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Some Distribution Free Tests for Exponential Distributions

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ABSTRACT

This paper considers two sided tests for exponential null distribution against NBUE or NWUE alternative in life testing. The main results concern the strong consistency of two proposed statistics, one being similar to Kolmogorov – Smirnov statistic, the other similar to Cramer-Von Mises statistic. Also obtained are the asymtotic null distribution and the exact Bahadur slope of the statistic similar to Kolmogorov-Smirnov.

I. INTRODUCTION

A non-degenerate life distribution with finite mean μ is said to be New Better than Used in Expectation (NBUE) if

(1.1) $$\int_y^\infty \bar{F}(x)\ dx \leq \bar{F}(y) \quad \text{for all} \quad 0<y<\infty,$$

or equivalently if

(1.2) $$\int_0^y \bar{F}(x)\ dx \geq \mu\ F(y) \quad \text{for all} \quad 0<y<\infty,$$

where $\bar{F}(y)=1-F(y)$, $0<y<\infty$,

Similarly, a life distribution of New Worse than Used in Expectation (NWUE) is defined by reversing the inequality in (1.1).

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If F is an exponential distribution, then equality is obtained in (1.1). However, if equality holds in (1.1), then F has a density f with

$$\bar{F}(y) = \mu f(y), \quad 0 < y < \infty,$$

Therefore, equality holds in (1.1) if and only if F is an exponential distribution.

Consider the following null and alternative hypothesis;

$$H_0: F \text{ is exponential,}$$
$$i.e., \bar{F}(x) = e^{-\lambda x} \text{ for some } \lambda > 0,$$

versus

$$H_1: F \text{ is NBUE or NWUE, not exponential.}$$

The tests for exponentiality against various alternatives have been studied. Proschan and Pyke (1967) defined the normalized spacings

$$D_j = (n-j+1)[X_{(j)} - X_{(j-1)}], \quad j = 1, \cdots, n,$$

where $X_{(1)} \cdots X_{(n)}$ denote order statistics of the random sample $X_1, \cdots, X_n$, and also suggested a test statistic for Increasing Failure Rate (IFR) alternative. Barlow (1968) considered likelihood ratio tests for IFR alternative, and also proposed the test statistic

$$W_{ni} = \sum_{j=1}^{i} D_j \Big/ \sum_{j=1}^{n} D_j$$

for Increasing Failure Rate Average (IFRA) alternative. Hollander and Proschan (1972) proposed a U-statistic for New Better than Used (NBU) alternative, a wider class than IFR and IFRA alternatives. Hollander and Proschan (1975) further suggested a test for NBUE alternative which contains NBU alternative. Koul (1978) compared the efficiency of the total time on test statistic

$$W = \sum_{i=1}^{n} W_{ni}$$

with the one-sided Kolmogorov-Smirnov statistic for NBUE alternative.

In this paper, the tests for exponentiality against two-sided alternative (*i.e.*, NBUE or NWUE, not exponential) are considered. The two types of test statistics are proposed, one of which is a two-sided Kolmogorov-Smirnov type statistic K_n (K-statistic) and the other a Cramer-von Mises type statistic C_n (C-statistic). We show that two proposed test statistics are strongly consistent. Next, for the statistic K_n, we obtain its asymptotic null distribution and the exact Behadur slope. Finally the values of exact slope are computed for Weibull distribution in the neighborhood of the shape parameter $\theta = 1$.

II. TEST STATISTICS

The main purpose of this section is to propose two types of test statistic and to find their limit.

2.1 Kolmogorov-Smirnov Type Statistic

For a non-degenerate life distribution F which is NBUE or NWUE with mean μ, $0 < \mu < \infty$, let

$$(2.1)\qquad K(F)=\sup_{0\le y<\infty}|\mu^{-1}L(y)-F(y)|,$$

where

$$L(y)=\int_0^y \bar{F}(x)\,dx.$$

It is well known that if F is exponential, then $K(F)=0$. On the other hand, if $K(F)=0$ and F is NBUE or NWUE, then F must be exponential. Hence $K(F)$ is a measure of exponenentiality of F. The more significant evidence for F H_1 is, if the value of $K(F)$ becomes larger. Therefore it is natural to use K_n as a test statistic, where

$$(2)\qquad K_n=\sup_{0\le y<\infty}|\bar{X}^{-1}L_n(y)-F_n(y)|.$$

Here, $L_n(y)=\int_0^y \bar{F}_n(x)\,dx$, F_n is the empirical distribution function of F,

and $\bar{X}=n^{-1}\sum_{i=1}^{n}X_i$.

We reject H_0 in favor of H_1, if the test statistic is large.

