〈研究論文〉

Journal of the Korean Institute of Industrial Engineers
Vol. 12, No. 2, Dec. 1986

A Lagrangean Relaxation Method for the Zero-One Facility Location Problem with Uniform Customer Demands and Facility Capacities

Dong-wan, Tcha*
Chun-beon, Yoo**

Abstract

Consider a capacitated facility location problem in which the demands of customers are all equal and so are the capacities of facilities. Under the restriction that each customer's uniform demand be met by exactly one facility, the objective is to select a set of facilities to open, and to assign customer's demand to them so as to minimize the total cost which includes fixed costs of opening facilities as well as variable assignment costs. The problem is modelled as a pure zero-one program which may be viewed as a variant of well-known capacitated facility location problems. The purpose of this study is to develop efficient computational procedures for solving the pure zero-one facility location problems. Due to the special structure of our zero-one location problem with uniform demand, it can be converted to a location problem with the unimodular property. A Lagrangean relaxation algorithm is developed to solve the location problem. The algorithm is made efficient by employing a device which exploits the special structure of a surrogate constraint. The efficiency of the algorithm is analyzed through computational experiments with some test problems.

1. Introduction

In this study, we deal with the problem of determining a subset of sites to establish facilities and satisfying customers' uniform demand from this predetermined facilities at minimum fixed and allocation costs. Each customer's demand must be met by exactly one of the facilities, and the total demand of customers assigned to any facility does not exceed its uniform capacity. The basic problem here is to answer three related questions. How many facilities should be established? Where shall they be located? Which customers should be assigned to which facilities? With respect to this problem, possible configurations are trees, stars, loops, etc. We assume that all customers will be connected to their facilities in a star fashion. This star connection will enable us to focus on the basic features of the problem and its algorithms.

One of the application areas is what is called "the concentrator location problem" in a computer

*Korea Advanced Institute of Science and Technology.

** Kyonggi University.

communication network design system. The minimum cost topological design of centralized networks is characterized by the problem of economically connecting many geographically dispersed terminals to one or more central computers. For better link utilization, higher-speed transmission and more cost effective links, points of concentration of data flows may be used. The network problem can then be conceived as a hierarchical design where terminals are connected to the concentrator and each concentrator is also connected to the central computer. One of the basic problems that arises in this design is the question of where and how many concentrators should be located, and how the terminals are assigned to concentrators. All terminals or concentrators will be connected to their concentrators of central computer in a star-star fashion. The fixed costs of placing a concentrator at a site includes the cost of the concentrator itself, link charges to the central computer. By considering central computer as a concentrator, this problem becomes a type of facility location problem. The concentrator location problem has been extensively studied in the literature in [7, 17, 21]. Recently, Mirzaian[17] researched this problem by the techniques based on linear programming relaxation and Lagrangean relaxation. Tang et al.[21] developed an optimization procedure for this problem without computational results. Other algorithms for this proglem usually used heuristic algorithms of facility location problem, e.g. "Add", "Drop", "Interchange" and "Clustering" algorithms (see [5, 7, 15].

In this study, we shall use a Lagrangean relaxation methods as a solution technique for the above pure zero-one location problem with the fixed uniform facility capacity. In recent years a number of papers applying Lagrangean relaxation to the problem have appeared. Useful survey papers on Lagrangean relaxation can be found in Fisher[8], Geoffrion[9], and Shapiro[20]. Geoffrion and McBride [10] applied Lagrangean relaxation to the capacited facility location problem with arbitrary additional constraints. Nauss[18] used the same Lagrangean relaxation augmented by a surrogate constraint which ensures that the total supply capacity meets total demand in any subproblem solution. He implemented subgradient optimization to obtain good Lagrangean multipliers and stronger lower bounds. Christofides and Beasley[4] developed a similar approach and obtained slightly better results than Nauss'. Recently, Barcelo and Casanovas[1] studied the application of Lagrangean relaxation to the facility location problem with the single assignment constraints. They designed a heuristic algorithm to solve a Lagrangean subproblem, and the heuristic defined a set of Lagrangean multipliers from the analysis of the dual of linear programming relaxation.

When the Lagrangean relaxation is used to solve our problem, its Lagrangean subproblem can be reduced to an easy-to-solve knapsack problem, and primal feasible solutions can be obtained by solving transportation problems successively. We also use the subgradient optimization technique to obtain good Lagrangean multipliers.

