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—Research Paper—

Sensitivity Analysis for Production Planning Problems with Backlogging*

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ABSTRACT

This paper addresses sensitivity analysis for a deterministic multi-period production and inventory model. The model assumes a piecewise linear cost structure, but permits backlogging of unsatisfied demand. Our approach to sensitivity analysis here can be divided into two basic steps;(1) to find the optimal production policy through a forward dynamic programming algorithm similar to the backward version of Zangwill [1966], and (2) to apply the penalty network approach by the author [1986] in order to derive sensitivity ranges for various model parameters. Computational aspects are discussed and topics of further research are suggested.

1. Introduction

We consider a determinisitic multi-period production and inventory model in which a single product is produced to satisfy known market requirements over a finite planning horizon of n periods. As in Zangwill [1966], the model permits backlogging of unsatisfied demand. Production, inventory holding, and backlogging cost functions for individual periods are assumed to be concave so that the overall cost functions for individual periods are assumed to be concave so that the overall cost function is piecewise concave[1]. The capacity of production facilities is large enough so that all requirements may be produced in any period. Zangwill characterized the structure of optimal production schedules and developed an efficient dynamic programming algorithm.

In this paper, sensitivity analysis for production planning problems with backlogging is studied under the additional assumption of piecewise linear cost stucture; *i.e.*, for each period, the production cost function consists of fixed (setup) and linear cost terms, and the inventory holding and backlogging cost functions are linear. An illustration is depicted in Figure 1. It is easy to see that any linear combination of these cost functions is a piecewise concave function.

*This paper constitutes the majority of Chapter 4 of the author's Ph.D. dissertation at UCLA.

**I wish to express my sincere gratitude to Professor Donald Erlenkotter for his criticism and guidance as my thesis advisor at UCLA.

[1]For definition and properies of piecewise concavity, see Zangwill [1967].

Our approach to sensitivity analysis here can be divided into two basic steps; (1) to find the optimal production policy through a forward dynamic programming algorithm similar to the backward version of Zangwill [1966], and (2) to apply the penalty network approach by the author [1986] in order to derive sensitivity ranges for model parameters such as setup costs, unit production costs, unit holding costs, unit stockout costs, and market requirements.

In Section 2, we formalize our production and inventory model. Section 3 reviews Zangwill's optimization approach, and devises its forward version which is compatible with the penalty network approach. The computational complexity of the forward version is discussed. In Section 4, we interpret the main results of the penalty network approach in connection with sensitivity analysis for the production and inventory model. Then, in Sections 5 and 6, we apply these results to derive the sensitivity ranges for various model parameters that are easy to implement. Finally, topics of furt her research are suggested in Section 7.

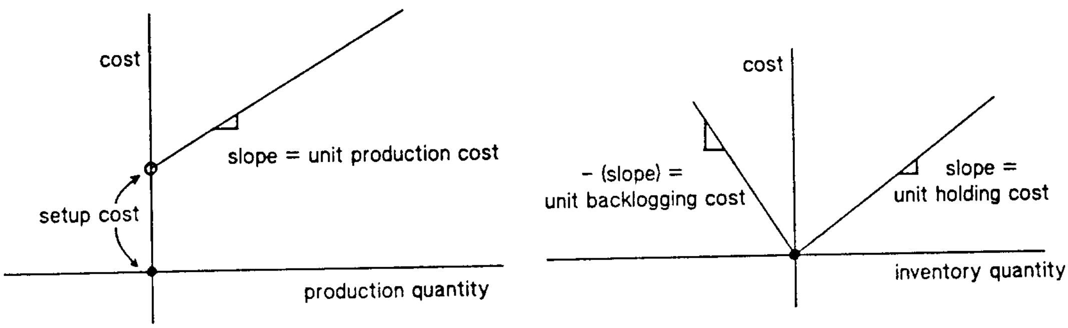

Figure 1: A Piecewise Linear Cost Structure

2. The Model

For period i $(i=1,\cdots,n)$, let r_i be the fixed market requirement and x_i the production amount. The vectors $r=(r_1,\cdots,r_n)$ and $x=(x_1,\cdots,x_n)$ conveniently summarize a market requirement schedule and production schedule, respectively. r and x are elements of R^n where R^n is $n-$dimensional Euclidean space. It is assumed that r_i, $x_i \geq 0$ for $i=1,\cdots,n$.

