NOTES ON THE PSEUDO-COMPLETE ALGEBRA

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1. Introduction

In [5], Rickart proved that, when $F$ is a Hermitian functional on the Banach $*$-algebra $A$, in order for $F$ to be representable, it is necessary and sufficient that

(i) $F$ is bounded,
(ii) $|F(x)|^2 \leq \mu F(x^*x)$, $x \in A$

where $\mu$ is a positive real constant independent of $x$. In this note, conditions for a functional to be admissible on a locally convex $*$-algebra are defined and sufficient conditions for a functional $F$ to be representable are also given in Theorem 4.2.

2. Preliminaries

Definition 2.1. By a locally convex algebra $A$ we shall mean an algebra $A$ over the complex field $C$, equipped with a topology $\tau$ such that

(i) $(A;\tau)$ is a Hausdorff locally convex topological vector space,
(ii) multiplication is separately continuous.

$A$ will be called a locally convex $*$-algebra if $A$ has a continuous involution.
Definition 2.2. Let $A$ be a locally convex algebra. An element $x$ of $A$ is said to be bounded if, for some nonzero complex number $\lambda$, the set $\{(\lambda x)^n : n \in \mathbb{N}\}$ is a bounded subset of $A$.

The set of all bounded elements of $A$ will be denoted by $A_0$.

Notation. By $B_1$ we denote the collection of all subsets $B$ of $A$ such that

(i) $B$ is convex and idempotent,
(ii) $B$ is bounded and closed.

If $B \subseteq B_1$, then $A(B)$ will denote the subalgebra of $A$ generated by $B$, i.e., $A(B) = \{\lambda x : \lambda \in C$ and $x \in B\}$, and the equation $||\hat{x}||_B = \inf \{\lambda > 0 : x \in \lambda B\}$ defines a norm which makes $A(B)$ a normed algebra.

Definition 2.3. The locally convex algebra $A$ is called pseudo-complete if each of the normed algebras $A(B)$ is a Banach algebra.

If $A$ is a locally convex algebra and $x \in A$, we define the radius of boundedness of $x$ by

$$\beta(x) = \inf \{\lambda > 0 : \{(\lambda^{-1}x)^n : n \in \mathbb{N}\}$ is bounded\}$$

with the usual convention that $\inf \emptyset = \infty$.

The following simple facts about $\beta(x)$ are obvious:
1°. $\beta(x) \geq 0$ and $\beta(\lambda x) = |\lambda|\beta(x)$ where $\lambda \in C$ and $0 \cdot \infty = 0$.
2°. $\beta(x) < \infty$ iff $x \in A_0$.
3°. In particular, if $A$ is pseudo-complete, then $\beta(x)$ equals to the spectral radius of $x$ [1].
Definition 2.4. Let $A$ be a locally convex $\ast$-algebra, and let $F$ be a linear functional on $A$. If $F(x^*) = (F(x))^\ast$ for all $x$ in $A$, then $F$ will be called Hermitian. If $F(x^*x) \geq 0$ for all $x$ in $A$, then $F$ will be called a positive functional.

Lemma 2.5. Let $A$ be a pseudo-complete locally convex $\ast$-algebra and let $x_0$ be any element of $A$ such that $\beta(x_0) < 1$. Then there exists an element $y_0$ of $A$ such that $2y_0 - y_0^2 = x_0$. In addition, if $x_0$ is Hermitian, so is $y_0$.

Proof. Consider the function $f$ defined in terms of the binomial series as follows:

$$f(z) = -\sum_{n=1}^{\infty} \binom{1/2}{n} (-z)^n.$$  

Then $f$ is well-defined and $2f(z) - [f(z)]^2 = z$ for all $|z| \leq 1$. Now consider the vector valued function $-\sum_{n=1}^{\infty} \binom{1/2}{n} (-x_0)^n$.

