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(s, S) Spare Part Inventory System*

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ABSTRACT

This paper deals with a continuous review (s,S) spare part inventory system. The distributions of service life of each part and the replenishment lead time are assumed to be exponential. Assuming that there is never more than a single order outstanding, we obtain the average annual cost of operating the inventory system. If the length of stockout period is small enough to be neglected compared to the length of operating period, the optimal operating policy variables minimizing the cost rate can be calculated iteratively. For the case of one-for-one ordering (that is, $s=S-1$), an exact cost rate, and a closed form decision rule minimizing the cost rate are obtained for a more general situation in which more than one order is allowed to be outstanding and the distribution of the replenishment lead time is general

I. INTRODUCTION

This paper deals with a spare part inventory system, in which stock depletion arises not from the external market demand but from internal demand resulting from failure of the part in use. Suppose that a piece of equipment is required to be used for an infinite time span. The equipment has a vital part which fails according to some probability law. If an order for spare parts is placed, the delivery takes place after a random lead time. Since having the equipment inoperative is a significant cost factor, it seems desirable to reorder spare parts before the stock level falls to zero. Further, because of fixed ordering cost, which is independent of the quantity ordered, quantity purchases might be desirable, but holding or storage cost furnishes a restraint on ordering too much. Thus, in order to minimize the total cost, we must determine when to order for spare parts and how many spare parts to purchase on each order.

This problem differs from the classical inventory problem in that the demand for the part never arises during stockout period, since the equipment remains inoperative when stockout occurs until the failed part is replaced by new one. Karlin [3] introduced (s,S) inventory model for this problem and derived some probability quantities associated with the spare part inventory model, but some of his results contain some errors. Falkner [1] treated a similar problem, but his study was a single-period model in

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which the procurement of spare parts was allowed once only at the beginning of planning horizon and the problem was to determine initial stock level. Park [6] considered $(S-1,S)$ spare part inventory model for fleet maintenance, which is a particular case of (s,S) inventory model with $s=S-1$, and presented a heuristic algorithm to determine the optimal stock level.

In this paper, we consider (s,S) spare part inventory problem treated by Karlin [3], and correct Karlin's results and present an iterative procedure to determine a pair of values (s,S) which minimizes the cost rate for an infinite time span. For the case of one-for-one ordering (that is, $s=S-1$), a closed form decision rule which minimizes the cost rate is obtained.

II. *(s,S)* Spare Part Inventory Model

In this model, the stock level (that is, the number of spare parts) is reviewed continuously, and an order for a quantity $Q(=S-s)$ is placed when the stock level including the part in use drops to a reorder point s. The following assumptions and notations are used.

Assumptions

1. The unit cost of a part is a constant independent of the order quantity.
2. The distribution of service life of a part is exponential.
3. The distribution of a replenishment lead time is exponential.
4. There is never more than a single order outstanding.

Notations

s	reorder point,
S	reorder level; inventory position (on hand plus on order) just after an order,
Q	S-s; order quantity per order,
λ	failure rate of a part,
x	random variable denoting a replenishment lead time,
$g(x)$	$\mu \exp(-\mu x)$; p.d.f. of a replenishment lead time,
$p(k)$	$(\lambda x)^k \exp(-\lambda x)/k!$; Poisson p.m.f. with parameter λx,
$\bar{P}(s)$	$\sum_{k=s}^{\infty} p(k)$; complementary Poisson c.d.f.,
$\Psi(v)$	probability that v units are on hand just after the arrival of an order,
c_0	fixed ordering cost per order,
c_h	holding cost of a part per unit time,
c_s	shortage cost per unit time of having the equipment idle.

Other notations are defined as needed.

Imbedded in the implementation of the (s,S) policy is a natural cycle which commences when an order is delivered and lasts until the next order is delivered. The special nature of the exponential lifetime distribution, which in effect implies that any conditional density is independent of the conditional statement (; the so-called "lack of memory"), enables us to start the process anew when delivery takes place, regardless of the age of the part in use [3].

We shall now derive the exact equantions for the (s,S) spare part inventory model. Our formulation here parallels the 'lot size reorder point models with stochastic demands' of Hadley and Whitin [2].