The inventory I_i on hand at the end of period i is the total amount of production completed through period i *less* the total market reqirements through period i. I_0 is assumed to be zero so that

$$I_i=\sum_{h=1}^{i}(r_h-x_h)$$

for $i=1,\cdots,n$. If $I_i \geq 0$, then stock is on hand or zero; otherwise, unsatisfied demand is backlogged.

It is assumed that customers are willing to wait α periods after the due date for their goods. However, all goods must be delevered within α periods or no later than period n. α is called the backlog limit. If $\alpha=0$, no backlogging is permitted. If $\alpha \geq n-1$, backlogging is without limit for all practical purposes since the time horizon is limited to n periods. Since all goods must be delivered no more than α periods late,

$$I_i \geq -\sum_{h=i-\alpha+1}^{i} r_h$$

where $r_h=0$ for $h \leq 0$.

Let $F(x)$ be the total cost of following production schedule $x=(x_1,\cdots,x_n)$. $F(x)$ is assumed to be the sum of production costs $P_i(x_i)$ and inventory costs $H_i(I_i)$ over periods $i=1,\cdots,n$. With the piecewise linear cost structure as shown in Figure 1, $F(x)$ can be written as

$$F(x)=\sum_{i=1}^{n}\{P_i(x_i)+H_i(I_i)\}$$

where for each i

$$P_i(x_i)=\begin{cases} p_i^1+p_i^2 x_i & \text{if } x_i>0 \\ 0 & \text{if } x_i=0 \end{cases}$$

and

$$H_i(I_i)=\begin{cases} h_i^1 I_i & \text{if } I_i \geq 0 \\ -h_i^2 I_i & \text{if } I_i<0. \end{cases}$$

We shall interpret p_i^1 as the setup cost, p_i^2 as the unit production cost, h_i^1 as the unit holding cost, and h_i^2 as the unit backlogging cost for period i. These parameters are required to be nonnegative.

In summary, a statement of the production planning problems is: Given certain fixed nonnegative market requirements $r=(r_1,\cdots,r_n)$, find a production schedule $x=(x_1,\cdots,x_n)$ that minimizes the overall cost function

$$F(x)=\sum_{i=1}^{n}\{P_i(x_i)+H_i(I_i)\} \tag{1}$$

subject to

$$I_i=\sum_{h=1}^{i}(x_h-r_h) \tag{2}$$

$$I_i \geq -\sum_{h=i-\alpha+1}^{i} r_h \tag{3}$$

$$x_i \geq 0 \tag{4}$$

for $i=1,\cdots,n$ and

$$I_n=0. \tag{5}$$

Note that $r_h=0$ for $h \leq 0$. There is no loss in generality in assuming that the final inventory I_n is zero as in equation (5). Intuitively, this is true since there is no advantage to overproduction.[2]

3. Zangwill's Algorithm

Following Zangwill [1966], the set of all feasible solutions that satisfy constraints(2)-(5) can be partitioned into disjoint subsets, called *basic sets,* which are characterized by whether I_i is nonnegative or negative in each of the periods $1, \cdots, n-1$. There are 2^{n-1} basic sets, all of which are compact and convex. Since the objective function $F(x)$ is concave on each basic set, it follows that for a fixed basic set, $F(x)$ attains its minimum at one of extreme points of the basic sets. Let D be the union of all extreme points of all basic sets. D is called the *dominant set*. The minimum of $F(x)$ occurs at a point in the dominant set D. The productin schedule corresponding to a point in the dominant set is called a *dominant schedule.*

A complete enumeration of all the points in D ensures finite calculation of an optimal production schedule. However, the size of the dominant set D grows exponentially at least with rate 2^{n-1} as the length of the planning horizon n increases. A less tedious approach has been provided by Zangwill, who characterized the structure of the dominant set D and developed a backward dynamic programming algorithm for efficiently (implicitly) enumerating D.

