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-Research Paper-

An One-for-One Ordering Inventory Policy with Poisson Demands and Losses with Order Dependent Leadtimes

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ABSTRACT

A stochastic model for an inventory system in which depletion of stock takes place due to random demand as well as random loss of items is studied under the assumption that the intervals between successive unit demands as well as those between successive unit losses, are independently and identically distributed random variables having negative exponential distributions with respective parameters μ and λ. It is further assumed that leadtime for each order is an outstanding-order-dependent random variable. The steady state probability distribution of the net inventory level is derived under the continuous review (S-1, S) inventory policy, from which the total expected cost expression is formulated.

1. Introduction

In certain real world situations it is appropriate to order units one at a time as demanded. This can be true, for example, if the demand for the item is very low or the item is very expensive.

This policy is called a continuous review (S-1, S) inventory (i.e., one-for-one-ordering) policy and fills demands on a first-come first-served basis. It means that a reorder is placed whenever a demand occurs and the inventory position (i.e., the amount on hand plus on order minus backorders) remains constant.

Innumerable papers have been written analyzing mathematical models for describing (S-1, S) inventory policies [2, 3, 4, 5, 10]. Invariably, it was assumed implicitly that once units enter into inventory, they last forever or else they expire after only a single planning period. Though this assumption is not altogether unreasonable or unjustifiable in cases where planning over a short run is essentially unaffected by obsolescence, there is a wide variety of situations that arise in food industry, drug industry and in the area of health administration in which the perishable nature of the inventory has to be taken into account in developing optimal ordering policies. Some researchers have considered this aspect of the inventory problem [7, 8, 9] but

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have assumed that the items, like foodstuffs, photographic films, drugs and pharmaceuticals and boold, have fixed life time. However there exists a significant class of problems for which the life time of the items in the inventory is not deterministic and the assumption of random life time of the items would be more appropriate (e. g., the electronic industry).

Recently, authors (Choi and Kim [1], Kumaraswamy and Sankarasubramanian [6])have considered a problem in which the depletion of inventory level is due to random demand as well as random failure (or loss) of items proportional to on hand inventory. But, in their papers, they assumed instantaneous replenishment of order under a continuous review (s, S) inventory policy.

In this paper, we consider a $(S\text{-}1, S)$ problem under the assumption that replenishment time is negative exponentially distributed with parameter r, that demands occur in a Poisson manner with parameter μ , and that the life time distribution of each item in inventory is negative exponential with parameter λ . We have derived the steady state probability distribution of the net inventory level, and the total cost expression.

2. Analysis of the Problem

Notations

μ : average demand rate

λ : average failure (loss) rate

r : average item-replenishment rate

S : stock level specified

h : holding cost per unit per unit time

π : backorder cost per unit backordered per unit time.

Let $H(t)$ be the net inventory level at an arbitrary time t. Let $P(n, t)$ denote the probability that there are exactly n units in stock at time t, i.e., $P(n, t)=Pr\{H(t)=n\}$.

Changes in the net inventory level are caused by demands, losses of units in inventory, and arrivals of on-order inventory. In an infinitesimal time interval Δt, the net inventory moves from state $n+1$ to state n if a demand or a loss occurs, while it moves from state n to $n+1$. if an on-order inventory arrives. This can be represented by a transition diagram as depicted in Figure 1. The transitions are described as the following mathematical expression:

For $n=S$ and $t \geqq 0$,

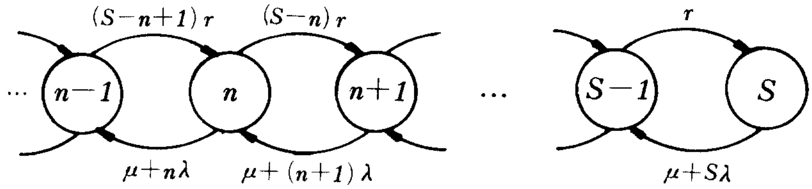
$$P(S,t+dt)=[1-(\mu+S\lambda)dt]\ P(S,t)+rdtP(S-1,t).$$

For $0 \leqq n \leqq S-1$ and $t \geqq 0$,

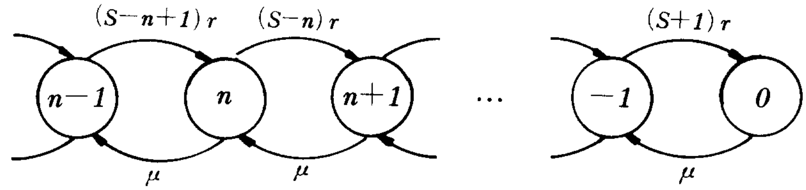
$$P(n,t+dt)=[1-(\mu+n\lambda)dt-(S-n)rdt]\ P(n,t)+[(S-n+1)rdt]\ P(n-1,t)+[\mu+(n+1)\quad]\ dtP(n+1,t).$$

For $n<0$ *and* $t \geqq 0$,

$$P(n,t+dt)=[1-\mu dt-(S-n)rdt]\ P(n,t)+[(S-n+1)rdt]P(n-1,t)+\mu dtP(n+1,t).$$



(a) For $0 \leqq n \leqq S-1$



(b) For $n<0$

Figure I. Transition diagram for the net inventory level.