The expected cost per cycle is the sum of the ordering, holding, and shortage costs. Since the number of orders per cycle is one, the ordering cost per cycle is c_0. The expected holding cost per cycle will be computed in two parts. First, for the time period up to the time the reorder point is reached, and second for the period of the exponential replenishment lead time x from the time the reorder point is reached until the next order arrives. Given that v units are on hand including the unit in use after an order arrives, the expected unit years of spares in stock held until the reorder point is reached is:

$$[(v-1)+(v-2)+\cdots\cdots+s]/\lambda=[v(v-1)-s(s-1)]/2\lambda\,. \tag{1}$$

The probability that v units are on hand just after the arrival of an order is:

$$\Psi(v)=\begin{cases}\int_0^\infty \bar{P}(s;\lambda x)g(x)dx, & if\ v=Q\\ \int_0^\infty p(Q+s-v;\lambda x)g(x)dx, & if\ Q<v\leq Q+s\,.\end{cases} \tag{2}$$

Averaging Equation (1) over the initial inventory v, the expected unit years stock held until the reorder point is reached is:

$$\sum_{v=Q}^{Q+s}\Psi(v)[v(v-1)-s(s-1)]/2\lambda$$

$$=[Q(Q-1)-s(s-1)]\int_0^\infty \bar{P}(s;\lambda x)g(x)dx/2\lambda$$

$$+\sum_{v=Q+1}^{Q+s}[v(v-1)-s(s-1)]\int_0^\infty p(Q+s-v;\lambda x)g(x)dx/2\lambda\,. \tag{3}$$

The expected unit years of spares in stock held from the time the reorder point is reached until the next order arrives is the integral from 0 to the replenishment lead time x of the expected amount of spares on hand at time t, *i.e.*,

$$\int_0^\infty\int_0^x\sum_{u=0}^{s-1}(s-u-1)p(u;\lambda t)g(x)\,dt\,dx$$

$$=(s/\lambda)\sum_{u=0}^{s-1}\int_0^\infty \bar{P}(u+1;\lambda x)g(x)dx-(1/\lambda)\sum_{u=0}^{s-1}\int_0^\infty(u+1)\bar{P}(u+1;\lambda x)g(x)\,dx\,. \tag{4}$$

On summing Equations (3) and (4), and making some algebraic manipulations, we find that the expected number of unit years of stock held per cycle is:

$$\int_0^\infty Q[(Q-1)/2\lambda+s/\lambda-x+x\bar{P}(s-1;\lambda x)-(s/\lambda)\bar{P}(s;\lambda x)]g(x)\,dx$$

$$=Q[(Q-1)/2\lambda+s/\lambda-1/\mu+\lambda^s/\mu(\lambda+\mu)^s]\,. \tag{5}$$

Thus the expected holding cost per cycle is:

$$c_h\,Q[(Q-1)/2\lambda+s/\lambda-1/\mu+\lambda^s/\mu(\lambda+\mu)^s]\,. \tag{6}$$

Let us now compute the expected shortage cost per cycle. If the system reaches an out of stock condition in the time interval between t and $t+dt$ after the reorder point is hit, this implies that in the time 0 to t, $s-1$ units have been demanded and the s-th one is demanded between t and $t+dt$, it will be out of stock for a length of time $x-t$ during the cycle. Hence the expected length of stockout per cycle is:

$$\int_0^{\infty}\int_0^{x} \lambda(x-t)\, p(s-1;\lambda t) g(x)\, dt\, dx = \lambda^s/\mu(\lambda+\mu)^s. \tag{7}$$

Thus the expected total cost per cycle is:

$$c_0 + c_h Q[(Q-1)/2\lambda + s/\lambda - 1/\mu + \lambda^s/\mu(\lambda+\mu)^s] + c_s \lambda^s/\mu(\lambda+\mu)^s. \tag{8}$$