We show that this series converges. Let $\varepsilon > 0$. Since $\beta(x_0) < 1$, there exists a $B \subseteq B_1$ by [1] such that $x_0 \in A(B)$ and $||x_0||_B < 1$. Since $f$ converges for $|z| \leq 1$, there exists an $n_0$ such that for $p, q > n_0$

$$\left( \sum_{n=p}^{n+q} \binom{1/2}{n} (-x_0)^n \right) < \varepsilon.$$  

Since $A(B)$ is complete, we have that vector valued series converges to an element $y_0$ of $A(B)$ such that $2y_0 - y_0^2 = x_0$.

Theorem 2.6. Let $A$ be a pseudo-complete locally convex $\ast$-algebra and let $F$ be any positive functional on $A$. Then
\[|F(u^*hu)| < \beta(h)F(u^*u) \] for all \( u \in A \) and \( h \) Hermitian.

**Proof.** By Lemma 2.5 and [5, Theorem 4.5.2], the above theorem is obvious.

Let \( F \) be a positive functional on \( A \) and define
\[
L_F = \{ x \in A : F(y*x) = 0 \text{ for all } y \text{ in } A \}.
\]
Then \( L_F \) is a left ideal of \( A ([3, p.288]) \). Now we define \( X_F = A/L_F \) and denote \( x + L_F \) by \( \overline{x} \).

**Definition 2.7.** A positive linear functional \( F \) which satisfies the following conditions will be called admissible:

1. \( \sup \{ F(x^*a^*ax)/F(x*x) : x \in A, a \in A \} < \infty \) for all \( a \in A \).
2. For each \( x \in A \), there is a \( x_0 \in A_0 \) such that \( \overline{x} = \overline{x_0} \).

**Corollary 2.8.** If \( A \) is a pseudo-complete locally convex \(*\)-algebra such that \( A = A_0 \), then any positive functional is admissible.

**Proof.** By Theorem 2.6 and 2°,
\[
\{ F(x^*a^*ax) : x \in A = A_0 \} \leq \beta(a^*a) < \infty \text{ for all } a \in A.
\]
Since \( A = A_0 \) for each \( x \in A \), there exists a \( x_0 (=x) \in A_0 \) such that \( \overline{x} = \overline{x_0} \).

3. **Topologically Cyclic Representation**

Let \( A \) be a \(*\)-algebra over the complex field \( C \) and \( X \) a vector space over \( C \). A \(*\)-homomorphism \( A \rightarrow L(X) \) is called a \(*\)-representation of \( A \) on \( X \), where \( L(X) \) is an algebra of all linear transformations of \( X \) into itself.
LEMMA 3.1. Let $A$ be a locally convex $\ast$-algebra and let $F$ be an admissible positive functional on $A$. If $a, b \in A$, then $(a + b)_F = (a_0 + b_0)$.

THEOREM 3.2. Let $F$ be an admissible positive Hermitian functional on the commutative locally convex $\ast$-algebra $A$. Then there exists a representation $a \mapsto T_a$ of $A$ on a Hilbert space $H$ such that $(T_a)^* = T_a$ for all $a \in A_0$.

Proof. Since $A$ is commutative, $L_F$ is a two-sided ideal and hence $X_F$ is an algebra. Let $\bar{x} = x + L_F$ and define a scalar product in $X_F$ by $(\bar{x}, \bar{y}) = F(y^*x)$, for $x, y \in A$. The completion of $X_F$ with respect to the inner product will be called $H$, and then $H$ is a Hilbert space.

