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NECESSARY AND SUFFICIENT CONDITIONS FOR A GENERALIZED TOPOLOGICAL SPACE TO BE A TOPOLOGICAL SPACE

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1. Introduction

Let X be any non-empty set and let $P(X)$ denote the set of all subsets of X. A function $c: P(X) \to P(X)$ is called a *semi-closure operator* on X if it satisfies the following four axioms:

(1) $c(\phi)=\phi$,

(2) $A \subset c(A)$ for each $A \in P(X)$,

(3) if $A \subset B$, then $c(A) \subset c(B)$ for each $A, B \in P(X)$,

(4) $c(c(A)) \subset c(A)$ for each $A \in P(X)$.

If a semi-closure operator c on X is given, then the pair (X, c) is called a *semi-closure space*. These concepts of a semi-closure operator and a semi-closure space are generalized forms of a Kuratowski closure operator and a topological space, respectively.

For our convenience, throughout this paper, C will mean $\{A \in P(X) \mid c(X-A)=X-A\}$ in a semi-closure space (X, c). Then for each semi-closure operator c on X, C satisfies the following two properties:

(S1) X, $\phi \in C$,

(S2) C is closed under arbitrary union.

But C is not closed under finite intersection, in general. A collection C of subsets of X satisfying the properties (S1) and (S2) listed above is named a *supratopology* ([2] and [3]) or a *pretopology* ([1]). In [1], [3], and [5], the authors have investigated several properties of this generalized topological space. Indeed, the concepts of supratopologies and semi-closure operators are equivalent. In [4], Netzer has introduced an operator $- : P(X) \to P(X)$ satisfying (1) $A \subset \overline{A}$ for each $A \in P(X)$ and (2) if $A, B \in P(X)$ and $A \subset B$, then $\overline{A} \subset \overline{B}$. Using pseudo-sequences (generalized nets), he has shown a necessary and sufficient condition

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for the operator $-$ to satisfy the following property: $\overline{A\cup B}\subset\bar{A}\cup\bar{B}$ for each $A, B\in P(X)$.

In this paper, we shall study necessary and sufficient conditions for a semi-closure space to be a topological space, using the concepts of stacks and grills.

2. Results

Let (X, c) be a semi-closure space and let A be a non-empty subset of X and $x\in X$. A is called a ***neighborhood*** of x in X if there is a B in C such that $x\in B\subset A$. Let $N(x)$ denote the set of all neighborhoods of x in X.

DEFINITION 1. *A* ***stack*** $\mathcal{S}$ on a set X is a non-empty collection of subsets of X with the properties:

(1) if $A\in\mathcal{S}$ and $A\subset B\in P(X)$, then $B\in\mathcal{S}$,

(2) if $A, B\in\mathcal{S}$, then $A\cap B\neq\phi$.

Then, clearly, for each $x\in X$, $N(x)$ is a stack on X. We call $N(x)$ the neighborhood stack at x in (X, c).

LEMMA 1. *Let (X, c) be a semi-closure space. Then $c(A)=\{x\in X\mid A\cap N\neq\phi$, for each $N\in N(x)\}$, for each $A\in P(X)$.*

Proof. If $x\notin c(A)$, then $x\in X-c(A)\in C$ and $A\cap(X-c(A))=\phi$ since $A\subset c(A)$. Hence $x\notin\{x\in X\mid A\cap N\neq\phi$, for each $N\in N(x)\}$. Conversely, if $x\notin\{x\in X\mid A\cap N\neq\phi$, for each $N\in N(x)\}$, then there is an N in $N(x)$ such that $A\cap N=\phi$ and thus there is a B in C such that $x\in B\subset N$ and $A\cap B=\phi$. Since $B\in C$, $c(X-B)=X-B$ and hence we have $c(A)\subset X-B$. Therefore $x\notin c(A)$.

DEFINITION 2. Let (X, c) be a semi-closure space. A stack $\mathcal{S}$ on X is said to ***converge to*** x (written $\mathcal{S}\to x$) if and only if $N(x)\subset\mathcal{S}$.