Since the time between successive replenishments is a cycle, the expected duration of a cycle is the expected time between the arrival of an order and the next order plus the expected replenishment lead time. Thus the expected cycle length is:

$$\sum_{v=Q}^{Q+s} \Psi(v)\cdot(v-s)/\lambda + \int_0^{\infty} x g(x)\, dx = Q/\lambda + \lambda^s/\mu(\lambda+\mu)^s. \tag{9}$$

From the renewal reward theorem [7, p. 52], the expected cost rate for an infinite time span is the expected cost per cycle divided by the expected cycle length. Hence, the expected cost rate is:

$$\frac{c_0 + c_h Q[(Q-1)/2\lambda + s/\lambda - 1/\mu + \lambda^s/\mu(\lambda+\mu)^s] + c_s \lambda^s/\mu(\lambda+\mu)^s}{Q/\lambda + \lambda^s/\mu(\lambda+\mu)^s} \tag{10}$$

III. THE ITERATIVE OPTIMIZATION PROCEDURE

Since having the equipment inoperative is a significant cost factor, it might be assumed that the length of stockout period is small enough to be neglected compared to the length of the operating period as Hadley and Whitin [2, p. 168]. Then, the expected cycle length in Equation (9) approximates to Q/λ and the expected cost rate in (10) approximates to the following convex function:

$$C(Q,s) = \lambda c_0/Q + c_h \lambda[(Q-1)/2\lambda + s/\lambda - 1/\mu + \lambda^s/\mu(\lambda+\mu)^s] + c_s \lambda^{s+1}/Q\,\mu(\lambda+\mu)^s. \tag{11}$$

A condition for Q and s being optimal is that they satisfy

$$\partial C/\partial Q = -[\lambda c_0 + c_s \lambda^{s+1}/\mu(\lambda+\mu)^s]/Q^2 + c_h/2 = 0 \tag{12}$$

and

$$\partial C/\partial s = c_h - (c_h + c_s/Q)\lambda^{s+1}/\mu(\lambda+\mu)^s\, l_n(1+\mu/\lambda) = 0\ . \tag{13}$$

Here we have two equations to be solved for Q and s. It is convenient to write Equations (12) and (13) as:

$$Q=\sqrt{2\lambda[c_0+c_s\lambda^s/\mu(\lambda+\mu)^s]/c_h} \tag{14}$$

and

$$s=l_n[(\lambda/\mu)(1+c_s/c_hQ)\,l_n(1+\mu/\lambda)]/l_n(1+\mu/\lambda) \tag{15}$$

Since $C(Q,s)$ is convex, the solutions Q^* and s^* obtained from Equations (14) and (15) yield an absolute minimum, and Q^*+s^* is the optimal value of S.

To find the optimal pair (Q^*, s^*), the following iterative procedure of Hadley and Whitin [2, pp. 169-172] can be used.

(1) The initial estimate for Q is $Q=\sqrt{2\lambda c_0/c_h}$ (Wilson's lot size formula). Call this value Q_1.

(2) Use Equation (15) with $Q=Q_1$ to find the reorder point s. Call this value s_1.
(3) Use Equation (14) with $s=s_1$ to find Q_2.
(4) Repeat Step (2) with $Q=Q_2$, etc. Convergence occurs when at iteration i, $Q_i=Q_{i-1}$ or $s_i=s_{i-1}$.

A test for the convergence of the iterative procedure can be carried out using the following argument as in Kim and Park's [4].
Equations (14) and (15) can be thought of as describing two curves in the (Q,s) plane. For the curve described by Equation (15), we note that

when $Q=0$, $s=\infty$

and

when $Q=\bar{Q}\equiv c_s(\lambda/\mu)\,l_n(1+\mu/\lambda)/c_h[1-(\lambda/\mu)\,l_n(1+\mu/\lambda)]$, $s=0$.

Furthermore, differentiating Equation (15) yields:

$ds/dQ < 0$.

For the curve described by Equation (14), note that

when $s=0$, $Q=\hat{Q}\equiv\sqrt{2\lambda(c_0+c_s/\mu)/c_h}$

and

when $s=\infty$, $Q=Q_1\equiv\sqrt{2\lambda c_0/c_h}$

Furthermore, differentiating Equation (14) yields:

$dQ/ds < 0$

or, equivalently

$ds/dQ < 0$.

