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A Two-Process Two-Product Inventory Model on a Single Facility with By-Product

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Abstract

This study is concerned with a single-facility production system in which two kinds of products, product 1 and 2, are produced by two different processes, process 1 and 2. Product 2 is produced either by process 2 or by process 1 as by-product, while product 1 is obtained only through process 1.

The repeating sequences of the two processes and their associated lot sizes are determined which minimize the total inventory related cost.

A solution algorithm is developed and is illustrated with numerical examples.

I. Introduction

This paper studies production lot sizes and repeating sequences for a production system which produces two kinds of products using two different processes on a single facility. The system we consider is shown in Figure 1. There are two products, product 1 and 2, and two production processes, process 1 and 2. Process 1 is capable of making both products, while process 2 produces product 2 only. When process 1 is in operation, (1-b) and b units, $0 \leq b \leq 1$, of products 1 and 2 are obtained respectively for each unit produced.

Deuermeyer and Pierskalla [1] approached a similar system with optimal control but their study was not based on the production on a single facility. Recently, Goyal [2] developed an Economic Production Quantity (EPQ) model for two-product single-machine system, and presented a search procedure which permits unequal batch quantities for the more frequently manufactured product.

The objective of this study is to determine the production lot size of each product and the schedule of the two processes on a single-facility that minimize the total variable costs.

II. The Development of the Model

The mathematical model presented is based on the following assumptions.

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1. Each process is set up instantaneously.
2. The production rate and the demand rate for each process are known and constant.
3. Stock-outs are not permitted.
4. By-product ratio "b" is known and constant.
5. Time horizon is infinite.
6. The demand rate, D_1, of product 1 is less than the production rate, $(1\text{-}b)P_1$, of process 1, i.e., $D_1 < (1\text{-}b)P_1$.
7. The demand rate, D_2, of product 2 is less than the production rate, P_2, of process 2, i.e., $D_2 < P_2$.

Figure 1 shows the block diagram and related parameters of the production system of this study.

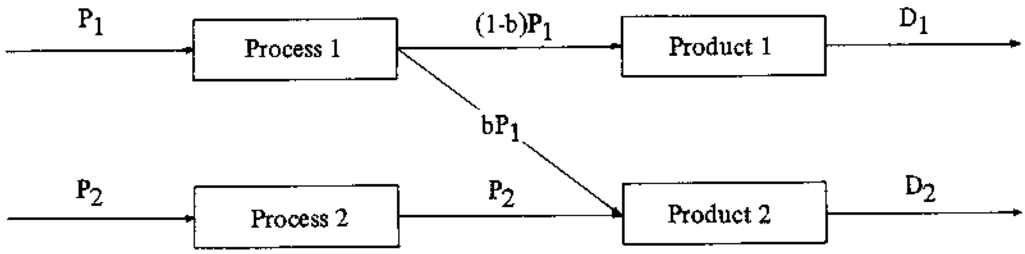


Figure 1. The By-product System

During an interval between two successive set-ups of a process, the other process can be scheduled more than one time. From this point of view, we can identify two systems, (K, 1) and (1, K).

1. (K,1) System

The (K, 1) system describes a single-facility two-product system in which process 1 is set up K times during the time interval between successive set-ups of the process 2.

Let T_1 be one batch production time interval and T be an interval between two successive productions of process 1. We call T basic period. Vemuganti [5] studied the feasibility condition for two products on one machine and this condition is applied to this system. Following his approach, the total quantities of product 1 produced during T_1 should be equal to the total demands during basic period T. Thus

$$T_1\,(1\text{-}b)P_1 = D_1 T,$$

$$T_1 = D_1 T \,/\, (1\text{-}b)P_1 = f_1 T \tag{1}$$

where $f_1 = D_1 \,/\, (1\text{-}b)P_1$.

Also, let T_2 be the duration of the production period of the process 2. Then the total demands of product 2 during a cycle time, KT, are equal to the sum of quantities of product 2 produced by process 2 and by process 1 during KT;

$$D_2 KT = P_2 T_2 + KbP_1 \{ D_1 T/(1\text{-}b)P_1 \} ,$$

$$T_2 = \{ D_2 - D_1 b / (1-b) \} \; KT / P_2 = f_2 KT, \tag{2}$$

where $f_2 = \{ D_2 - D_1 b / (1-b) \} / P_2$.