Throughout this study we use the following notations: if Q is an optimization problem, then v(Q) is its optimal solution value, and $\bar{Q}$ is the linear programming relaxation of Q obtained by relaxing all the integrality restrictions on the variables.

2. Properties of the Problem

In this section, we formulate the facility location problem with the fixed uniform capacity as a pure zero-one integer programming problem. This problem can be considered as the special case of the capacitated facility location proglem with single source constraints. As noted in the preceding section, this special case has been extensively studied and it has practical significance in itself.

Let

$I = \{1, 2, \ldots, n\}$: a set of customer indices,

$J = \{1, 2, \ldots, n\}$: a set of facility indices,

c_{ij} : the cost for supplying customer i's demand from the facility j,

f_j : the fixed cost of establishing facility j,

q : the fixed uniform capacity of any facility,

d : the customer's uniform demand requirement,

and as the decision variables,

$$x_{ij} = \begin{cases} 1 & \text{if customer i's demand is supplied from the facility j} \\ 0 & \text{otherwise;} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if a facility is located at site j,} \\ 0 & \text{otherwise.} \end{cases}$$

With these notations, facility location problem can be formulated as the following zero-one integer programming problem:

$$\text{(P)} \quad \text{Min} \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} f_j y_j, \tag{1}$$

$$\text{s.t.} \quad \sum_{j \in J} x_{ij} = 1 \qquad \forall i \in I, \tag{2}$$

$$\sum_{i \in I} d x_{ij} \geqslant q y_j \qquad \forall j \in J, \tag{3}$$

$$x_{ij} \in 0, 1 \qquad \forall i \in I, \ j \in J, \tag{4}$$

$$y_j \in 0, 1 \qquad \forall j \in J. \tag{5}$$

Constraints (2) and (4) require that each customer's demand be met by exactly one facility; constraints (3) ensure that the total demand of customers assigned to any facility can not exceed its capacity. The objective (1) is to minimized total cost which includes fixed costs for opening facilities and allocation costs from facilities to customers.

The model P, unlike the usual capacitated facility location problem, has one outstanding property: single assignment of a customer to a facility. As observed by Barcelo and Casanovas[1], this characteristic generally increases the complexity of the problem. However, the problem can be reformulated using the single assignment restriction cleverly, and the resulting alternative formulation is easier to solve than P.

For each $j \in J$, divide constraints (3) by d, obtaining

$$\sum_{i \in I} x_{ij} \geqslant (q/d) y_j \qquad \forall j \in J.$$

From the integrality of the variables x_{ij}, the above constraints are equivalent to

$$\sum_{i \in I} x_{ij} \geqslant \lfloor q/d \rfloor y_j \qquad \forall j \in J. \tag{3'}$$

where $\lfloor * \rfloor$ means the largest integer less than or equal to $*$.

When replacing constraints (3) by (3') in P, the problem P is unimodular in x_{ij} for fixed values of

y_j. Thus the variables x_{ij} don't have to the specified as the zero-one variables, that drastically decreases the complexity of the original proglem P.

By using this reformulation scheme, i.e., by letting $b = [q/d]$ in constraints (3), P can be reformulated as follows:

$$\text{(P)} \quad \text{Min} \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{j\in J} f_j y_j, \tag{6}$$

$$\text{s.t.} \quad \sum_{j\in J} x_{ij} = 1 \qquad \forall i \in I, \tag{7}$$

$$\sum_{i\in I} x_{ij} \geqslant by_j \qquad \forall j \in J, \tag{8}$$

$$x_{ij} \geqslant 0 \qquad \forall i \in I,\ j \in J, \tag{9}$$

$$y_j \in 0,1 \qquad \forall j \in J. \tag{10}$$

The problem P is unimodular in x_{ij} for fixed values of y_j. Thus the variables x_{ij} don't have to be specified as the 0-1 variables. A number of equivalent type formulation can be found in the literature (see [17, 21]).

