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SOME PROPERTIES OF SCHRÖDINGER OPERATORS

HAN-SOO KIM and LEE-CHAE JANG

1. Introduction

The aim of this note is to study some properties of Schrödinger operators, the magnetic case,

$H_0(a)=\frac{1}{2}(-i\nabla-a)^2$;

$H(a)=H_0(a)+V$, where $a=(a_1, \cdots\cdots, a_n)\in L^2_{loc}$ and V is a potential energy.

Also, we are interested in solutions, ϕ, of $H(a)\phi=E\phi$ in the sense that $(\phi, e^{-tH(a)}\Psi)=e^{-tE}(\phi, \Psi)$ for all $\Psi\in C_0^\infty(R^n)$ (see B. Simon [1]).

In section 2, under some conditions, we find that a semibounded quadratic form of $H(a)$ exists and that the Schrödinger operator $H(a)$ with Re $V\geq 0$ is accretive on a form domain $Q(H_0(a))$. But, it is well-known that the Schrödinger operator $H=\frac{1}{2}\Delta+V$ with Re $V\geq 0$ is accretive on $C_0^\infty(R^n)$ in N Okazawa [4].

In section 3, we want to discuss L^p estimates of Schrödinger semigroups.

2. Schrödinger operators, the magnetic case

A quadratic form is a map $q: Q(q)\times Q(q)\to C$ where $Q(q)$ is a dense linear subset of a Hilbert space H called the form domain such that $q(\cdot, \Psi)$ is semilinear and $q(\phi, \cdot)$ is linear for $\phi, \Psi\in Q(q)$.

If $q(\phi, \Psi)\geq -M\|\phi\|^2$ for some M, we say that q is semibounded.

THEOREM 2.1. *Let* $a\in L^2_{loc}$ *and let* $H_0(a)$ *denote the self-adjoint operator whose form domain is* $\{\phi\in L^2(R^n)\mid (i\nabla+a)\phi\in L^2(R^n, R^n)\}$ *with* $(\phi, H_0(a)\phi)=\|(i\nabla+a)\phi\|^2$.

Then for any real valued function $V\in L^\infty(R^n)$, *there exists the semibounded quadratic form* q *satisfying* $q(\phi, \Psi)=(\phi, H(a)\Psi)$, *for all*

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$\phi, \Psi \in Q(q) \equiv Q(H_0(a))$.

Proof. Since $Q(q) = Q(H_0(a))$, $(\phi, H_0(a)\Psi)$ exists. And we have,

$$|(\phi, V\Psi)| = \left|\int \overline{\phi(x)}\, V(x)\Psi(x)dx\right|$$
$$\leq \|V\|_\infty \int |\overline{\phi(x)}\Psi(x)|dx < \infty$$

So $q(\phi, \Psi) = (\phi, H_0(a)\Psi) + (\phi, V\Psi)$ exists, i.e. q is well-defined. Therefore it is easy to show that q is the quadratic form on $Q(q)$. Also, we have

$$\begin{aligned} q(\phi, \phi) &= (\phi, H(a)\phi) \\ &= (\phi, H_0(a)\phi) + (\phi, V\phi) \\ &= \|(i\nabla + a)\phi\|^2 + \int_{R^n} |\phi(x)|^2 V(x) d^n x \\ &= \|(i\nabla + a)\phi\|^2 + \int_{R^n} |\phi(x)|^2 V^+(x) d^n x - \int_{R^n} |\phi(x)|^2 V^-(x) d^n x \\ &\geq -\int_{R^n} |\phi(x)|^2 V^-(x) d^n x \geq -\|V\|_\infty \int_{R^n} |\phi(x)|^2 d^n x \\ &= -\|V\|_\infty \|\phi\|_2^2 \end{aligned}$$

This completes the proof.

REMARK. Since $a\phi \in L^2_{loc}$ it defines a distribution and the symbol $(i\nabla + a)\phi \in L^2$ means the distributional sum lies in L^2.