Since T_2 in (2) must be a positive value,

$$D_2 > D_1 b / (1-b). \tag{3}$$

The sum of production time of process 1 and that of process 2 is

$$T_1 + T_2 = f_1 T + f_2 KT. \tag{4}$$

A feasible production schedule must satisfy $T_1 + T_2 \leqslant T$, and this relation can be expressed as

$$K \leqslant L \tag{5}$$

$$\text{where } L = (1-f_1) / f_2. \tag{6}$$

Note that for the feasibility of the production schedule, L must be equal to or greater than one. If K (the number of set-up for process 1) does not satisfy condition (5), unequal lot size rule can be considered following Goyal's [2].

The change of the inventory level of product 2 during T_1 depends on the relationship between D_2 and bP_1. The inequality $D_2 \geqslant bP_1$ means that demand rate of product 2 is not less than by-product rate of process 1, so difference has to be satisfied from the inventory of product 2. On the other hand, inequality $D_2 < bP_1$ means that the demand rate of product 2 can be satisfied by the by-product of process 1 when process 1 is in operation.

Based on the above observations, four cases can be identified as depicted in Figure 2.

conditions	$D_2 \geqslant bP_1$	$D_2 < bP_1$
$K \leqslant L$	CASE 1	CASE 3
$K > L$	CASE 2	CASE 4

Figure 2. Cases of (K, 1) system

Since unit production cost may be considered as constant and shortage is not permitted, only the set-up and inventory holding costs are included in the total cost. Then the cost components of (K, 1) system are as follows.

For product 1 Set up cost = S_1 / T (7)

Holding cost = $H_1 A_i T / 2$ (8)

For product 2 Set up cost = S_2 / KT (9)

Holding cost = $H_2 B_i T / 2$ (10)

where S_j = set up cost of process j, j = 1,2,
H_j = unit holding cost of product j per unit time, j = 1,2,
and A_i and B_i are to be determined for each case.

Then the total cost, $CF_i(K, T)$, per unit time of case i is given by

$$CF_i(K, T) = S_1 / T + H_1 A_i T/2 + S_2/KT + H_2 B_i T/2$$
$$= (S_1 + S_2/K)/T + (H_1 A_i + H_2 B_i)T/2. \quad (11)$$

CASE 1

Figure 3 shows the inventory levels of product 1 and 2 of case 1. The condition of $K \leqslant L$ (or $T_2 \leqslant T-T_1$) permits process 1 to be scheduled in equal lot size. Whenever process 1 is in operation, the by-products are added to the inventory of product 2. For each product, the average inventory level is calculated and from (8) and (10),

$$A_1 = D_1 \{ 1 - D_1 / (1-b)P_1 \} = D_1(1 - f_1), \quad (12)$$

$$B_1 = - \{D_2 - D_1 b / (1-b)\}^2 K/P_2 - \{1 - D_1 / (1-b)P_1\} D_1 b/(1-b)$$
$$+ \{D_2 - D_1 b / (1-b)\} K$$
$$= P_2 f_2(1-f_2)(K - c_1 D_1 (1-f_1) \quad (13)$$

where $c_1 = b / (1-b)$.

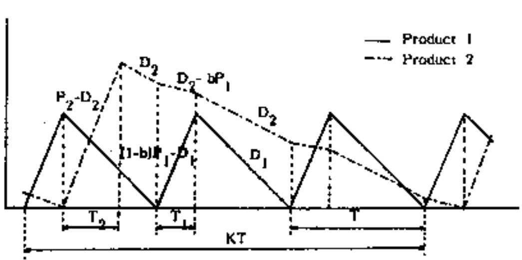


Figure 3. Behavior of inventory levels of CASE 1.