An interesting extension of constraints may be considered by adding the following constraints

$$\sum_{j\in J} g_{hj} y_j \geqslant r_h \qquad \forall h \in H \tag{11}$$

If we set $h = 1$, $g_{lj} = 1\ \forall j \in J$ and $r_l = M$, then constraints (11) becomes a median constraint which limits the number of facility to be opened to a maximum M facilities. And if we set $h = 1$, $g_{lj} = f_j$ and r_l = available budget(F), then constraints (11) become $\sum_{j\in J} f_j y_j \geqslant F$ which limits the budget of establishing facilities. This type of constraints can be found in some location models (see Guignard and Spielberg[12], Barcelo and Casanovas[1]). Though all the results in this chapter can accomodate the variations equally effectively with minor modifications, their adjustment will be given briefly in our procedure without computational results.

3. Solution via Lagrangean Relaxation

3.1 Lower Bounds

Many hard discrete optimization problems can be viewed as easy problems complicated by a relatively small set of side constraints. Dualizing the constraints produces a Lagrangean problem which is easy to solve and whose optimal value is a lower bound on the optimal value of the original minimization problem. The obtained lower bound is at least as strong as the linear programming lower bound, and furthermore useful strong penalties are yielded by the Lagrangean relaxation (see Geoffrion [9]). Lagrangean relaxation is a highly flexible device based on the formal application of Lagrangean duality theory. Its application must be tailored to fit the particular problem structure at hand in order to achieve full effectiveness.

In the special context of problem P, we define the Lagrangean relaxation of P relative to Lagrangean multipliers u_i ($\geqslant 0$, $i = 1, \ldots, n$) for the constraints (7) as:

$$\text{(LR)} \quad \text{Min} \sum_{i\in I}\sum_{j\in J}(c_{ij}\text{-}u_i)\,x_{ij} + \sum_{j\in J} f_j y_j + \sum_{i\in I} u_i, \tag{12}$$

$$\text{s.t.} \quad \sum_{i\in I} x_{ij} \geqslant b y_j \qquad \forall j \in J, \tag{13}$$

$$x_{ij} \geqslant 1 \qquad \forall i \in I, j \in J, \tag{14}$$

$$x_{ij} \geqslant 0 \qquad \forall i \in I, j \in J, \tag{15}$$

$$y_j \in 0, 1 \qquad \forall j \in J. \tag{16}$$

Constraints (14) are redundant in P, but they are included in this Lagrangean subproblem(LR) to ensure the primal feasibility and then to enhance lower bounds. It is easily seen that $v(P) \geqslant v(LR) \geqslant v(\overline{P})$ if optimal Lagrangean multipliers, u from the $\overline{P}$ are used (see Geoffrion[9]). Unfortunately, in practice sometimes $v(LR) = v(\overline{P})$ and then the Lagrangean relaxation may be not so strong as $\overline{P}$. However, there are at least two improvements which have empirically tightened the relaxation (see Nauss [18]).

The first is to find a good set of Lagrangean multipliers. They can be obtained from the subgradient optimization technique which will be described in the later section.

The second improvement is to append the constraint:

$$\sum_{j\in J} b y_j \geqslant n \qquad \text{to (LR).}$$

This surrogate constraint forces feasibility of any facility design by requiring that sufficient facilities are opened to handle the total demand of customers. Of course, the constraints (7) may still be violated in LR since they have been relaxed. From now on, we assume that the surrogate constraint is included in LR.

Consider the effect of setting y_j to one, then the best set of allocations of customer j to this facility is given by

$$\text{(LR1)} \quad \text{Min} \sum_{i\in I}(c_{ij} - u_i)\,x_{ij}, \tag{17}$$

$$\text{s.t.} \quad \sum_{i\in I} x_{ij} \geqslant b \qquad \forall j \in J, \tag{18}$$

$$x_{ij} \geqslant 1 \qquad \forall i \in I, j \in J, \tag{19}$$

$$x_{ij} \geqslant 0 \qquad \forall i \in I, j \in J. \tag{20}$$

LR1 has the similar structure of a knapsack problem, but the coefficient matrix of the constraint has the unit weight. Hence this problem can be easily solved by the procedure outlined below (see Christofides and Beasley[4]).

(a) Order x_{ij} according to the increasing values of $(c_{ij}\text{-}u_i)$, i.e. by the increasing order of the cost from facility j to customer i.

(b) Set each $x_{ij} = 1$ (i.e. customer i is assigned to facility j) in this order, provided that there is enough capacity left at j to assign customer i taking into account the other customers already assigned to facility j and $(c_{ij}-u_i) \geqslant 0$. When we encounter a customer i for whom there is not enough spare capacity at j to assign i, set x_{ij} to zero.