A linear operator A with domain $D(A)$ and range $R(A)$ in $L^2(R^n)$ is said to be accretive if Re $(\phi, A\phi) \geq 0$ for all $\phi \in D(A)$. Here (ϕ, Ψ) is semilinear in ϕ and linear in Ψ.

THEOREM 2.2. *Let $H_0(a)$ be as theorem 2.1. Then for any $V \in L^\infty(R^n)$ with Re $V \geq 0$, $H(a)$ is accretive on a form domain $Q(H_0(a))$.*

Proof. For $\phi \in Q(H_0(a))$, we have

$$\begin{aligned} (\phi, H(a)\phi) &= (\phi, H_0(a)\phi) + (\phi, V\phi) \\ &= \|(i\nabla + a)\phi\|^2 + (\phi, \text{Re } V\phi) + i(\phi, \text{Im } V\phi) \end{aligned}$$

So $\text{Re }(\phi, H(a)\phi) = \|(i\nabla + a)\phi\|^2 + \int_{R^n} |\phi(x)|^2 \text{Re } V(x) d^n x \geq 0.$

3. L^p estimates of Schrödinger semigroups

A real-valued measurable function V in R^n is said to lie in K_n if and only if

(a) If $n\geq 3$

$$\lim_{\alpha\downarrow 0}[\sup_x\int_{|x-y|\leq\alpha}|x-y|^{-(n-2)}|V(y)|d^ny]=0$$

(b) If $n=2$

$$\lim_{\alpha\downarrow 0}[\sup_x\int_{|x-y|\leq\alpha}\ln\{|x-y|^{-1}\}|V(y)|d^2y]=0$$

(c) If $n=1$

$$\sup_x\int_{|x-y|\leq 1}|V(y)|dy<\infty$$

We say V is in K_n^{loc} if and only if $V\chi_r\in K_n$ for all r, where χ_r is the characteristic function of $\{x|\ |x|\leq r\}$.

THEOREM 3.1. *Let* $V\in K_n^{\text{loc}}$ *with* $V\geq 0$. *If* $H_0(a)$ *is a positive self-adjoint operator on a form domain* $Q(H_0(a))$ *and if there is a function* λ *in* $C^\infty\cap L^\infty$ *such that* $a_j=i\partial_j\lambda$ *for* $j=1,2,\cdots,n$, *then*

(1) *There exist some constants* $M>0$ *and* $b>0$ *such that*
$|(e^{-tH(a)}\Psi)(x)|\leq Me^{-bx^2}$ *for all* $\Psi\in C_0^\infty(R^n)$.

(2) *For all* $\Psi\in C_0^\infty(R^n)$, $(\phi,\ e^{-tH(a)}\Psi)$ *makes sense if* ϕ *obeys* $|\phi(x)|\leq C(1+|x|)^N$ *for some* C *and* N.

Proof. (1) By the second hypothesis and [2, p. 191, Remark 3],

$$e^{-t(i\partial_j-a_j)^2}=e^\lambda e^{t\partial_j^2}e^{-\lambda}\text{ for } j=1,2,\cdots,n.$$

By the first hypothesis and the trotter product,

$$\begin{aligned}e^{-tH(a)}&=s-\lim_{m\to\infty}[e^{-tH_0(a)/m}e^{-tV/m}]^m\\&=s-\lim_{m\to\infty}[e^{-t(i\partial_1-a_1)^2/m}\cdots e^{-t(i\partial_n-a_n)^2/m}e^{-tV/m}]^m\\&=e^\lambda s-\lim_{m\to\infty}[e^{t\partial_1^2/m}\cdots e^{t\partial_n^2/m}]^m e^{-\lambda}\\&=e^\lambda e^{-tH}e^{-\lambda}\end{aligned}$$

