ON FINITELY GENERATED SEMIPRIME ALGEBRA
OVER COMMUTATIVE RINGS

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1. Introduction

Let $R$ be a commutative ring. E. P. Armendariz studied in [2] that a semiprime finitely generated $R$-algebra when $R$ is a regular ring and that by combining the above fact with the results of [6, 7], $R$ is semiprime and every f.g. semiprime $R$-algebra $A$ is Azumaya if the ring $R$ is regular.

In this paper, we prove converse of Armendariz's theorem and we get a necessary and sufficient condition on which a regular ring $R$ is $\pi$-regular.

That is, we have the following results;

1) Let $R$ be a commutative ring. Then the following are equivalent;
   i) $R$ is von Neumann regular.
   ii) $R$ is semiprime and every f.g. semiprime $R$-algebra is Azumaya.

2) Let $R$ be a commutative ring. Then the following are equivalent;
   i) $R$ is von Neumann regular.
   ii) Every integral extension of $R$ is $\pi$-regular.

An algebra $A$ is called Azumaya if $R$ is both central and
separable. The ring $R$ is said to be $P. I. \ ring$ if $R$ satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible. All other notations and terminologies will follow from [2] and [4].

2. Preliminaries

Kaplansky made the following conjecture in [4]: A ring $R$ is von Neumann regular if and only if $R$ is semiprime and each prime factor ring of $R$ is von Neumann regular. That the conjecture fails to hold in general was shown by a counter example of J.W. Fisher and R.L. Snider.

**Theorem 2.1** [4]. A ring $R$ is von Neumann regular if and only if $R$ is semiprime, the union of any chain of semiprime ideals of $R$ is a semiprime ideal of $R$ and each prime factor ring of $R$ is von Neumann regular.

Since any finitely generated algebra over a commutative ring satisfies a polynomial identity (is a $P. I. -$algebra), this leads to consideration of semiprime $P. I. -$algebra with regular center.

**Theorem 2.2** [2]. Let $A$ be a semiprime finitely generated algebra over a commutative regular ring $R$. Then $A$ is a regular ring.

The ring $R$ is finitely generated as a ring over its center $Z(R)$, if $R$ is an epimorphic image of a free (non commutative) ring over $Z(R)$ generated by finitely many indeterminates $[x_1,x_2,...,x_n]$ which only commute with elements of $Z(R)$. Following C.Proces, the ring $R$ is called an affine ring if $R$ is finitely generated over its center $Z(R)$. 


**Theorem 2.3** [7]. Let $R$ be an affine ring. Then the following properties are equivalent:

1) Every simple right $R$-module is injective.
2) $R$ is von Neumann regular.
3) $R$ is biregular.

**Theorem 2.4**[2]. Let $A$ be an algebra over a regular ring with center of $A$ being $R$. $A$ is Azumaya over $R$ if and only if $A$ is a biregular ring which is finitely generated over $R$.

Combining Theorems 2.2, 2.3 and 2.4, we have the following result.

**Theorem 2.5** [2]. Let $A$ be a finitely algebra over a regular ring. The following conditions on $A$ are equivalent:

1) $A$ is semiprime.
2) $A$ is regular.
3) $A$ is biregular.
4) $A$ is semiprime Azumaya algebra.

The following theorem was shown by Storrer.

**Theorem 2.6** [4]. Let $R$ be a P.I. ring. Then the following are equivalent:

1) $R$ is $\pi$-regular.
2) Each prime ideal of $R$ is primitive.
3) Each prime ideal of $R$ is maximal.
4) $R$ is left (right) $\pi$-regular.
5) $R/\text{rad}(R)$ is $\pi$-regular, where $\text{rad}(R)$ is prime radical.
6) Each prime factor ring of $R$ is von Neumann regular.

3. Main results

**Lemma 3.1.** Let $R$ be a commutative prime ring and $0 \neq a$
\[ \mathcal{R} \text{. If } A = \begin{pmatrix} a^R & a \mathcal{R} \\ a \mathcal{R} & \mathcal{R} \end{pmatrix} \text{ is Azumaya, then } a \text{ is invertible in } \mathcal{R}. \]

**Proof.** It is easily checked that \( \mathcal{R} \) coincides with the center \( Z(A) \). Now if \( A \) is Azumaya, \( A \otimes_{\mathcal{R}} A^{\ast} \cong \text{Hom}_{\mathcal{R}}(A, A) \). In this case \( \sigma(a \otimes b)(x) = axb \) for \( x \in \mathcal{A} \).

Consider \( f \in \text{Hom}_{\mathcal{R}}(A, A) \) such that \( f \left( \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).

Then since \( A \) is Azumaya, there are \( \begin{pmatrix} x_i & ay_i \\ az_i & w_i \end{pmatrix} \) and \( \begin{pmatrix} x_i' & ay_i' \\ az_i' & w_i' \end{pmatrix} \) in \( A \), \( 1 \leq i \leq n \) for some \( n \) such that

\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} x_i & ay_i \\ az_i & w_i \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_i' & ay_i' \\ az_i' & w_i' \end{pmatrix}.
\]

By this relation, we have \( 1 \in a^2 \mathcal{R} \) and so \( a \) is invertible in \( \mathcal{R} \).

**Theorem 3.2.** Let \( \mathcal{R} \) be a commutative ring. Then the following are equivalent.

1) \( \mathcal{R} \) is von Neumann regular.