(c) Let the set of values for x_{ij} chosen as in (b) $\bar{x}_{ij}$ (all x_{ij} not given an explicit value in (b) being zero).

Let e_j represent the contribution to the Lagrangean dual objective function resulting from setting y_j to one, then

$$e_j = f_j + \sum_{i \in I} (c_{ij} - u_i) \bar{x}_{ij} \quad \forall j \in J. \tag{21}$$

Computing e_j $\forall j \in J$ as above the Lagrangean relaxation program reduces to

$$\text{(LR2)} \quad \text{Min} \sum_{j \in J} e_j y_j + \sum_{i \in I} u_i, \tag{22}$$

$$\text{s.t.} \sum_{j \in J} b y_j \geqslant n, \tag{23}$$

$$y_j \in \ 0, 1 \qquad \forall j \in J. \tag{24}$$

This program LR2 is a zero-one knapsacke problem. But, since it holds that b has the same value for all j, the LR2 can be solved by inspection as follows:

(a) By letting $R = \lceil n/b \rceil$ where $\lceil * \rceil$ means the smallest integer greater than or equal to $*$, the constraints (23) can be written as:

$$\sum_{j \in J} y_j \geqslant R. \tag{25}$$

(b) Form a list of the e_j arranged in ascending order.

(c) Set each $y_j = 1$ until both

   (i) have opened R facilities and

   (ii) the e_j values become nonnegative.

(d) Let the set of values for the y_j chosen as above be $\bar{y}_j$ (all j not picked above having $\bar{y}_j = 0$).

Then the optimal solution to the Lagrangean subproblem is ($\bar{y}_j$, $\forall j \in J$ and $\bar{x}_{ij}$, $\forall i \in I, j \in J$) with the solution value v(LR) given by

$$v(\text{LR}) = \sum_{j \in J} e_j \bar{y}_j + \sum u_i \tag{26}$$

For any set of Lagrangean multipliers, the LR can be easily solved by the preceeding procedures. The equation (26) presents a lower bound on the optimal objective value of P.

3.2. Determination of a Feasible Solution

A feasible solution to the P is not automatically available from the solution to the Lagrangean problem given above, since the solution x_{ij} of the LR1 may have the value of one for a closed facility j. The advantage in searching for a good feasible solution is that, if an improved feasible solution is found, the better Lagrangean multipliers are obtained within the small number of iterations in the subgradient optimization. For a solution $\bar{y}_j$ of the LR2, the P can be rewritten as:

$$v(TP) = \sum_{j\in J} f_j\bar{y}_j + \text{Min} \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}, \tag{27}$$

$$\text{s.t.} \sum_{j\in J} x_{ij} = 1 \qquad \forall i \in I, \tag{28}$$

$$\sum_{i\in I} x_{ij} \geqslant b\bar{y}_i \qquad \forall J \in J, \tag{29}$$

$$x_{ij} \geqslant 0 \qquad \forall i \in I, j \in J. \tag{30}$$

Since the coefficient matrix of problem TP is totally unimodular and $a_i \; \forall \; i \in I'$ where $I' = I \cup (n+1)$ and $b\bar{y}_j \; \forall j \in J$ be considered as a supply and a demand respectively, the problem can be reduced to the ordinary balanced transportation problem by setting $a_i = 1 \; \forall \; i \in I$, $a_{n+1} = \sum_{j\in J} by_j - n$ and $c_{n+1,j} = 0 \; \forall \; j \in J$. To solve TP, we employ the out-of-kilter algorithm among several efficient solution methods for the transportation problem. In constructing a circulating form for applying the out-of-kilter method, several procedures should be followed (se Phillips and Garcia-Diaz[19]). An optimal solution of TP becomes a feasible solution of P and also is an upper bound of the optimal objective value of P.

3.3. The Subgradient Procedure

Our primary goal in selecting Lagrangean multipliers u_i is to find one providing the greatest lower bound, or in other words, one which is optimal in the dual problem

$$D = \text{Max } LR(u_i) \tag{31}$$

$$\text{s.t. } u_i \geqslant 0 \qquad \forall i \in I \tag{32}$$

An algorithm for D, the subgradient method has been popular in Lagrangean ralaxation applications. Fisher[8], Geoffrion[9] and Held et al.[15] discussed generally on the solution of D within the context of Lagrangean relaxation. The subgradient method is a brazen adaptation of the gradient method in which gradients are replaced by subgradients (see Fisher[8]). The subgradient procedure for obtaining better Lagrangean multipliers u_i is as follows:

(1) Choose an initial value of u_i as follows:

$$u_i = \min_{j\in J} \{c_{ij}\} \qquad \forall i \in I \tag{33}$$

(2) Determine an initial value for $v^*(TP)$ – the upper bound on the problem P. This can be obtained from the transportation problem in Section 3.2.