For $i=1,\dots,n$, denote

$$
\begin{aligned}
(2.3)\qquad W_{ni}&=\bar{X}^{-1}L_n(X_{(i)})\\
&=\bar{X}^{-1}n^{-1}\sum_{j=1}^{n}\min\{X_{(i)},X_{(j)}\}\\
&=(n\bar{X})^{-1}[\sum_{j=1}^{i}X_{(j)}+(n-i)X_{(i)}]\\
&=\sum_{j=1}^{i}D_j/\sum_{j=1}^{n}D_j,\quad D_j=(n-j+1)[X_{(j)}-X_{(j-1)}].
\end{aligned}
$$

By rewriting K_n as a function of W_{ni}, we propose the following proposition which is useful in §3.1.

Proposition 2.1

$$(2.4)\qquad K_n=\max\{\max_{0\le i\le n-1}[W_{n\,i+1}-\frac{i}{n}],\ \max_{0\le i\le n-1}[\frac{i}{n}-W_{ni}]\}.$$

(Proof.) For $i=0,\dots,n$, note that

$$F_n(X)=\frac{i}{n},\ X_{(i)}\le x<X_{(i+1)}\quad\text{with}\quad X_{(0)}=0\quad\text{and}\quad X_{(n+1)}=\infty.$$

Then,

$$
\begin{aligned}
\sup_{0\le y<\infty}\{\bar{X}^{-1}L_n(y)-F_n(y)\}&=\max_{0\le i\le n}\ \sup_{x_{(i)}\le y<x_{(i+1)}}\{\bar{X}^{-1}L_n(y)-F_n(y)\}\\
&=\max_{0\le i\le n-1}\{\bar{X}^{-1}L_n(X_{(i+1)})-F_n(X_{(i)})\}\\
&=\max_{0\le i\le n-1}\{W_{n\,i+1}-\frac{i}{n}\}\qquad\text{from (2.3).}
\end{aligned}
$$

Similarly,

$$\sup_{0\le y<\infty} \{F_n(y) - \bar{X}^{-1} L_n(y)\} = \max_{0\le i\le n} \sup_{x_{(i)}\le y< x_{(i+1)}} \{F_n(y) - \bar{X}^{-1} L_n(y)\}$$

$$= \max_{0\le i\le n} \{\frac{i}{n} - \bar{X}^{-1} L_n(X_{(i)})\}$$

$$= \max_{1\le i\le n-1} \{\frac{i}{n} - W_{ni}\} \quad \text{from (2.3)}.$$

Hence,

$$K_n = \max\{\max_{0\le i\le n-1} [W_{n\ i+1} - \frac{i}{n}],\ \max_{1\le i\le n-1} [\frac{i}{n} - W_{ni}]\}.$$

Next, we show that K_n converges to $K(F)$ almost surely as n increases.

Proposition 2.2. Let $F(0) = 0$. Then

(2.5) $$K_n \to K(F) \text{ a.s. } [F] \text{ as } n\to\infty.$$

(Proof.) Note that

$$|K_n - K(F)| = |\sup_y |\bar{X}^{-1} L_n(y) - F_n(y)| - \sup_y |\mu^{-1} L(y) - F(y)||$$

$$\le \sup_y ||\bar{X}^{-1} L_n(y) - F_n(y)| - |\mu^{-1} L(y) - F(y)||$$

$$\le \sup_y |\bar{X}^{-1} L_n(y) - F_n(y) - \mu^{-1} L(y) + F(y)|$$

$$\le \sup_y |\bar{X}^{-1} L_n(y) - \mu^{-1} L(y)| + \sup_y |F_n(y) - F(y)|.$$

Two terms of the right hand side of the last inequality converge to 0 almost surely as n increases, by Lemma 2.1 of Koul (1978) and the Glivenko-Cantelli lemma.

2.2 Cramer-von Mises Type Statistic

For a non-degenerate life distribution F which is NBUE or NWUE with mean μ, $0<\mu<\infty$, let

(2.6) $$C(F) = \mu^{-1} \int_0^\infty [\mu^{-1} L(y) - F(y)]^2 \bar{F}(y)\, dy.$$

By the same argument as in Kolmogorov-Smirnov type statistic in §2.1, we propose

(2.7) $$C_n = \bar{X}^{-1} \int_0^\infty [\bar{X}^{-1} L_n(y) - F_n(y)]^2 \bar{F}_n(y)\, dy.$$

And we reject H_0 in favor of H_1, if the test statistic C_n is large. We propose the following proposition which is parallel to Proposition 2.1 in §2.1.