If $e^{-tH(a)}(x,\ y)$, $e^{-tH}(x,\ y)$ are integral Kernels of $e^{-tH(a)}$, e^{-tH}, respectively, then we have $e^{-tH(a)}(x,\ y)=e^{\lambda(x)}e^{-tH}(x,\ y)e^{-\lambda(y)}$.
So, for $\Psi\in C_0^\infty(R^n)$,

$$\begin{aligned}|(e^{-tH(a)}\Psi)(x)|&=|\int_{R^n}e^{\lambda(x)}e^{-tH}(x,y)e^{-\lambda(y)}\Psi(y)d^ny|\\&=e^{\lambda(x)}|[e^{-tH}(e^{-\lambda}\Psi)](x)|\\&\leq\|e^\lambda\|_\infty C_1e^{-bx^2},\ \text{for some } C_1 \text{ and } b,\end{aligned}$$

by [1, p. 349, (1, 6)].

This completes the proof of (1).

(2) Let $\Psi\in C_0^\infty(R^n)$. Since $|\phi(x)|\leq C(1+|x|)^N$,

$$|(\phi,\ e^{-tH(a)}\Psi)|\leq\int_{R^n}|\phi(x)||(e^{-tH(a)}\Psi)(x)|d^nx$$
$$=CM\int_{R^n}(1+|x|)^N e^{-bx^2}d^nx<\infty,\quad \text{by (1)}.$$

Let $L_\delta^2(R^n)=\{\phi\,|\,(1+x^2)^{\delta/2}\phi\in L^2(R^n)\}$ for any δ, positive or negative with $\|\phi\|_{2,\delta}=\|(1+x^2)^{\delta/2}\phi\|_2$ and let $\|A\|_{2,\delta;2,\delta}$ be the norm of a map A from $L_\delta^2(R^n)$ to $L_\delta^2(R^n)$.

PROPOSITION 3.2. *Let* $V\in K_n^{\text{loc}}$ with $V\geq 0$. *If* $H_0(a)$ *is as theorem* 3.1 *and if there exists a function* λ *in* L^∞ *such that* $a_j=i\partial_j\lambda$ *for* $j=1, 2, \cdots, n$, *then* $e^{-tH(a)}: L_\delta^2(R^n)\to L_\delta^2(R^n)$ *is a bounded map obeying* $\|e^{-tH(a)}\|_{2,\delta;2,\delta}\leq Ce^{At}$ *for some* C *and* A.

Proof. Let $\phi\in L_\delta^2(R^n)$. Then, we have

$$\|e^{-tH(a)}\phi\|_{2,\delta}=\left[\int_{R^n}(1+x^2)^\delta|(e^{-tH(a)}\phi)(x)|^2d^nx\right]^{\frac{1}{2}}$$
$$=\left[\int_{R^n}(1+x^2)^\delta|e^{\lambda(x)}|^2|[e^{-tH}(e^{-\lambda}\phi)](x)|^2d^nx\right]^{\frac{1}{2}}$$
$$\leq\|e^\lambda\|_\infty\left[\int_{R^n}(1+x^2)^\delta|[e^{-tH}(e^{-\lambda}\phi)](x)|^2d^nx\right]^{\frac{1}{2}}$$
$$=\|e^\lambda\|_\infty\|e^{-tH}(e^{-\lambda}\phi)\|_{2,\delta}$$
$$\leq\|e^\lambda\|_\infty C_1e^{At}\|e^{-\lambda}\phi\|_{2,\delta},\quad \text{for some } C_1 \text{ and } A,$$

by [1, proposition 2.1].
Hence $\|e^{-tH(a)}\|_{2,\delta;2,\delta}\leq\|e^\lambda\|_\infty\|e^{-\lambda}\|_\infty C_1e^{At}$, since $\|e^{-\lambda}\phi\|_{2,\delta}\leq\|e^{-\lambda}\|_\infty\|\phi\|_{2,\delta}$.
This completes the proof.

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Kyungpook National University
Daegu 635, Korea