(3) For the current set of Lagrangean multipliers u_i, determine the optimal Lagrangean dual solution $v^*(LR)$ with the set of open facilities and x_{ij} the associated allocation values. Update the best lower bound found $v(LR)$ accordingly.

(4) Use a transportation algorithm of Section 3.2 to derive a feasible solution from the set of open facilities. Update $v(TP)$ accordingly.

(5) Stop if $v(TP) = v(LR)$, i.e. the best lower bound and the upper bound coincide, else go to (6).

(6) Define the subgradient vector s by

$$si = 1 - \sum_{j\in J} x_{ij} \qquad \forall i \in I \tag{34}$$

Stop if $s_i = 0 \ \forall i \in I$, else go to (7).

(7) Calculate the step size t for use in updating Lagrangean multipliers by

$$t = \pi \, [v(TP) - v^*(LR)] \, / \, \|s\|^2 \tag{35}$$

where π a constant $(0 > \pi \geqslant 2)$ and $\|s\|$ any norm of the subgradient vector – we used the Euclidean norm $(\sum_{i \in I} s_i^2)^{1/2}$

(8) Update the multipliers by

$$u_i = \max \{ \min (c_{ij}), u_i + t \cdot s_i \} \ \forall i \in I \tag{36}$$

(9) Go to (3) unless sufficient subgradient iterations have been done. Otherwise stop.

Determining the Lagrangean multipliers and step size have influence on the solution quality of the Lagrangean problem. Usually the method of selecting the Lagrangean multipliers in the above is the most natural choice but in some cases one can implement different methods (see Hogan et al. [14]).

Camerini et al.[3] proved that the efficiency of the relaxation technique is further improved by selecting the modified gradient direction

$$s_i^k = u_i^k + h_k s_i^{k-1} \tag{37}$$

where k is the iteration number and h_k is a suitable scalar. Note that equation (37) is in fact equivalent to a weighted sum of all preceding subgradient direction, which has been successfully used by Crowder [6] in order to avoid some possible troublesome effects due to the "subgradient's alternating components". Bazaraa and Sherali[2] have researched on the choice of step size in subgradient optimizaiton. Their method has two phase, the first phase is designed to accelerate the solution procedure towards an optimal solution and the step size are determined according to equation (35) where $\pi = 1$ and v(TP) is periodically updated by some weighted average. And the second phase helps the procedure computationally in repidly closing in on the optimal objective value.

Karkazis[16] developed a "principal direction search" method for uncapacitated facility location problem to overcome the main drawback of the subgradient method, namely the possibility of the values of v(LR) decreasing at some steps. Computational performance and theoretical convergence properties of the subgradient method are discussed in Held at al. [13] and Geoffrion [9]. The fundamental theoretical results is that optimal Lagrangean solution is obtained if $t_k \to 0$ and $\sum_k t_k \to \infty$.

There exist many approaches for choosing a value π : various methods can be found in Graves[11] and Held et al.[13]. In choosing a value for π we followed the similar approach of Held et al.[13] in letting $\pi = 1$ initially and π was then halved every 5 iteration.

Unless we obtain an optimal Lagrangean multipliers for which v(LR1) equals the cost of a known feasible solution, there is no way of proving optimality in the subgradient procedure. To resolve this difficulty, the method is usually terminated upon reaching an arbitrary iteration limit or the subgradient procedure is then carried out until either of (i), (ii) or (iii) below is satisfied (see Christofides and Beasley [4]).

(i) All subgradients are zero,

(ii) The upper bound v(TP) and the highest lower bound v(LR1) attained coincide,

(iii) π fall below 0.000001.

The above approach (i), (ii), (iii) and an iteration limit rule have been implemented in our procedure.