Let $x_0$ be a fixed element of $X_F$. Since $F$ is admissible, we may assume that $x_0 \in A_0$. Let $\bar{z} \in H$ and assume that $\bar{z}_n \to \bar{z}$ with $\bar{z}_n \in X_F$. Then

$$
||\bar{x}_0 \bar{z}_n - \bar{x}_0 \bar{z}_m||^2 = (\bar{x}_0 \bar{z}_n - \bar{x}_0 \bar{z}_m, \bar{x}_0 \bar{z}_n - \bar{x}_0 \bar{z}_m) = F((x_0 z_n - x_0 z_m)^*(x_0 z_n - x_0 z_m)) = F((z_n - z_m)^* x_0^* x_0 (z_n - z_m))
$$

and

$$
||\bar{z}_n - \bar{z}_m||^2 = F((z_n - z_m)^*(z_n - z_m)).
$$

Since $F$ is admissible,

$$
||\bar{x}_0 \bar{z}_n - \bar{x}_0 \bar{z}_m||^2 \leq M ||z_n - z_m||^2 \text{ with } M > 0.
$$

Thus $\{\bar{x}_0 \bar{z}_n\}$ is a Cauchy sequence with respect to the inner product norm, and hence the sequence converges to an element $\bar{y}$ of $H$. Similarly we can show that if $\bar{y}_n \to \bar{z}$ with
respect to the inner product norm, then \( \{ \bar{x}_n \w_n \} \) converges to \( \bar{y} \). Now we define the mapping \( a \rightarrow T_a \) of \( A \) on \( H \) by

\[
T_a \bar{x} = \bar{a}_0 \bar{x}, \quad \bar{x} \in H \text{ where } \bar{a}_0 = \bar{a}.
\]

Then, if \( a, b \in A \),

\[
T_{ab} \bar{x} = (ab)^{-1} \bar{x} = (ab)^{-1} \bar{a} \bar{b} \bar{x} = \bar{a}_0 \bar{b}_0 \bar{x} \\
= (a_0(b_0x))^{-1} = T_a(b_0x)^{-1} \\
= T_a T_b \bar{x} \quad \text{for all } \bar{x} \in H.
\]

Similarly \( T_{a+b} = T_a + T_b \) and \( T_{\lambda a} = \lambda T_a \) for all \( \lambda \in C \). Thus \( a \rightarrow T_a \) defines a representation of \( A \) on \( H \).

Consider the restriction of the representation to \( A_0 \). Let \( a \in A_0 \). Since \( F \) is admissible, we have

\[
||T_a(x)||^2 = ||\bar{a} \bar{x}||^2 = (\bar{a} \bar{x}, \bar{a} \bar{x}) \\
= F(a^*a^*ax) \\
\leq M ||\bar{x}||^2 \text{ for some } M > 0, \quad \bar{x} \in X_F.
\]

Hence \( T_a \) is a continuous mapping on \( X_F \). Since \( X_F \) is dense in \( H \), \( T_a \) can be uniquely extended to a continuous mapping \( \hat{T}_a \) on \( H \). However if \( x \in H - X_F \), let \( \{x_n\} \) be a subset of \( X_F \) such that \( x_n \rightarrow x \). Then

\[
\hat{T}_a(x) = \lim \hat{T}_a(x_n) = \lim T_a(\bar{x}_n) = \lim \bar{a} \bar{x}_n \\
= \bar{a} \bar{x} = T_a(\bar{x}).
\]

Thus \( \hat{T}_a = T_a \) and \( T_a \) is a continuous function on \( H \) for \( a \in A_2 \). Since \( T_a \) is continuous, we can show that \( (T_a)^* = T_a^* \) by proving that \( (T_a)^*(x) = T_a^*(x) \) for all \( x \in X_F \).

Let \( \bar{x} \) and \( \bar{y} \) be elements of \( X_F \), then
\[(T_x \bar{z}, \bar{y}) = F(y^*ax) = F((y^*a)x) = (x, (\bar{a}^*)\bar{y}) = (x, T_x \bar{y}).\]

Thus for \(a \in A_0\), we have \((T_x)^* = T_x^*\).

**Corollary 3.3.** If \(A_1\) is also an algebra e.g., the product of bounded sets of \(A\) is bounded, then the restriction of the above representation to \(A_0\) is a *-representation of \(A_0\) on \(H\).

