CERTAIN SUBCLASSES OF ANALYTIC P-VALENT FUNCTIONS

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1. Introduction

Let $A_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^\infty a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, \ldots\})$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. For $f(z)$ and $g(z)$ being in the class $A_p$, $f(z)$ is said to be subordinate to $g(z)$ if there exists a Schwarz function $w(z)$, $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$, such that $f(z) = g(w(z))$. We denote by $f(z) \prec g(z)$ this relation. In particular, if $g(z)$ is univalent in the unit disk $U$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

A function $f(z)$ belonging to $A_p$ is said to be in the class $S^*[a, b]$ if it satisfies

$$\frac{zf'(z)}{pf(z)} < \frac{1+az}{1+bx}$$

for some $a$ and $b$ with $-1 \leq b < a \leq 1$, and for all $z \in U$.

Further, a function $f(z)$ belonging to $A_p$ is said to be in the class $K_p[a, b]$ if it satisfies $zf'(z)/p \in S^*[a, b]$.

The class $S^*[a, b]$ was introduced by Goel and Mehrok ([1], [2]), and Janowski [3]. Further, the class $K[a, b]$
was introduced by Silverman and Silvia [5].

Let $S_{\rho}(a, b)$ be the subclass of $A_{\rho}$ consisting of functions which satisfy the condition

$$(1.3) \quad \frac{zf''(z)}{pf(z)} - a < b$$

for some $a$ and $b$ with $a \geq b$, and for all $z \in U$. Furthermore we denote by $K_{\rho}(a, b)$ the subclass of $A_{\rho}$ consisting of functions which satisfy the condition $zf''(z)/p \in S_{\rho}(a, b)$.

The classes $S_{\rho, 1}(a, b)$ and $K_{1}(a, b)$ were introduced by Silverman [4] and Silverman and Silvia [5], respectively.

2. Some Properties

We begin with the statement and the proof of the following result.

**Theorem 1.** If $-1 < b < a \leq 1$, then

$$(2.1) \quad S_{\rho}[a, b] = S_{\rho}\left(\frac{1-ab}{1-b^2} , \frac{a-b}{1-b^2}\right).$$

Further, if $a \geq b$, then

$$(2.2) \quad S_{\rho}(a, b) = S_{\rho}\left(\frac{b^2-a^2+b}{b}, \frac{1-a}{b}\right).$$

**Proof.** We employ the same manner by Silverman and Silvia [5]. Let $f(z) \in S_{\rho}[a, b]$ with $-1 < b < a \leq 1$, that is,

$$(2.3) \quad \frac{zf'(z)}{pf(z)} < F(z) = \frac{1+az}{1+bz}.$$ 

By using the result due to Singh and Goel [6], we have

$$(2.4) \quad \left|F(z) - \frac{1-ab}{1-b^2} \frac{|z|^2}{|z|^2} \right| \leq \frac{(a-b)|z|}{1-b^2} \frac{|z|^2}{|z|^2} (z \in U).$$

It follows from (2.4) that $F(z)$ maps the circle $|z|=1$ onto
a circle

$$(2.5) \quad \left| w - \frac{1-ab}{1-b^2} \right| = \frac{a-b}{1-b^2}. $$

This implies that

$$(2.6) \quad \left| \frac{zf'(z)}{pf(z)} - \frac{1-ab}{1-b^2} \right| < \frac{a-b}{1-b^2} (-1 < b < a \leq 1; z \in U).$$

Noting $(1-ab)/(1-b^2) \geq (a-b)/(1-b^2)$ for $-1 < b < a \leq 1$, we obtain (2.1).