2) \( \mathcal{R} \) is semiprime and every \( f, g \)-semiprime \( \mathcal{R} \)-algebra \( A \) is Azumaya.

**Proof.** Assume that \( \mathcal{R} \) is von Neumann regular. By Theorem 2.5, \( A \) is Azumaya algebra.

For the opposite direction, let \( \mathcal{P} \) be a prime ideal of \( \mathcal{R} \). We will show that \( \mathcal{P} \) is a maximal ideal of \( \mathcal{R} \). Now, take \( a \in \mathcal{R} \) and consider an \( \mathcal{R} \)-algebra \( A = \begin{pmatrix} a^R & a \mathcal{R} \\ a \mathcal{R} & \mathcal{R} \end{pmatrix} \). Then \( A \) is a finitely generated semiprime algebra over \( \mathcal{R} \). In this case, the center \( Z(A) = \{ \begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix} \mid (x-w)a = 0 \} \). By our assumption, \( A \) is separable over \( Z(A) \).
Consider the mapping \( \sigma: A \to \left( \frac{R/P}{aR/P} \right) \) with \( \sigma \left[ \begin{array}{c} x \\ ay \\ az \\ w \end{array} \right] = \left( \begin{array}{c} x + \bar{p} \\ ay + \bar{p} \\ az + \bar{p} \\ w + \bar{p} \end{array} \right) \), where \( \bar{a} = a + \bar{p} \). Then since \( a \not\in P \), we have that \( \text{Ker} \sigma = \{ \left( \begin{array}{c} x \\ ay \\ az \\ w \end{array} \right) | x, y, z, w \in P \} = PA \). Therefore \( A/PA \cong \left( \begin{array}{cc} R/P & \bar{a}R/P \\ \bar{a}R/P & R/P \end{array} \right) \).

Now since \( PZ(A)A=PA \) and \( A \) is Azumaya, we have \( PA \cap Z(A)=PZ(A) \). So \( A/PA \) is Azumaya over \( Z(A)/PZ(A) \). Also in this case \( Z(A/PA)=Z(A)/PZ(A) \) [1]. But since \( A/PA \cong \left( \begin{array}{cc} R/P & \bar{a}R/P \\ \bar{a}R/P & R/P \end{array} \right) \), we have \( Z(A/PA) \cong R/P \), So \( \left( \begin{array}{cc} R/P & \bar{a}R/P \\ \bar{a}R/P & R/P \end{array} \right) \) is Azumaya over \( -R/P \). Therefore, by our Lemma 3.1, \( \bar{a} \) is invertible in \( R/P \). Hence \( R/P \) is a field. Thus \( R \) is a von Neumann regular ring.

**Corollary 3.3.** Let \( R \) be a commutative ring, then the following are equivalent:

1) \( R \) is von Neumann regular.

2) \( R \) is semiprime and for every finitely generated \( R \)-algebra \( A \), \( J(A) \) is nilpotent and \( A/J(A) \) is Azumaya.

**Proof.** In [2], E. P. Armendariz proved that if \( R \) is von Neumann regular then \( J(A) \) is nilpotent and \( A/J(A) \) is a regular ring.

Conversely, let \( P \) be a prime ideal and \( a \not\in P \). Then \( A=\left( \begin{array}{cc} R & aR \\ aR & R \end{array} \right) \) is finitely generated semiprime \( R \)-regular.

But since \( A \) is a normalizing finite extension of \( R \), we have
0 = J(R) = R ∩ J(A) and so \( R \cong A/J(A) \). This shows that \( A/J(A) \) is \( R \)-algebra.

Now since \( A \) is semiprime and \( J(A) \) is nilpotent, \( J(A) = 0 \). Therefore \( A \) is Azumaya. By Theorem 3.2, \( R \) is von Neumann regular.

Let \( A \) be a ring with identity. Consider the condition (*) the ring \( A \) satisfies a polynomial identity \( f(x_1, x_2, \ldots, x_n) = 0 \) for which \( f \) has coefficient in \( C \), the center of \( A \), and for which at each prime ideal \( P \) of \( A \), \( f \) induces a nontrivial polynomial identity on \( A/P \).

**Theorem 3.4** [5]. Let \( A \) be a ring with identity which is integral over unital subring \( B \) of \( C \), the center of \( A \), suppose further that \( B \) satisfies (*), then; If \( P \) is prime ideal of \( A \), \( P \) is maximal ideal of \( A \) if and only if \( P \cap A \) is maximal ideal of \( B \).

**Theorem 3.5.** Let \( R \) be a commutative ring. Then the following are equivalent;
1) \( R \) is von Neumann regular.
2) Every integral extension of \( R \) is \( \pi \)-regular.

**Proof.** Suppose that \( R \) is von Neumann regular and \( A \) is integral extension of \( R \). To show that \( A \) is \( \pi \)-regular, let \( P \) be a prime ideal of \( A \). Then \( A/P \) is integral over \( R/P \cap R \). Since \( P \) is a maximal ideal of \( A \), \( P \cap R \) is maximal ideal of \( R \). Therefore \( R/P \cap R \) is a field. By Theorem 2.6, \( A/P \) is \( \pi \)-regular. Thus \( A \) is \( \pi \)-regular. Conversely, since \( A = \begin{pmatrix} R & R \\ R & R \end{pmatrix} \) is integral extension of \( R \), it is \( \pi \)-regular. It follows that \( R \) is von Neumann regular.
References


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