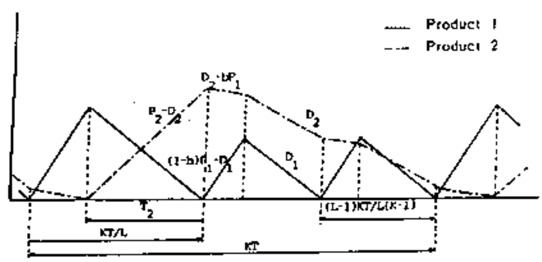


Figure 4. Behavior of inventory levels of CASE 2.

CASE 2

Figure 4 shows the inventory levels of product 1 and 2 of case 2. Since $K > L$, unequal lot size is adapted. The effect of the condition $D_2 \geqslant bP_1$ to the inventory level of product 2 is similar to that of case 1. The related inventory elements of case 2 are as follows.

$$A_2 = KD_1 (1 - f_1)(1 + c_2) / L^2, \quad (14)$$

where $c_2 = (L-1)^2 / (K-1)$.

$$B_2 = K\,[\,(P_2 - D_2)\,f_2{}^2 L^2 + (-bP_1 c_2 + D_2 - bP_1)f_1{}^2 + 2(D_2 - bP_1)(L-1)f_1 + D_2c_2 + (K-2)\,c_2P_2f_2]\,/\,L^2 . \tag{15}$$

The similar analysis can be applied to the other cases and the following results are obtained.

CASE 3

$$A_3 = D_1\,(1 - f_1), \tag{16}$$

$$B_3 = [-KbP_1f_1{}^2 + (K^2-K+2)P_2f_2 + 2(bP_1 + Kc_1D_1 - D_2K)f_1 - P_2f_2{}^2K^2 + (K-2)D_2\,]\,/\,K . \tag{17}$$

CASE 4

$$A_4 = D_1(1 - f_1)\,(1 + c_2)K\,/\,L^2 , \tag{18}$$

$$B_4 = K\,[\,(P_2-bP_1-bP_1c_2)f_1{}^2 + 2(bP_1-P_2)f_1 + (P_2 - D_2) + c_2D_2 + (K-2)c_2P_2f_2\,]\,/\,L^2 . \tag{19}$$

2. (1, K) System

The (1, K) system describes a single-facility two-product system in which process 2 is set up K times during the time interval between successive set-ups of the process 1.

Let T_1 be the production time interval of process 1. Then we can easily see that the total quantities of product 1 produced during a common cycle KT are equal to the total demand during KT;

$$(1 - b)P_1T_1 = D_1KT ,$$
$$T_1 = f_1KT . \tag{20}$$

Also, let T_2 be the production time interval of process 2. Then total demands of product 2 during KT are equal to the sum of quantities of product 2 produced by process 1 and by process 2 during KT.

$$KP_2T_2 + bP_1T_1 = KP_2T_2 + bP_1\,\{\,D_1\,/\,(1 - b)P_1\}\,KT = D_2KT,$$
$$T_2 = f_2T. \tag{21}$$

The sum of the production time of process 1 and that of process 2 is

$$T_1 + T_2 = f_1KT + f_2T. \tag{22}$$

Hence, the feasibility criterion for (1, K) system can be obtained by the similar procedure of (K, 1) system. Let

$$M = (1 - f_2)\,/\,f_1. \tag{23}$$

(1, K) system is different from (K, 1) system in that the relation of D_2 and bP_1 is not significant since by-product of process 1 occurs only once during KT.

Thus, two cases can be identified as shown in Figure 5.

condition	$K \leqslant M$	$K > M$
	CASE 5	CASE 6

Figure 5. Cases of (1, K) system

The cost components of each case of (1, K) system are as follows.