Let \(X\) be a vector space over \(C\) and let \(K\) be a subalgebra of \(L(X)\). Let \(z\) be a fixed vector in \(X\) and let \(X_z = \{T(z): T \in K\}\). Then \(X_z\) is an invariant subspace of \(X\) with respect to \(K\). If there exists an element \(z\) of a normed space \(X\) such that \(X_z = X\), then \(K\) is said to be *topologically cyclic* and the vector \(z\) is called a *topologically cyclic vector*. A representation \(x \to T_x\) of \(A\) on \(X\) is said to be *topologically cyclic* if, when \(K = \{T_x: x \in A\}\), there is a vector \(z\) in \(X\) such that \(X_z = X\).

With these definitions we state the following corollary to Theorem 3.2.

**Corollary 3.4.** Let \(A\) be a commutative locally convex \(*\)-algebra with identity. Let \(F\) be an admissible positive Hermitian functional on \(A\). Then the representation obtained above is topological cyclic with a cyclic vector \(h_0\) such that \(F(x) = \langle T_x h_0, h_0 \rangle\), \(x \in A\).

**Proof.** Let \(h_0 = \bar{1} = 1 \oplus X_f\). Then by definition \(T_x h_0 = \bar{x}_0\), so that the set \(\{T_x h_0: x \in A\} = X_f\) and hence is dense in \(H\). Thus \(h_0\) is a topologically cyclic vector. Now let \(x \in A\), then there exists \(x_0 \in A_0\) such that \(\bar{x} = \bar{x}_0\). Thus
\[ F(1^*(x-x_0)) = F(x-x_0) = F(x) - F(x_0). \]

By the way, \[ F(1^*(x-x_0)) = ((x-x_0)^-, 1) = (x, 1) - (x_0, 1) = 0. \]

Consequently \( F(x) = F(x_0). \) Therefore \( (T_a h_0, h_0) = (x_0 h_0, h_0) = (x_0 1, 1) = F(x_0) = F(x) \) for all \( x \in A. \)

4. Representable Functional

Let \( F \) be a linear functional on the locally convex \(*\)-algebra \( A \) and let \( a \to T_a \) be a representation of \( A \) on a Hilbert space \( H \) such that the restriction of the representation to \( A_0 \) is a \(*\)-representation of \( A_0 \) on \( H \). Then \( F \) is said to be representable by \( a \to T_a \) provided there exists a topologically cyclic vector \( h_0 \in H \) such that

\[ F(a) = (T_a h_0, h_0) \] for all \( a \in A. \)

Let \( a \to T_a \) be a representation of \( A \) on \( H \) and let

\[ M = \{ h \in H : T_a h = 0 \text{ for all } a \in A \}. \]

If \( M = \{0\} \), we say that the representation is essential.

**Lemma 4.1.** If the representation \( a \to T_a \) is essential, then each of the subspaces \( H_h = \{ T_a h : a \in A \} \) is cyclic with \( h \) as a cyclic vector.

**Proof.** [5, p. 206].

**Theorem 4.2.** Let \( F \) be a Hermitian functional on the pseudo-complete commutative locally convex \(*\)-algebra \( A \). Then in order for \( F \) to be representable, it is sufficient that
(1) for each $x \in A$, there is a $x_0 \in A_0$ such that $\bar{x} = x_0$,
(2) $|F(x)|^2 \leq \mu F(x^* x)$, $x \in A$,
where $\mu$ is a positive real constant independent of $x$.

Proof. Assume that $F$ satisfies the conditions and denote by $A_1$ the pseudo-complete locally convex $*$-algebra obtained by adjoining the identity element to $A$. Extend the functional $F$ to $A_1$ by the definition,

$$F(x + \alpha) = F(x) + \mu \alpha$$

for $x \in A$ and $\alpha$ a scalar.