Proposition 2.3

(2.8) $$C_n = \sum_{i=1}^{n-1} (W_{ni} - W_{n\ i-1})(W_{ni} - \frac{i}{n})^2$$

(Proof.)

$$C_n = \bar{X}^{-1} \int_0^\infty \bar{X}^{-1} L_n(y) - F_n(y)]^2 \bar{F}_n(y)\, dy$$

$$= \bar{X}^{-1} \sum_{i=1}^{n} \int \{X_{(i-1)} < y \leq X_{(i)}\} [\bar{X}^{-1} L_n(y) - F_n(y)]^2 dL_n(y)$$

$$= \bar{X}^{-1} \sum_{i=1}^{n} [\bar{X}^{-1} L_n(X_{(i)}) - F_n(X_{(i)})]^2 [L_n(X_{(i)}) - L_n(X_{(i-1)})]$$

$$= \sum_{i=1}^{n-1} (W_{ni} - \frac{i}{n})^2 (X_{ni} - W_{n\ i-1}), \quad \text{by (2.3).}$$

The following proposition is parallel to Proposition 2.2 in §2.1.

Proposition 2.4. Let $F(0) = 0$. Then

(2.9) $$C_n \to C(F) \quad \text{a.s.} \quad [F] \quad \text{as} \quad n \to \infty.$$

(Proof.) Note that

$$|C_n - C(F)|$$

$$= |\bar{X}^{-1} \int_0^\infty [\bar{X}^{-1} L_n(y) - F_n(y)]^2 \bar{F}_n(y)\, dy - \mu^{-1} \int_0^\infty [\mu^{-1} L(y) - F(y)]^2 \bar{F}(y)\, dy|$$

$$= |\bar{X}^{-1} \int_0^\infty [\bar{X}^{-1} L_n(y) - F_n(y)]^2 \bar{F}_n(y)\, dy - \mu^{-1} \int_0^\infty [\mu^{-1} L(y) - F(y)]^2 \bar{F}_n(y)\, dy$$

$$+ \mu^{-1} \int_0^\infty [\mu^{-1} L(y) - F(y)]^2 \bar{F}_n(y)\, dy - \mu^{-1} \int_0^\infty [\mu^{-1} L(y) - F(y)]^2 \bar{F}(y) dy|$$

$$\leq \sup_y |\bar{X}^{-1} [\bar{X}^{-1} L_n(y) - F_n(y)]^2 - \mu^{-1} [\mu^{-1} L(y) - F(y)]^2|$$

$$+ |\mu^{-1} \int_0^\infty [\mu^{-1} L(y) - F(y)]^2 \, d[L_n(y) - L(y)]|.$$

Two terms of the R.H.S. of the inequality converge to 0 almost surely as n increases, by Lemma 2.1 of Koul (1978) and the strong law of large numbers. Thus the proposition follows.

III. EXACT BAHADUR SLOPE OF K_n

In order to discuss the exact Bahadur slope of K-test, we first find the asymptotic null distribution. The concept and definition of exact slope can be found in Bahadur (1971).

Since the proposed statistics K_n and the considered problem are scale invariant, we may assume that the scale parameter λ in H_0 to be 1 without loss of generality. For $i = 1, \ldots, n\text{-}1$, let U_i be i-th order statistic of a random sample of size n-1 from the Uniform (0,1) distribution. A distributional property of statistic W_{ni} in (2.3) under H_0 is the following.

Lemma 3.1. For $i = 1, \ldots, n\text{-}1$, W_{ni} and U_i have the same distribution, under H_0.

(Proof.) For an exponential distribution, it is known that the distributions of the normalized spacings

$$D_j = (n - j + 1)[X_{(j)} - X_{(j-1)}], \quad j = 1, \cdots, n$$

are the same as the original distribution of X. Thus W_{ni} has the Beta $(i, n\text{-}i)$ distribution for $i = 1, \ldots,$

n-1. Hence the lemma follows.

By combining Proposition 2.1 and Lemma 3.1, we obtain the following theorem.

Theorem 3.2 Under H_0, K_n has the asymptotically same distribution as the Kolmogorov-Smirnov statistic

$$A_n = \sup_{0 \le y \le 1} |G_n(y) - y|, \tag{3.1}$$

where G_n is the empirical distribution function based on $U_i, \cdots, U_{n-1}$.

(Proof.) For $i=0, \cdots, n-1$, note that

$$G_n(y) = \frac{i}{n-1}, \quad U_i \le y < U_{i+1} \text{ with } U_0 = 0 \text{ and } U_n = 1.$$

Then,

$$A_n = \sup_{0 \le y \le 1} |G_n(y) - y|$$

$$= \max\{\sup_{0 \le y \le 1} [G_n(y) - y], \sup_{0 \le y \le 1} [y - G_n(y)]\}.$$

Here,

$$\sup_{0 \le y \le 1} \{G_n(y) - y\} = \max_{0 \le i \le n-1} \sup_{u_i \le y \le u_{i+1}} \{G_n(y) - y\}.$$

$$= \max_{1 \le i \le n-1} \{\frac{i}{n-1} - U_i\}.$$

Similarly,

$$\sup_{0 \le y \le 1} \{y - G_n(y)\} = \max_{0 \le i \le n-1} \sup_{u_i \le y \le u_{i+1}} \{y - G_n(y)\}$$

$$= \max_{0 \le i \le n-1} \{U_{i+1} - \frac{i}{n-1}\}.$$

Thus it is obtained that

$$A_n = \max\{\max_{1 \le i \le n-1} [\frac{i}{n-1} - U_i], \max_{0 \le i \le n-1} [U_{i+1} - \frac{i}{n-1}]\}.$$

From Proposition 2.1 and Lemma 3.1, the theorem follows.