For product 1 Set up cost = S_1 / KT (24)

Holding cost = $H_1 A_i T / 2$ (25)

For product 2 Set up cost = S_2 / T (26)

Holding cost = $H_2 B_i T / 2$ (27)

Then the total cost per unit time of each case is given by

$$\begin{aligned} CF_i(K, T) &= S_1/KT + H_1 A_i T / 2 + S_2 / T + H_2 B_i T / 2 \\ &= (S_1/K + S_2) / T + (H_1 A_i + H_2 B_i) T / 2. \end{aligned} \tag{28}$$

CASE 5

Figure 6 shows the inventory levels of two products and the related inventory elements are as follows.

$$A_5 = D_1(1 - f_1)K, \tag{29}$$

$$B_5 = - P_2 f_2^2 - 2D_1 c_1 f_2 - D_1 c_1 f_1 + D_2 + (K - 1)D_1 c_1. \tag{30}$$

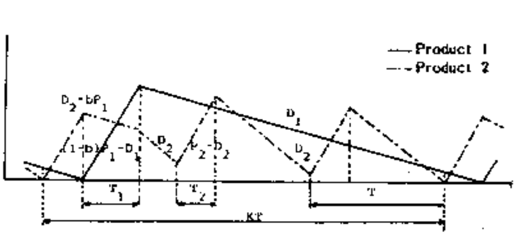


Figure 6. Behavior of inventory levels of CASE 5.

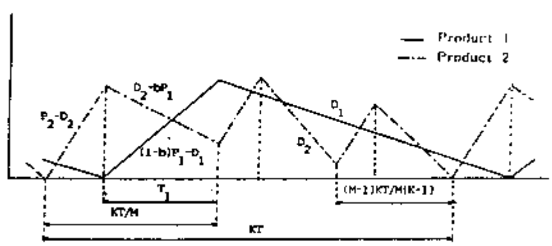


Figure 7. Behavior of inventory levels of CASE 6.

CASE 6.

Figure 7 shows the inventory levels of the two products and

$$A_6 = D_1(1 - f_1)K, \tag{31}$$

$$B_6 = K[\{-P_2 + bP_1 - c_3P_2\} f_2^2 + 2(P_2 - bP_1)f_2 - (D_2 - bP_1) + c_3D_2 + (K-2)D_1c_1c_3] / M^2, \tag{32}$$

where $c_3 = (M-1)^2/(K-1)$.

III. Algorithm

Note that the basic period T, $0 < T < \infty$, is a real variable and the multiple K, K = 1,2,3,.., is an integer variable. Kumin [3] studied the optimization procedure in which the objective function is composed of a discrete variable and a continuous variable. Park [4] proposed the following property.

Property In the problem of $\min_{u,s} f(u,s)$ where $0 < u < \infty$, $0 \leqslant s \leqslant \infty$, if the following conditions are satisfied, f is unimodal function;

i) For any finite s, f is positive infinite when u = 0 or ∞.
ii) f is bounded below and of the function type to "hold-water".
iii) f is continuously differentiable, $\partial^2 f/\partial s^2 > 0$ and $\partial^2 f/\partial u^2 > 0$.

Each $CF_i(K, T)$ goes to infinite when T goes to either infinite or to zero and is bounded below. Also the partial second derivatives with respect to K and T become positive. Following the above observations, $CF_i(K, T)$ is a unimodal function for every i. Thus, differenciating $CF_i(K, T)$ in equation (11) with respect to T and setting the result equal to zero, we obtain

$$T_i(K) = \sqrt{2(S_1 + S_2/K)/(H_1A_i + H_2B_i)}, \quad i = 1,2,3,4. \tag{33}$$

For any given K, the minimum total cost $CG_i(K)$ becomes

$$CG_i(K) = CF_i(K, T(K)) = \sqrt{2(S_1 + S_2/K)(H_1A_i + H_2B_i)}, \quad i = 1,2,3,4. \tag{34}$$