Next, let $f(z) \in S_*(a,b)$ with $a \geq b$, that is,

$$$(2.7) \quad \left| \frac{zf'(z)}{pf(z)} - a \right| < b. $$$$\]

Then we have to find $A$ and $B$ such that $-1 < B < A \leq 1$ such that $a = (1-AB)/(1-B^2)$ and $b = (A-B)/(1-B^2)$ for $a \geq b$. Because we find such $A$ and $B$, then we have

$$(2.8) \quad \left| \frac{zf'(z)}{pf(z)} - \frac{1-AB|z|^2}{1-B^2|z|^2} \right| \leq \frac{(A-B)|z|}{1-B^2|z|^2}. $$

which implies

$$(2.9) \quad \frac{zf'(z)}{pf(z)} < \frac{1+Ax}{1+Bx}, $$

or $f(z) \in S_*(A,B)$.

Letting $A = B + b(1-B^2)$ for $-1 < B < A \leq 1$, we obtain

$$(2.10) \quad bB^3 + (a-1)B^2 - bB + 1 - a = 0. $$

The above equation (2.10) has the solutions $B = \pm 1, (1-b)/a$. Since $-1 < B < A \leq 1$, we only take $B = (1-b)/a$. Thus we have $B = (1-b)/a$ and $A = (b^2 - a^2 + a)/b$. This completes the proof of (2.2).

**Theorem 2.** If $-1 < b < a \leq 1$, then
(2.11) \[ K_\rho[a, b] = K_\rho \left( \frac{1-ab}{1-b^2}, \frac{a-b}{1-b^2} \right). \]

Further, if \( a \geq b \), then

(2.12) \[ K_\rho(a, b) = K_\rho \left[ \frac{b^2-a^2+a}{b}, \frac{1-a}{b} \right]. \]

**Proof.** Note that \( f(z) \in K_\rho[a, b] \) if and only if \( zf'(z)/\rho \in S^*_\rho[a, b] \), and that \( f(z) \in K_\rho(a, b) \) if and only if \( zf'(z)/\rho \in S^*_\rho(a, b) \). Therefore the proof of Theorem 2 follows from Theorem 1.

Next we prove

**Theorem 3.** \( S^*_\rho(a_1, b_1) \subseteq S^*_\rho(a_2, b_2) \) if and only if \( |a_2-a_1| \leq b_2-b_1 \). Furthermore, \( S^*_\rho[a_1, b_1] \subseteq S^*_\rho[a_2, b_2] \) if and only if \( |a_2b_1-a_1b_2| \leq (a_2-a_1)-(b_2-b_1) \).

**Proof.** Since \( S^*_\rho(a_1, b_1) \subseteq S^*_\rho(a_2, b_2) \) if and only if

\[ \{ w : |w-a_1| < b_1 \} \subseteq \{ w : |w-a_2| < b_2 \}, \]

or, if and only if \( a_2-b_2 \leq a_1-b_1 \) and \( a_1+b_1 \leq a_2+b_2 \), we have \( S^*_\rho(a_1, b_1) \subseteq S^*_\rho(a_2, b_2) \) if and only if \( |a_2-a_1| \leq b_2-b_1 \). In view of Theorem 1, we note that \( S^*_\rho[a_1, b_1] \subseteq S^*_\rho[a_2, b_2] \) if and only if

\[ \{ w : |w-A_1| < B_1 \} \subseteq \{ w : |w-A_2| < B_2 \}, \]

where \( A_1 = (1-a_1b_1)/(1-b_1^2) \) and \( B_1 = (a_1-b_1)/(1-b_1^2) \), which equivalent to \( |A_2-A_1| \leq B_2-B_1 \). Thus we have \( S^*_\rho[a_1, b_1] \subseteq S^*_\rho[a_2, b_2] \) if and only if \( |a_2b_1-a_1b_2| \leq (a_2-a_1)-(b_2-b_1) \).

Finally, we have

**Theorem 4.** \( K_\rho(a_1, b_1) \subseteq K_\rho(a_2, b_2) \) if and only if \( |a_2-a_1| \leq b_2-b_1 \). Furthermore, \( K_\rho[a_1, b_1] \subseteq K_\rho[a_2, b_2] \) if and only if \( |a_2b_1-a_1b_2| \leq (a_2-a_1)-(b_2-b_1) \).
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Received February 1, 1987