Transposing, dividing by dt, and letting $dt \rightarrow 0$, the following differential-difference equations are derived:

$$\frac{dP(S,t)}{dt} = -(\mu+S\lambda)P(S,t)+rP(S-1,t), \qquad n=S. \tag{1}$$

$$\frac{dP(n,t)}{dt} = -[(\mu+n\lambda)+(S-n)r]P(n,t)+(S-n+1)rP(n-1,t) + [\mu+(n+1)\lambda]P(n+1,t), \qquad 0 \leqq n \leqq S-1. \tag{2}$$

$$\frac{dP(n,t)}{dt} = -[\mu+(S-n)r]P(n,t)+(S-n+1)rP(n-1,t) + \mu P(n+1,t), \qquad n<0. \tag{3}$$

Let P_n be the probability that in the steady state there are exactly n net inventory, that is

$$Pn = \lim_{t\to\infty} P(n,t) = \lim_{t\to\infty} P\{H(t)=n\}.$$

The following steady state difference equations are obtained from (1), (2), and (3).

$$-(\mu+S\lambda)P_S+rP_{S-1}=0, \qquad n=S. \tag{4}$$

$$-[(\mu+n\lambda)+(S-n)r]P_n+(S-n+1)rP_{n-1} + [\mu+(n+1)\lambda]P_{n+1}=0, \qquad 0 \leq n \leq S-1. \tag{5}$$

$$-[\mu+(S-n)r]P_n+(S-n+1)rP_{n-1} + \mu P_{n+1}=0, \qquad n<0. \tag{6}$$

Solving recursively the system of Eqs. (4) and (5), the follwing relations are found:

$$P_{S-1}=\frac{\mu+S\lambda}{r}P_S,$$

$$P_{S-2}=\frac{\mu+(S-1)\lambda}{2r}P_{S-1}=\frac{\mu+(S-1)\lambda}{2r}\,\frac{\mu+S\lambda}{r}P_S.$$

In general, we find that

$$P_n=\frac{\prod_{i=1}^{S-n}\{\mu+[S-(i-1)]\lambda\}}{(S-n)!\,r^{S-n}}P_S,\quad 0\le n\le S-1. \tag{7}$$

Solving recursively the system of Eqs. (5) and (6), we have

$$P_{-1}=\frac{\mu}{(S+1)r}P_0,$$

$$P_{-2}=\frac{\mu}{(S+2)r}P_{-1}=\frac{\mu^2}{(S+2)(S+1)r^2}P_0.$$

In general, we find that

$$Pn=\frac{\mu^{-n}}{(S-n)(S-n-1)\dots(S+1)r^{-n}}P_0$$

$$=\frac{\mu^{-n}\prod_{i=1}^{S}\{\mu+[S-(i-1)]\lambda\}}{(S-n)!\,r^{S-n}}P_S,\qquad n<0. \tag{8}$$

The value of P_s is determined by recorgnizing that

$$\sum_{n=-\infty}^{S}P_n=1. \tag{9}$$

Equivalently,

$$P_S+\sum_{n=0}^{S-1}P_n+\sum_{n=-\infty}^{-1}P_n=1. \tag{10}$$

Thus, substituting Eqs. (7) and (8) into Eq. (10),

$$P_S+\sum_{n=0}^{S-1}\frac{\prod_{i=1}^{S-n}\{\mu+(S-(i-1))\lambda\}}{(S-n)!\,r^{S-n}}P_S+\sum_{n=-\infty}^{-1}\frac{\mu^{-n}\prod_{i=1}^{S}\{\mu+(S-(i-1))\lambda\}}{(S-n)!\,r^{S-n}}P_S=1. \tag{11}$$

Solving Eq. (11) with respect to P_s,

$$P_S=\{1+\sum_{n=0}^{S-1}\frac{\prod_{i=1}^{S-n}[\mu+(S-(i-1))\lambda]}{(S-n)!\,r^{S-n}}+\mu^{-S}\prod_{i=1}^{S}[\mu+(S-(i-1))\lambda]e^{u/r}$$

$$
- \sum_{n=0}^{S} \frac{\mu^{-n} \prod_{i=1}^{S} \left[\mu + (S-(i-1))\lambda\right]}{(S-n)!\, r^{S-n}} \Big\}^{-1} \tag{12}
$$