For the purpose of sensitivity analysis, we develop a forward version of Zangwill's algorithm. In subsequent sections, we shall discuss sensitivity analysis based on this forward algorithm.

One of the characteristics of the dominant set D is that *the entering inventory for any period, say i, of a dominant schedule can be expressed in terms of an integer s as follows:*[3]

$$I_{i-1} = \sum_{h=i}^{s} r_h = \sum_{h=0}^{s} r_h - \sum_{h=0}^{i-1} r_h. \tag{6}$$

If $s=i-1$ or $I_{i-1}=0$, we know that requirements $r_1, \cdots, r_{i-1}$ are exactly satisfied. If $s \geq i$ or $I_{i-1} \geq 0$, the integer s specifies that stock is on hand at the start of period i to satisfy requirements from periods i to s. On the other hand, if $s \leq i-2$ or $I_{i-1} \leq 0$, the integer s specifies that requirements from periods s to $i-1$ are backlogged to period i.

When the entering inventory can be expressed as in equation (6), the entering inventory level of period i is said to be s, or the state is said to be (i, s). State (i, s) represents the situation in which the entering inventory level of period i is s.[4]

The entering inventory level s of period i is said to be *permissible* if state (i, s) insures feasibility and prevents excess backlogging. Let J_i be the index set of permissible inventory levels for period i. Mathematically, J_i can be defined as

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[2] For example, if $p_1^1=p_1^2=0$ *and* $h_i^1=0$ for $i=1, \cdots, n$, it could be possible to overproduce. However, this is a trivial and uninteresting case.

[3] See Theorem 1 in Zangwill [1966].

[4] Therefore, the state for our model consists of the stage variable i (period) and the state variable s (entering inventory level of period i). Although both i and s for state (i, s) refer to period indices, s can be defined only with period i.

$$J_i=\begin{cases}\{0\} & \text{if } i=1\\ \{s \mid \max[0,\ i-1-\alpha] \le s \le n\} & \text{if } 1<i\le n\\ \{n\} & \text{if } i=n+1.\end{cases} \tag{7}$$

A transition from state (i,s) to state $(i+1,t)$, $s \le t$, represents the activity of producing $\sum_{h=s+1}^{t} r_h$ at period i and storing (or backlogging if $t \le i-1$) the amount $\sum_{h=i+1}^{t} r_h$ to next period. Let $C_i(s,\ t)$ be the transition cost from state $(i,\ s)$ to state $(i+1,\ t)$. Then $C_i(s,\ t)$ is the sum of production and inventory charges; that is,

$$C_i(s,\ t) \equiv P_i(\sum_{h=s+1}^{t} r_h) + H_i(\sum_{h=i+1}^{t} r_h). \tag{8}$$

Given the piecewise linear structure, the transition cost $C_i(s,\ t)$ is expressed as the sum of

$$P_i(\sum_{h=s+1}^{t} r_h) = \begin{cases} p_i^1 + p_i^2\,\{R(t)-R(s)\} & \text{if } R(t)-R(s)>0\\ 0 & \text{if } R(t)-R(s)=0\end{cases} \tag{9}$$

and

$$H_i(\sum_{h=i+1}^{t} r_h) = \begin{cases} h_i^1\,\{R(t)-R(i)\} & \text{if } t \ge i\\ h_i^2\,\{R(i)-R(t)\} & \text{if } t<i\end{cases} \tag{10}$$

where $R(t) \equiv \sum_{h=1}^{t} r_h$ represents the cumulated requirements up to (and including) period t.

Another useful characteristic of the dominant set D is that *if $s \ge i$ for state (i,s), then x_i in the dominant schedule must be zero.*[5] In other words, if we have positive inventory on hand, we will produce nothing. Therefore, if $s \ge i$, it suffices to consider only the transition from state $(i,\ s)$ to $(i+1,\ s)$. This property saves considerable computational effort as the size of D grows.

