose simple left modules are flat;
(3) $A$ is a strongly right bounded ring whose simple right modules are flat.

**PROOF.** Obviously, (1) implies (2).

(2) implies (3) by [2, Lemma 1.2] and Proposition 6.

Assume (3). We first prove that $J = 0$. Suppose there exists $0 \neq b \in J$. Then $bA$ contains a non-zero ideal $T$. If $l(T) + bA \neq A$, let $R$ be a maximal right ideal containing $l(T) + bA$. Since $A/R_A$ is flat, and $b \in R$, then $b = db$ for some $d \in R$. Now $1 - d \in (b) \subseteq l(T) \subseteq R$ implies that $1 \in R$, which contradicts $R \neq A$. Now suppose that $l(T) + (bA) = A$. Then $1 = bc + u$, $c \in A$, $u \in l(T)$ and for any $0 \neq t \in T$, $t = bct$. Since $bc \in J$, there exists $v \in A$ such that $v(1 - bc) = 1$ which yields $t = v(1 - bc)t = v(t - bct) = 0$, a contradiction again! This proves that $J = 0$. Now if $0 \neq u \in A$ such that $u^2 = 0$, since $uA$ contains a non-zero ideal $W$, we
have $W^2-uAW=uW^2u^2A=0$, which implies that $W\subseteq J=0$. This contradiction proves that $A$ is reduced. It is well-known that a reduced ring whose simple right modules are flat is strongly regular and hence (3) implies (1).

If $T$ is a reduced ideal of $A$, then any idempotent in $T$ is central in $A$. Since a left continuous strongly regular ring is right continuous, the next result then follows from Propositions 1, 2 and the proof of Theorem 7.

**Theorem 8.** The following conditions are equivalent:

1. $A$ is left and right continuous strongly regular;
2. $A$ is left non-singular left $E$-injective whose complement left ideals are ideals of $A$;
3. $A$ is left non-singular left $E$-injective whose complement right ideals are ideals of $A$;
4. $A$ is semi-prime strongly right bounded left $E$-injective;
5. $A$ is reduced left $E$-injective.

**Corollary 8.1.** If $A$ is reduced with a classical left quotient ring $Q$, then $Q$ is continuous strongly regular iff $Q$ is a left $E$-injective ring.

**Remark 11.** $A$ is simple Artinian iff $A$ is a prime right GFC ring satisfying any one of the following conditions: (a) every simple right $A$-module is flat; (b) every simple left $A$-module is flat; (c) $A$ is right $p$-injective; (d) $A$ is left $p$-injective.

Finally, we note that the proof of "(9) implies (1)" in Theorem 5 shows the validity of the next remark.

**Remark 12.** A MERT fully left idempotent ring is von Neumann regular.

**References**

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