Thus, the expected on hand inventory in the steady state is

$$
E(H) = \sum_{n=1}^{S} nPn = \sum_{n=1}^{S-1} n \frac{\prod_{i=1}^{S-n} \{\mu + [S-(i-1)]\lambda\}}{(S-n)!\, r^{S-n}} P_S + SP_S. \tag{13}
$$

The expected backorder quantity in the steady state is

$$
E(B) = \sum_{n=-\infty}^{1} (-n) P_n = \sum_{n=-\infty}^{1} (-n) \frac{\mu^{-n} \prod_{i=1}^{S} \{\mu + [S-(i-1)]\lambda\}}{(S-n)!\, r^{S-n}} P_S. \tag{14}
$$

From a practical point of view, the $(S\text{-}1, S)$ inventory model represents the situations where both carrying costs and out-of-stock costs are very high but the order cost is so relatively small that it can be ignored. Hence, the optimal policy will usually be obtained by basing the total cost equation only on the carrying and out-of-stock costs. The expected holding cost per unit time is $hE(H)$, and the expected backorder cost per unit time is $\pi E(B)$. From Eqs. (13) and (14), the total expected cost per unit time is given by

$$
C(S) = hE(H) + \pi E(B) = h \sum_{n=1}^{S-1} n \frac{\prod_{i=1}^{S-n} \{\mu + [S-(i-1)]\lambda\}}{(S-n)!\, r^{S-n}} P_S + hSP_S
$$

$$
+ \pi \sum_{n=-\infty}^{1} (-n) \frac{\mu^{-n} \prod_{i=1}^{S} \{\mu + [S-(i-1)]\lambda\}}{(S-n)!\, r^{S-n}} P_S. \tag{15}
$$

The above expression is to be minimized with respect to S.

3. Solution Search

Eq.(15) shows that the total cost function is not easy for us to verify its convexity. Therefore, we shall solve a numberical example for a suggestion of the general cost function trend. Consider a system with the following parameter values ; $n = \$20$ per unit per unit time, $\pi = \$2200$ per unit per unit time, $\mu = 10$, $\lambda = 2$ and $r = 15$, We find that the optimal value of S is $S^* = 4$. From Eq.(15), $C(S^* = 4) = \$68.13$. Figure 2 depicts the total expected cost per unit time varying with S. Figure 2 may indicate that the following conditions are necessary for an optimum value of S :

$$
C(S^*) - C(S^* - 1) \leq 0 \text{ and } C(S^*) - C(S^* + 1) \leq 0.
$$

The effect of failure (loss) rate is also tested, and the results are tabulated in Table 1. The test shows that both the total cost and the optimal value of S are sensitive to the failure (loss) rate.

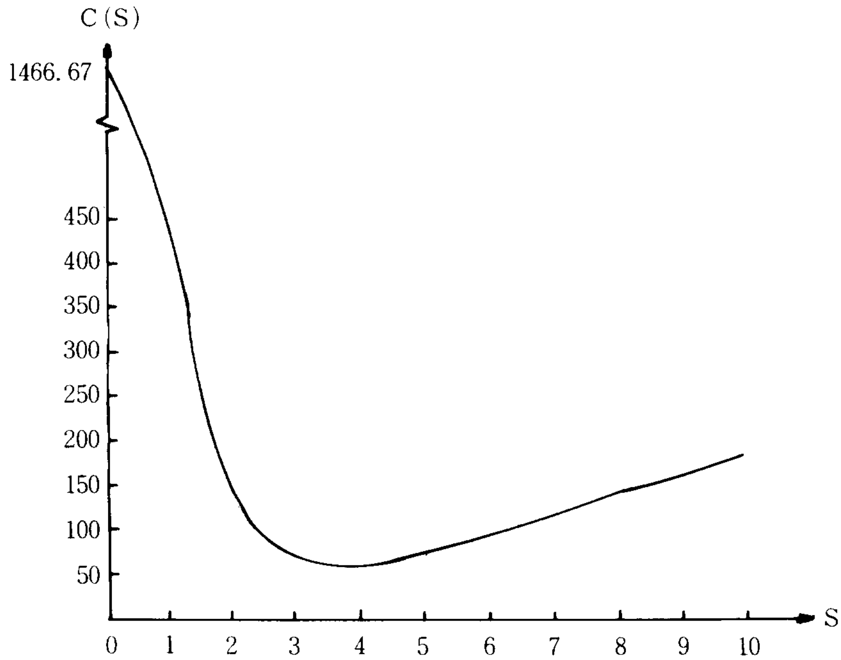


Fig. 2. Total expected cost per unit time varying with S.

Table 1. Optimal value of S varying with λ.

λ	S^*	$C(S^*)$
0	3	$ 58.97
1	3	$ 63.89
2	4	$ 68.13
3	4	$ 69.34
4	4	$ 71.47
5	4	$ 74.52
6	4	$ 78.51
7	5	$ 80.53
8	5	$ 82.14
9	5	$ 84.53
10	6	$ 87.47

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