4. Computational Results

The proposed Lagrangean relaxation procedure was implemented in FORTRAN IV and run on the CYBER 174-016 at KAIST. Since we have no available set of standard test problems for P in the literature, a number of randomly generated problems were set up to evaluate the effectiveness of our solution method. The distance between facilities and customers, which are considered to be the cost of assigning a customer to a facility, were randomly generated in the range of 10 mile to 100 mile, whereas the assigning cost from a customer to a facility was considered to be \$1/mile. The fixed facility costs were randomly generated in the range of \$100 to \$200. The set of test problems has been considered with the combination of n = {20, 40, 50}, m = {10, 15, 20} so that its scale is comparable to that of the cases in Mirzaian[17]. For each selected combination, several feasible values of b were considered namely 3, 5, 6, 7 and 9. The cost matrics for all of test problems are 100% dense, that is regarded as an unrealistically difficult situation.

Table 1 presents a brief summary of the 21 test problems and computational results with our code. Unfortunately, we have been unable to find other published studies on the same problem as ours for comparison. The one closely related studies are that by Mirzaian[17]. Since his test data and results are not available to obtain in detail, we just generate the problems with the same dimension as his for comparison. For our computational tests, we had the best experience on the test problem set using π from the following sequence : $\pi = 1$ for iteration(k) = 1,2,...,5 ; $\pi = 0.5$ for k = 6,7,...,10 ; $\pi = 0.25$ for k = 11,12,...,15 ; $\pi = 0.125$ for k = 16,17,...,20. For each problem we terminate the subgradient procedure after 20 iterations. The results showed that this number of iteration is enough to give a small duality gap measured by [(UB − LB) / UB] * 100%. From the Table 1, it is apparent that our Lagrangean relaxation procedure produces better bounds than Mirzaian[17] did. This improvement may be obtained by appending the surrogate constraint to P as Nauss[18] did. Note that Mirzaian[18] did not mention a word on the computing time, and thus we can not compare the computing time directly. It is perhaps worth here nothing that most of the time needed to solve the problem in our procedure was taken up with the solution of transportation time. By developing an efficient feasible solution procedure, the computing time may be reduced.

5. Concluding Remarks

In this study, we have dealt with a facility location problem with the single assignment constraints. While the single assignment restriction increases problem complexity, it can reflect the commonly occuring real world situation that a customer's demand should be met from only one facility. However, by taking the special structure of the proposed model, the location problem can be transformed to a variant of the usual capacitated facility location problem. We have developed a Lagrangean relaxation procedure with the reformulation scheme in Section 2.2 for solving this problem. The procedure was made efficient by employing a device which exploits the special structure of a surrogate constraint. The computational results of our procedure on 21 test problems are reported on. The relatively small duality gap within an acceptable computing time clearly demonstrate the efficiency of our Lagrangean relaxation method.

Table 1. Computational Results

n x m	b^+	Lower Bound	Upper Bound	CPU* Time	e(%)	Mirzaian** Heuristic e(%)
20x10	3	1386.6	1391	2.76	0.32	0.00
	5	993.0	993	0.91	0.00	0.00
	7	884.0	884	0.79	0.00	0.00
	9	884.0	884	0.73	0.00	–
40x10	5	1897.1	1906	14.69	0.48	–
	7	1635.4	1646	9.89	0.65	–
	9	1537.4	1539	9.01	0.10	–
40x20	3	2530.8	2533	9.16	0.08	2.79
	5	1686.6	1688	12.39	0.08	2.42
	7	1472.4	1473	9.68	0.04	0.86
	9	1380.0	1402	10.93	1.56	–
50x10	6	2233.9	2245	18.34	0.49	–
	7	2084.2	2099	18.77	0.71	–
	9	1848.7	1854	15.48	0.29	–
50x15	5	2289.1	2299	24.97	0.43	–
	7	2023.8	2037	24.21	0.65	–
	9	1797.8	1889	21.67	4.83	–
50x20	3	3201.3	3208	44.51	0.21	0.54
	5	2151.9	2158	29.32	0.28	2.35
	7	1905.0	1914	21.72	0.46	0.72
	9	1707.6	1709	18.03	0.08	–

\+ The capacity given fixed for all facilities.

* CPU time in seconds on CYBER 174-016 excluding I/O time.

** The magnitude of accuracy measure(%) that Mirzaian has shown for his test problem of the same dimension in his paper[17]. Dashes indicate that corresponding information is not available.

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