Ploting of the two curves is shown in Figure 1. It is obvious that the iterative procedure discussed above must converge to Q^* and s^*. Since $C(Q,s)$ is convex, the solutions Q^* and s^* to Equations (14) and (15) are always unique.

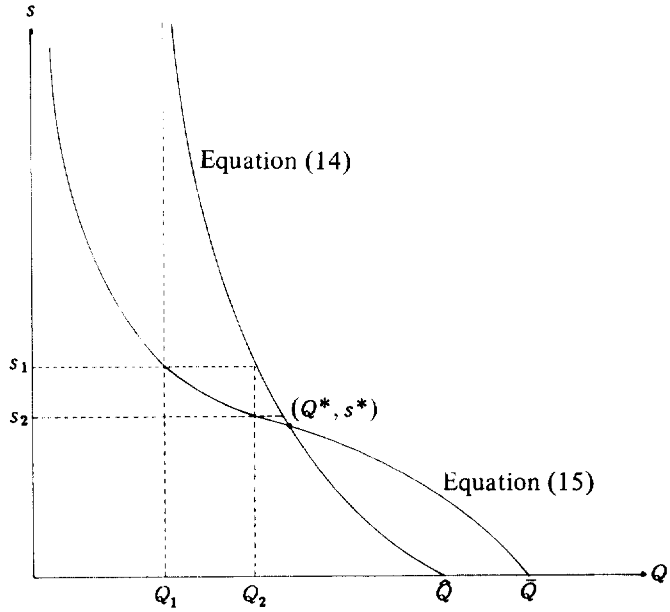

Figure 1. Convergence of the solution

IV. A PARTICULAR CASE: (S-1,S) SPARE PART INVENTORY MODEL

In this section, we consider a special case of (s,S) model in which $s=S-1$. In the $(S-1,S)$ policy, a reorder is placed whenever a demand occurs (*i.e.*, one-for-one ordering) and the inventory position remains constant. This particular policy is commonly used and considered suitable in low-demand or high-cost item inventory such as aircraft spare parts inventory [8].

Since there exists no demand during stockout period in the state-dependent demand model, the steady-state probabilities for on-hand inventory (including the operating unit) are the same as the steady-state probabilities in the lost sales case of Hadley and Whitin [2, pp. 211-212]. Thus the steady-state probabilities that the on-hand inventory is k $(k=0,1,\cdots,S)$ are:

$$\pi(k)\equiv p(S-k)/[1-\bar{P}(S+1)], \tag{16}$$

known as the truncated Poisson distribution.

In the derivation of the steady-state probabilities, more than one order was allowed to be outstanding. Further, exponential replenishment lead time was assumed in the derivation, but the same result holds for arbitrary replenishment lead time [5].

The expected cost rate for an infinite time span, $C(S)$, is:

$$C(S)=c_0\lambda[1-\pi(0)]+c_h\sum_{k=1}^{S}(k-1)\pi(k)+c_s\pi(0)$$

$$=c_hS+[c_h(1+\lambda/\mu)+c_s-c_0\lambda]\pi(0)+[c_0\lambda-c_h(1+\lambda/\mu)]. \tag{17}$$

Note that $\pi(0)(=p(S)/[1-\bar{P}(S+1)])$ is the Erlang loss formula and it is decreasing convex in S [9]. If $c_s-c_0\lambda<0$, there should not be any one-for-one ordering policy, since it implies that incurring the cost of shortage all the lifetime (c_s/λ) is cheapter than the ordering cost (c_0). Thus, in order to warrant one-for-one ordering, $c_s-c_0\lambda>0$. Therefore the cost rate function in (17) is a convex function, since it is the sum of a linear increasing function, a decreasing convex function, and a constant. Hence, the optimal stock level S^* is the smallest S that satisfies:

$$C(S)-C(S+1)<0$$

or

$$p(S)/[1-\bar{P}(S+1)]-p(S+1)/[1-\bar{P}(S+2)]<c_h/[c_h(1+\lambda/\mu)+c_s-c_0\lambda].$$

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