Following the similar way for (1, K) system,

$$T_i(K) = \sqrt{2(S_1/K + S_2)/(H_1A_i + H_2B_i)} \tag{35}$$

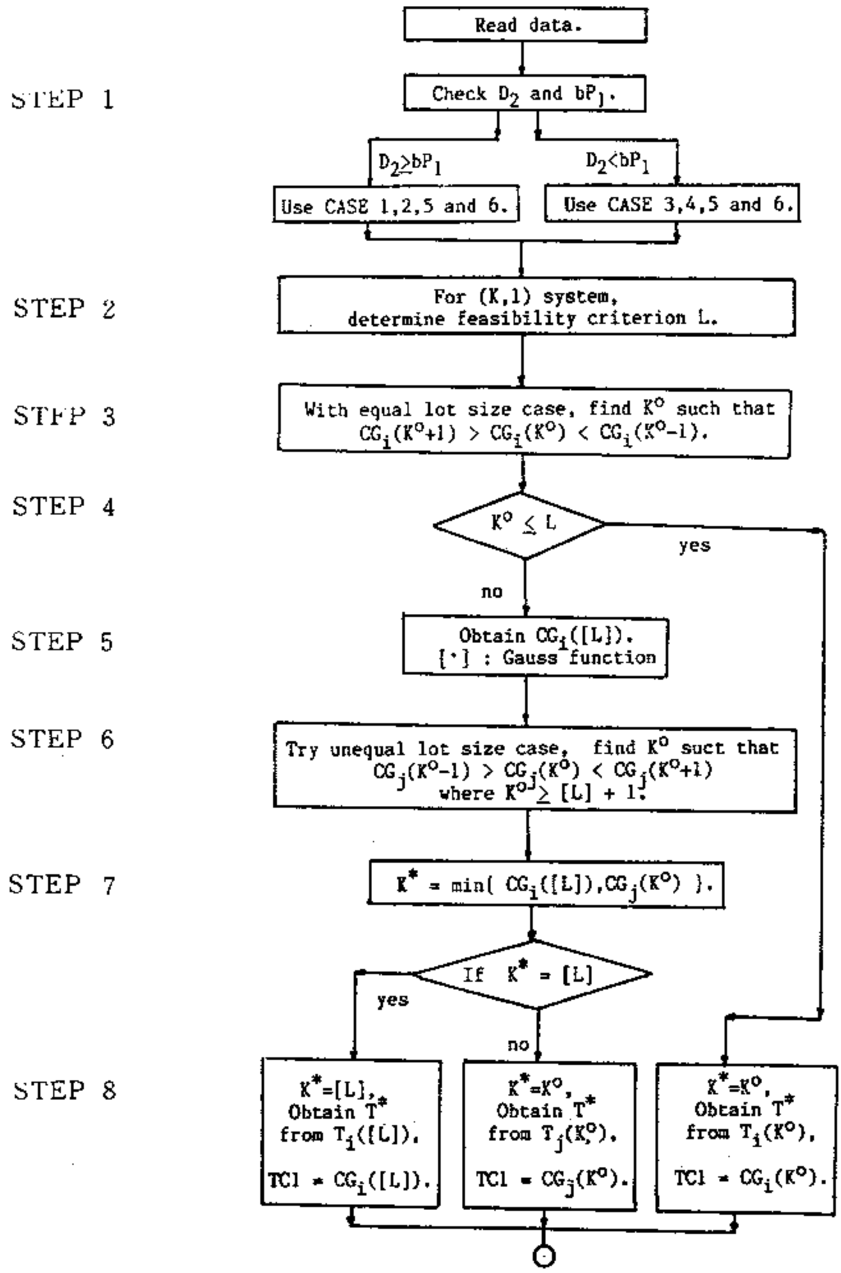
and

$$CG_i(K) = CF_i(K, T(K)) = \sqrt{2(S_1/K + S_2)(H_1A_i + H_2B_i)} \quad \text{for } i = 5,6. \tag{36}$$

Note that $CG_i(K)$ is also unimodal and the optimal solution K* must satisfy

$$CG_i(K^* - 1) > CG_i(K^*) < CG_i(K^* + 1).$$

A solution procedure is developed and shown in the flow diagram in the Figure 7.