Let $A_{i+1}(t)$ be the set of transitions that result in state $(i+1,t)$. If we let arc $((i,s),(i+1,t))$ represent the transition from state (i,s) to state $(i+1,t)$, $A_{i+1}(t)$ is simply the set of arcs that are incident to node $(i+1,t)$. Based on the definition of J_i's by equation(7), we formally define $A_{i+1}(t)$ as follows:

Case 1:

For $i=1$ and $t \in J_2$,

$$A_2(t)=\{((1,0),(2,t))\} \tag{11}$$

since $J_1=\{0\}$.

Case 2:

For $1<i<n$ and $t \in J_{i+1}$, any transition from state $(i,\ s)$ to state $(i+1,t)$ can be partitioned into four cases: $s \le t < i$, $s < i \le t$, $i \le s = t$, and $i \le s < t$ where $s \in J_i$. However, the transition$((i,s), (i+1,t))$ with $i \le s < t$ can be eliminated since if there is positive inventory on hand we will produce nothing. Thus,

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[5]See Theorem 2 in Zangwill [1966].

$$A_{i+1}(t)=\{((i,\ s),\ (i+1,\ t))\mid s\in J_i,\ s\leq t<i,\ s<i\leq t,\ i\leq s=t\}. \tag{12}$$

Case 3:

Finally, for $i=n$,

$$A_{n+1}(n)=\{((n,s),(n+1,n))\mid s\in J_n\} \tag{13}$$

since $J_{n+1}=\{n\}$.

In summary, the set of states J and the set of transitions A can be given as follows:

$$J=\bigcup_{i=1}^{n+1}\{(i,s)\mid s\in J_i\} \tag{14}$$

and

$$A=\bigcup_{i=1}^{n}\{A_{i+1}(t),\ t\in J_{i+1}\} \tag{15}$$

where J_i and $A_{i+1}(t)$ are defined by equations (7) and (11)–(13), respectivety. Figure 2 exemplifies the definitions of states and transitions.

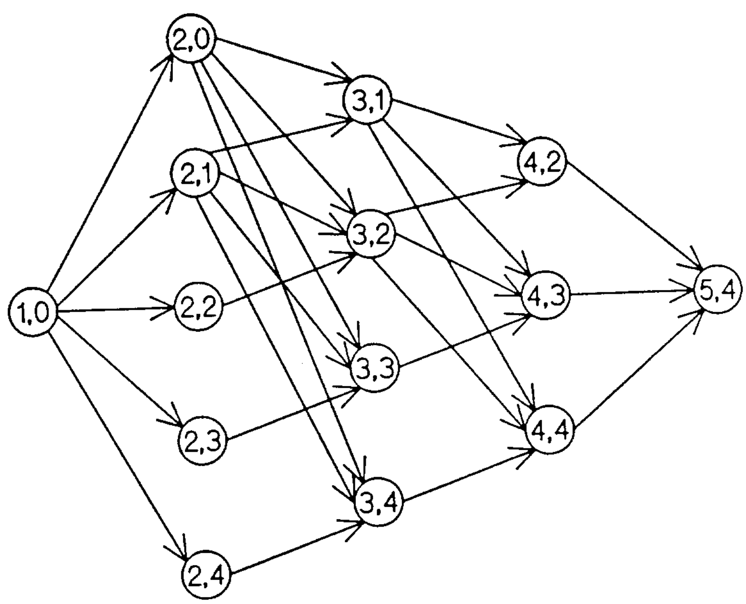

Figure 2: Transition Diagram for $n=4$ and $\alpha=1$

Let $F_i(t)$ be the minimum cost from period 1 through $i-1$ when following an optimal production schedule in periods 1 through $i-1$ such that the entering inventory level of period i is t.[6] Equivalently, $F_i(t)$ is defined to be the minimum transition cost from state (1, 0) to state $(i,\ t)$.

[6] $F_i(t)$ is the optimal value function associated with state $(i,\ t)$

Graphically, if we think of $C_i(s, t)$ as the length of arc $((i, s), (i+1, t))$, $F_i(t)$ represents the length of the shortest path from node $(1, 0)$ to node (i, t) on the network $G=(J, A)$ where J and A are defined by equations (14) and (15).