LEMMA 2. *Let (X, c) be a semi-closure space. If $A\in P(X)$, then $x\in c(A)$ if and only if there is a stack $\mathcal{S}$ on X such that $A\in\mathcal{S}$ and $\mathcal{S}\to x$.*

Proof. If $x\in c(A)$, $A\cap N\neq\phi$ for each $N\in N(x)$ by Lemma 1. Let $\mathcal{S}=\{B\in P(X)\mid A\subset B$ or $N\subset B$ for some $N\in N(x)\}$. Then it is easily verified that $\mathcal{S}$ is a stack on X and $A\in\mathcal{S}$ and $N(x)\subset\mathcal{S}$.

Conversely, suppose that there is a stack $\mathcal{S}$ on X such that $A\in\mathcal{S}$ and $\mathcal{S}\to x$. Then since $\mathcal{S}\to x$, $N(x)\subset\mathcal{S}$ and hence we have $A\cap N\neq\phi$ for each $N\in N(x)$ by property (2) of stacks. Therefore, by Lemma 1,

$x \in c(A)$.

DEFINITION 3. A *grill* $\mathcal{G}$ on a set X is a stack on X with the property: if $A \cup B \in \mathcal{G}$, then $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

THEOREM. *In a semi-closure space* (X, c), *the following statements are equivalent*:

(1) $c(A \cup B) \subset c(A) \cup c(B)$, *for each* $A, B \in P(X)$.

(2) *For each* $x \in X$ *and each stack* $\mathcal{S}$ *on* X *such that* $\mathcal{S} \to x$, *there is filter* $\mathcal{F}$ *on* X *such that* $N(x) \subset \mathcal{F} \subset \mathcal{S}$.

(3) *For each* $A \in P(X)$, *if* $x \in c(A)$ *then there is a grill* $\mathcal{G}$ *on* X *such that* $N(x) \subset \mathcal{G}$ *and* $A \in \mathcal{G}$.

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and $\mathcal{S}$ a stack on X such that $\mathcal{S} \to x$. Then $N(x) \subset \mathcal{S}$ by Definition 2. If a semi-closure operator c on X satisfies (1), c is a Kuratowski closure operator on X, i.e., (X, c) is a topological space. Hence for each $x \in X$, the neighborhood stack $N(x)$ at x is a neighborhood filter at x and therefore $N(x)$ is a desired filter on X.

(2) $\Rightarrow$ (3) If $A \in P(X)$ and $x \in c(A)$, then there is a stack $\mathcal{S}$ on X such that $A \in \mathcal{S}$ and $N(x) \subset \mathcal{S}$ by Lemma 2. By hypothesis (2), there is a filter $\mathcal{F}$ on X such that $N(x) \subset \mathcal{F} \subset \mathcal{S}$. Let $\mathcal{B} = \{A \cap F \mid F \in \mathcal{F}\}$. Then, since $\mathcal{S}$ is a stack on X, $A \cap F \neq \phi$ for each $F \in \mathcal{F}$ and thus $\mathcal{B}$ is a filterbase on X. If $\mathcal{F}'$ is the filter generated by the filterbase $\mathcal{B}$, then $N(x) \subset \mathcal{F}'$ and $A \in \mathcal{F}'$. Let $\mathcal{M}$ be an ultrafilter finer than $\mathcal{F}'$. It is easily proved that every ultrafilter on X is a grill on X and this ultrafilter $\mathcal{M}$ satisfies $A \in \mathcal{M}$ and $N(x) \subset \mathcal{M}$. Hence $\mathcal{M}$ is a desired grill on X.

(3) $\Rightarrow$ (1) Let $x \in c(A \cup B)$. Then by (3), there is a grill $\mathcal{G}$ on X such that $N(x) \subset \mathcal{G}$ and $A \cup B \in \mathcal{G}$. Since $\mathcal{G}$ is a grill on X and $A \cup B \in \mathcal{G}$, $A \in \mathcal{G}$ or $B \in \mathcal{G}$ and therefore, by Lemma 2, we have $x \in c(A)$ or $x \in c(B)$ since every grill on X is a stack.

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