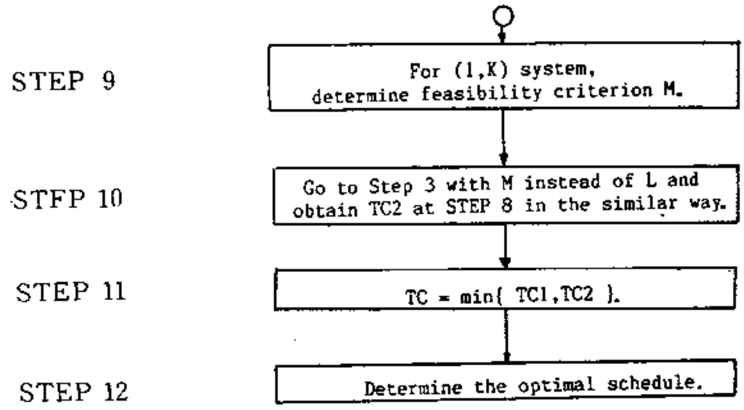


Figure 7. A Solution Procedure

IV. Numerical Example.

To illustrate the use of the above model developed, the example problem is solved using the solution procedure mentioned above.

Table 1 presents the input data for the two processes and two products.

Table 1. Input data of example 1

	P_j	b	S_j	D_j	H_j	
Process 1	7000	0.1	15000	3500	5	Product 1
Process 2	10000	0	25000	2000	1	Product 2

STEP 1 : Since $D_2 \geqslant bP_1$, use the case 1,2 and 5,6.

STEP 2 : For (K, 1) system, feasibility criterion L = 2.76.

STEP 3 : Minimum cost value of $CG_1(K^\circ) = 23326.2$ at $K^\circ = 3$.

STEP 4 : Since $K^\circ = 3 > L = 2.76$, Go to STEP 5.

STEP 5 : $[L] = [2.76] = 2$, and $CG_1([L]) = 23810.5$.

STEP 6 : $K^\circ = 3$, and corresponding cost $CG_2(K^\circ) = 23326.4$.

STEP 7 : $TC1 = \min\ \{CG_1([L]), CG_2(K^\circ)\} = 23326.4$.

STEP 8 : $K^* = K^\circ = 3$ (unequal lot size),
$T^* = T_2(3) = 2.000$ and $CG_2(3) = 23326.4$.

STEP 9 : For (1, K) system, the feasibility criterion M = 1.51.

STEP 10: Minimum value of $CG_5(K^\circ) = 27095.0$ at $K^\circ = 1$.
Since $K^\circ = 1 < M = 1.51$, minimum total cost of (1, K), $TC2 = CG_5(1) = 27095.0$.

STEP 11: $TC = \min\ \{TC1, TC2\} = \min\ \{23326.4, 27095.0\}$
$= 23326.4$.

STEP 12: Therefore, the optimal policy is (K, 1) system
where $(K^*, T^*) = (3, 2.000)$,

and use "Unequal Lot Size" (CASE 2).

Suppose the manager of a manufacturing company is operating the single facility two product system which has to deal with by-product. Due to the complexity caused by the by-product, if he opts to use Goyal's model [2], it is interesting to know how much the inventory cost is increased. To compare the two models on a compatible basis, we assume that D_1 is modified as $D_1/(1\text{-}b)$ and D_2 as $D_2\text{-}D_1 b/(1\text{-}b)$.

To answer the above question we solve two examples. The imput data of the problems and the results are shown in Table 2. As expected, the additional cost burden is increasing as b increases. And also R increases as the inventory holding cost of product 2 increases.

Table 2. Comparison of the two models

	Example 1			Example 2		
	P_1 = 7000 D_1 = 3500 S_1 = 15000 H_1 = 5	P_2 = 10000 D_2 = 2000 S_2 = 25000 H_2 = 1		P_1 = 7000 D_1 = 3500 S_1 = 15000 H_1 = 5	P_2 = 10000 D_2 = 2000 S_2 = 25000 H_2 = 3	
b	The model	Goyal's	R(%)	The model	Goyal's	R(%)
0.	25308.1	25308.1	0.	31768.7	31768.7	0.
0.1	23326.4	23890.9	2.4	29073.5	30038.4	3.3
0.2	20753.8	21478.9	3.5	25224.6	27286.7	8.2
0.3	16766.3	17648.4	5.3	19611.4	22199.9	13.2

$$R = \frac{\text{cost of Goyal's model - cost of the model}}{\text{cost of the model}} * 100$$

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