It is easily seen that G is an acyclic network defined by the evolution of transitions over time (see Figure 2). Also easily seen that $F_{n+1}(n)$ is the minimum of objective function(1) subject to constraints (2)−(5). Therefore, the optimal production schedule can be obtained by solving the shortest path from node $(1,0)$ to node $(n+1, n)$ on the acyclic network $G=(J, A)$.

Any shortest path algorithm now can be applied to find the value of $F_{n+1}(n)$. A simple forward recursion is defined as follows:

$$F_1(0) \equiv 0$$

$$F_2(t) = F_1(0) + C_1(0, t) \qquad \text{for } t \in J_2$$

$$F_{i+1}(t) = \min_{((i,s),(i+1,t)) \in A_{i+1}(t)} \{F_i(s) + C_i(s, t)\} \quad \text{for } 1 < i < n \text{ and } t \in J_{i+1}$$

$$F_{n+1}(1) = \min_{s \in J_n} \{F_n(s) + C_n(s, n)\}.$$

We call recursion *Zangwill's forward algorithm* in contrast with the backward algorithm in Zangwill's paper [1966]. Both algorithms are essentially the same and equally efficient.

It is clear that the forward recursion is run in $O(|A|)$ time where $|A|$ is the number of arcs in A. Since $|A|$ is at most $\frac{1}{3}n^3 + O(n^2)$ as shown in Proposition 1, the computational effort for Zangwill's forward algorithm is polynomially bounded.[7]

Proposition 1. For $G=(J, A)$ of Zangwill's forward algorithm,

$$|J| = \begin{cases} \frac{1}{2}n(n+2\alpha+1) - \frac{1}{2}(\alpha-1)(\alpha+2) & \text{if } 0 \le \alpha \le n-2 \\ n^2+1 & \text{if } \alpha \ge n-1 \end{cases}$$

and

$$|A| = \begin{cases} \frac{1}{2}n^2(\alpha+2) + \frac{3}{2}n\alpha - \frac{1}{6}\alpha(\alpha+1)(\alpha+8) & \text{if } 0 \le \alpha \le n-2 \\ \frac{1}{3}n(n^2+3n-1) & \text{if } \alpha \ge n-1. \end{cases}$$

Proof: It is left as an exercise for the reader.

4. The Penalty Network Approach

In this section, we review and interpret the main results of the penalty network approach proposed by the author [1986] in connection with sensitivity analysis for the production and inventory model.

[7]For the motivation for determining the computatinal complexity of an algorithm, see Garey and Johnson [1979].

In oredr to exploit the penalty network approach as a post-optimality tool, we need more definitions and notation. First, we define the penalty for transition $((k, s), (k+1, t))$ by

$$e_k(s, t)=F_k(s)+C_k(s, t)-F_{k+1}(t) \tag{16}$$

where $C_k(s, t)$ is available from the cost data as in equations(8)$-$(10), and $F_k(s)$ and $F_{k+1}(t)$ are obtained by solving the shortest path problem on $G=(J, A)$ through Zangwill's forward algorithm.

Physically, the penalty $e_k(s, t)$ measures the minimum possible *regret* if the transition $((k, s), (k+1, t)$ is to be included in a production subplan $\hat{x}=(x_1,\cdots,x_k)$ such that the entering inventory level of period $k+1$ is t.[8] It is easy to see that $e_k(s, t)=0$ if the transition $((k, s),(k+1, t))$ is included in an optimal production subplan $\hat{x}=(x_1,\cdots,x_i)$ where $i \geq k$.

We constuct the penalty network G_p from $G=(J, A)$ by replacing $C_k(s, t)$ with $e_k(s, t)$ for the attribute of every transition $((k, s), (k+1, t))$. An important observation is that the shortest path from node $(1, 0)$ to node $(n+1, n)$ on G is the same as that on G_p.