Remark. Since the statistic A_n is distribution free, K_n has the asymtotically same distribution as the Kolmogorov-Smirnov statistic for an arbitrary distribution function.

To compute the exact slope of K_n, consider the behavior of large diviation probability of K_n, $P(K_n \geqq a)$, $0 < a < 1$. Then

$$\lim_n n^{-1} \ln P(K_n \geqq a) = \lim_n n^{-1} \ln P(A_n \geqq a) \tag{3.2}$$

$$= -g(a) \text{ for all } a, 0 < a < 1,$$

where

$$g(a) = \inf_{0 \le t \le 1} f(a, t)$$

with

$$f(a,t)=(a+t)\ \text{in}\ \frac{a+t}{t}+(1-a-t)\ \text{in}\ \frac{1-a-t}{1-t},\quad 0\leq t\leq 1-a,$$

$$=\infty \qquad ,t>1-a$$

and

$$g(a)=2a^2+0(a^3)\ \text{as}\ a\rightarrow 0,$$

by Example 5.3 of Bahadur (1971).

Further, from Proposition 2.2, K_n converges to $K(F)$ almost surely as n increases, and $0<K(F)<1$ for all $F\in H_1$. Therefore, by Theorem 7.2 of Bahadur, the exact slope of the statistic K_n is approximated by

(3.3) $$S_k(F)=4[K(F)]^2\ \text{as}\ K(F)\rightarrow 0\quad \text{for all}\quad F\varepsilon H_1.$$

IV. An Example for Weibull Alternative

Consider the Weibull distribution given by

$$\bar{F}(x)=e^{-x^\theta},\ x>0,\theta\neq 1.$$

Note that this distribution is IFR for $\theta\geq 1$ and Decreasing Failure Rate (DFR) for $\theta\leq 1$. Thus it is NBUE for $\theta\geq 1$ and NWUE for $\theta\leq 1$. It is well known that

$$\mu=\Gamma[(\theta+1)^{-1}]=\theta^{-1}\Gamma(\theta^{-1}).$$

Then, from (2.1),

$$K(\theta)=K(F)$$

$$=\sup_y|\mu^{-1}L(y)-F(y)|,$$

where $L(y)=\int_0^y e^{-x\infty}\,dx.$

for $\theta>1$, since $\mu^{-1}L(y)>F(y)$,

$$K(\theta)=\sup_y\{\mu^{-1}L(y)-F(y)\},$$

we can find $y=y_\theta$ at which the supremum is obtained.

Observe that

$$\frac{d}{dy}\{\mu^{-1}L(y)-F(y)\}=\mu^{-1}\bar{F}(y)-f(y)$$

$$=\mu^{-1}\bar{F}(y)-\theta y^{\theta-1}\bar{F}(y)$$

$$=\bar{F}(y)(\mu^{-1}-\theta y^{\theta-1}).$$

Then $\{\mu^{-1}L(y)-F(y)\}$ is non-decreasing for $y\leq(\theta\mu)^{(1-\theta)^{-1}}$ and non-increasing for

$y \geqq (\theta\mu)^{(1-\theta)^{-1}}$. For $\theta < 1$, Similarly,

$$K(\theta) = \sup_y \{F(y) - \mu^{-1} L(y)\}$$

$$= F(y_\theta) - \mu^{-1} L(y_\theta).$$

Thus $K(\theta)$ is obtained at

$$y_\theta = [\Gamma(\theta^{-1})]^{(1-\theta)^{-1}} \text{ for } \theta \neq 1.$$

Therefore,

$$K(\theta) = |\mu^{-1} L(y_\theta) - F(y_\theta)| \tag{4.1}$$

$$= |\theta[\Gamma(\theta^{-1})]^{-1} \int_0^y \theta_e^{-x^\theta} dx - [1 - e^{-(y_\theta)^\theta}]|,$$

where $y_\theta = [\Gamma(\theta^{-1})]^{(1-\theta)^{-1}}$

Further, the local slope of K_n,

$$S_k(\theta) = 4K(\theta)^2 \text{ as } \theta \to 1, \text{ from (3.3).} \tag{4.2}$$

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