Then

$$F((x + \alpha)^*(x + \alpha)) = F((x^* + \bar{\alpha})(x + \alpha))$$

$$= F(x^* x + x^* \alpha + \bar{\alpha} x + \bar{\alpha} \alpha)$$

$$\geq F(x^* x) - 2|\alpha||F(x)| + \mu |\alpha|^2$$

$$\geq F(x^* x) - 2|\alpha|\mu^1 F(x^* x)' + \mu |\alpha|^2$$

$$= (F(x^* x) - |\alpha|\mu^1)^2.$$

Thus $F$ is a positive linear functional on $A_1$ and Theorem 2.6 guarantees that the first condition of admissibility is satisfied on $A_1$. To show that the second condition is satisfied, let $x + \alpha \in A_1$. Then by hypothesis there exists $x_0 \in A_0$ such that $\bar{x}_0 = \bar{x}$. Consider $x_0 + \alpha$. Then since $x_0 = \bar{x}$ and $(x - x_0) \in L_F$,

$$|F[\gamma(x + \beta)(x_0 + \alpha) - (x + \alpha)]|^2$$

$$= |F[\gamma(x + \beta)(x_0 - x)]|^2$$

$$= |F[\gamma(x - x_0) + F(\bar{\beta}(x_0 - x))]|^2$$

$$= |\bar{\beta} F(x_0 - x)|^2$$

$$\leq |\beta|^2 F[(x_0 - x)^* (x_0 - x)] = 0.$$

Consequently $(x_0 + \alpha)^- = (x + \alpha)_0^-$.

Therefore $F$ is an admissible positive Hermitian func-
tional on $A_i$. Hence by Corollary 3.4 there exists a representation $x \rightarrow T_x$ of $A_i$ on $H$ defined by $T_{(a, \alpha)}x = (a + \alpha)x_\alpha$ and such that

$$F(a + \alpha) = \langle T_{(a, \alpha)}h_0, h_0 \rangle$$

for some $h_0 \in H$.

Now let $N = \{h \in H : T_\alpha h = \theta \text{ for all } \alpha \in A\}$. Consider the restriction of $a \rightarrow T_\alpha$ to the space $N^1$, where

$$N^1 = \{h \in H : (h, n) = 0 \text{ for all } n \in N\}.$$

Since $\{h \in N^1 : T_\alpha h = \theta \text{ for all } \alpha \in A\} = \{0\}$, the restriction is essential.

Let $h_0 = h_0' + h_0''$ where $h_0' \in N^1$ and $h_0'' \in N$. Then for all $\alpha \in A$ we have

$$F(a) = \langle T_\alpha h_0, h_0 \rangle = \langle T_\alpha (h_0' + h_0''), h_0' + h_0'' \rangle = \langle T_\alpha h_0', h_0' + h_0'' \rangle = \langle h_0', T_\alpha (h_0' + h_0'') \rangle = \langle h_0', T_\alpha h_0' + h_0'' \rangle = \langle T_\alpha h_0', h_0' \rangle.$$

Thus there exists $h_0' \in N^1$ such that $F(a) = \langle T_\alpha h_0', h_0' \rangle$ for all $\alpha \in A$. Let $H_0 = \{T_\alpha h_0' : \alpha \in A\}$. Then, since the restriction of the representation to $N^1$ is essential, by Lemma 4.1 $H_0$ is cyclic with $h_0$ as a cyclic vector.

**Corollary 4.3.** If $A$ has an identity element, then every positive functional which implies condition (1) is representable.

**Proof.** If $A$ has an identity element, then by the Cauchy-Schwarz inequality, we have

$$|F(x)|^2 \leq F(1)F(x^*x)$$

for any positive functional $F$. Thus, condition (2) is au-
tomatically satisfied.

**Corollary 4.4.** Let $F$ be an admissible positive Hermitian functional on the pseudo-complete commutative locally convex $*$-algebra $A$. Then there exists a $*$-representation of $A_0$ on a Hilbert space $H$.

**Proof.** If $A$ is commutative and pseudo-complete, then $A_0$ is an subalgebra of $A$ [1]. Therefore by Theorem 3.2 and Corollary 3.3, the proof is obvious.

**References**


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