Now, suppose there have been changes in transition costs:

$$C_k(s, t) \rightarrow C_k(s, t)+\triangle_k(s, t) \tag{17}$$

for every transition $((k, s), (k+1, t))$ due to variation in the value of a model parameter. Note that a scalar quantity $\triangle_k(s, t)$ may be possibly negative. We want to find the condition these $\triangle_k(s, t)$'s must satisfy in order for the optimal production policy to remain unchanged. This necessary and sufficient condition is called the *optimality condition* on $\triangle_k(s, t)$'s from which the range of the model parameter called the *sensitivity range* is determined such that for any value of the parameter in the range, the optimal policy remains unchanged.

Let $V^*_k(t)$ be the minimum penalty for transition from (k, t) to $(n+1, n)$. The values of $V^*_k(t)$'s are available when a shortest path problem is solved on the penalty network G_p. Obviously, $V^*_1(0)=0$. Let $\tilde{G}_p$ be a perturbed penalty network that has the pseudo-penalty of $e_k(s, t)+\triangle_k(s, t)$ as the attribute of each arc $((k, s), (k+1, t))$ on $G=(J, A)$.

The following is a variant of the Sensitivity Theorem that is rephrased for easier interpretation:

Theorem 1. Lee [1986]. *The sum of pseudo-penalties of the shortest path form node $(1, 0)$ to node $(n+1, n)$ on $\tilde{G}_p$ is the sum of $\triangle_k(s, t)$'s over the basic arcs if and only if the optimal production policy remains unchanged*

where the set of *basic* arcs constitutes a shortest path form node $(1, 0)$ to node $(n+1, n)$ on the original notwork $G=(J, A)$.

Theorem 1 provides an useful logic for deriving the optimality condition on $\triangle_k(s, t)$'s which then will produce sensitivity ranges for model parameters. Practical ways of finding the optimality

[8]A subplan $\hat{x}=(x_1,\cdots,x_k)$ is a truncated production schedule that produces x_h in period $h-1,\cdots,k$ so that the ending inventory of period k is $I_k=\sum_{h=1}^{k}(x_h-r_h)$.

condition on $\triangle_k(s, t)$'s depend heavily on a given cost structure.

5. Sensitivity Ranges for Cost Parameters

In this section, we derive sensitivity ranges for cost parameters p_k^1, p_k^2, h_k^1, and h_k^2, while sensitivity ranges for requirements r_k will be discussed in next section.

Let s^*_k be the optimal entering inventory level of period k so that a path

$$(1, s^*_1)\rightarrow(2, s^*_2)\rightarrow\cdots\rightarrow(n, s^*_n)\rightarrow(n+1, s^*_{n+1})$$

corresponds to a shortest path from node $(1,0)$ to node $(n+1, n)$ on the network $G=(J, A)$. Note that $s^*_1=0$ and $s^*_{n+1}=n$.

Since any change in the values for cost parameters in period k affects only the costs of transitions that start in period k, Theorem 1 can be specialized to finding sensitivity ranges for cost parameters as follows:

Proposition 2. *Given a parametric change in period k,*

$$\triangle_k(s^*_k, s^*_{k+1})-\triangle_k(s, t)\leq e_k(s, t)+V^*_{k+1}(t) \tag{18}$$

for every transition $((k, s), (k+1, t))$ *in* $A_{k+1}(t)$, $t\in J_{k+1}$ *if and only if the current optimal policy remains unchanged.*

Proof: The sum of pseudo-penalties of a path from node $(1,0)$ to node $(n+1, n)$ that includes transition $((k, s), (k+1, t))$ is no less than

$$e_k(s, t)+\triangle_k(s, t)+V^*_{k+1}(t)$$

since the minimum penalty of a path form $(1,0)$ to (k, s) is always zero. In particular, we see that $e_k(s^*_k, s^*_{k+1})=0$ and $V^*_{k+1}(s^*_{k+1})=0$ for the basic arc $((k, s^*_k), (k+1, s^*_{k+1}))$. Hence, according to Theorem 1, we conclude

$$\triangle_k(s^*_k, s^*_{k+1})\leq\triangle_k(s, t)+e_k(s, t)+V^*_{k+1}(t)$$

for every feasible transition that starts in period k if and only if the current optimal policy remains unchanged.

The right-hand-side (RHS) of inequality (18) is obtainable by solving a shortest path problem on the penalty network G_p. On the other hand, the left-hand-side (LHS) varies over cost parameters, but we will see that only a simple algebraic operation is needed to calculate the LHS.

5.1 Sensitivbity Ranges for Setup Costs

Let $\triangle p^1$ be change in the setup cost of period k such that

$$p_k^1 \rightarrow p_k^1 + \triangle p^1.$$

Then, by definition, $G_k(s, t)$ is increased by $\triangle_k(s, t)$ By equation (9), we have

$$\triangle_k(s, t) = \begin{cases} \triangle p^1 & \text{if } R(t) - R(s) > 0 \\ 0 & \text{if } R(t) - R(s) = 0 \end{cases}$$

so that the LHS of inequality (18) is given by

$$\triangle_k(s^*_k, s^*_{k+1}) = -\triangle_k(s, t) \begin{cases} \triangle p^1 & \text{if } R(s^*_{k+1}) - R(s^*_k) > 0 \text{ and } R(t) - R(s) = 0 \\ -\triangle p^1 & \text{if } R(s^*_{k+1}) - R(s^*_k) = 0 \text{ and } R(t) - R(s) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

In addition, the nonnegativity of parameters in the model requires $\triangle p^1 \geq -p_k^1$.

Hence, depending on whether $R(s^*_{k+1}) - R(s^*_k) > 0$ or $R(s^*_{k+1}) - R(s^*_k) = 0$, Proposition 2 concludes

Proposition 3. *If* $R(s^*_{k+1}) - R(s^*_k) > 0$, *then* $\triangle p^1$ for p_k^1 satisfies $\triangle p^1 \geq -p_k^1$ *and*

$$\triangle p^1 \leq e_k(s, t) + V^*_{k+1}(t)$$

for every transition $((k, s), (k+1, t)) \in A_{k+1}(t)$, $t \in J_{k+1}$ *with* $R(t) - R(s) = 0$ *if and only if the optimal policy remains unchanged.*

Proposition 4. *If* $R(s^*_{k+1}) - R(s^*_k) = 0$, *then* $\triangle p^1$ *for* p_k^1 *satisfies* $\triangle p^1 \geq -p_k^1$ *and*

$$-\triangle p^1 \leq e_k(s, t) + V^*_{k+1}(t)$$

for every transition $((k, s), (k+1, t)) \in A_{k+1}(t)$, $t \in J_{k+1}$ *with* $R(t) - R(s) > 0$ *if and only if the optimal policy remains unchanged.*

These propositions imply:

Corollary 2. *If the optimal policy is to make products in period* $k > 1$, *the sensitivity range for the setup cost in period k has an upper bound; otherwise, a lower bound.*

If $k=1$ and $\alpha\geq 1$, Corollary 2 is still valid; however, for $k=1$ and $\alpha=0$, the setup cost for period 1 can be arbitrarily increased.

5.2 Sensitivity Ranges for Production Costs

Let $\triangle p^2$ be a change in the unit production cost of period k such that

$$p_k^2 \rightarrow p_k^2 + \triangle p^2$$

By equation (9), we have

$$\triangle_k(s,\ t) = \triangle p^2 \{R(t) - R(s)\}$$

The LHS of inequality (18) is given by

$$\triangle_k(s^*_k,\ s^*_{k+1}) - \triangle_k(s,\ t) = \triangle p^2\{\{R(s^*_{k+1}) - R(s^*_k)\} - \{R(t) - R(s)\}\}$$

so that Proposition 2 concludes

Proposition 5. *$\triangle p^2$ for p_k^2 satisfies $\triangle p^2 > -p^2$ and*

$$\triangle p^2\{\{R(s^*_{k+1}) - R(s^*_k)\} - R(t) - R(s)\}\} \leq e_k(s,\ t) + V^*_{k+1}(t)$$

for every transition $((k,\ s),\ (k+1,\ t)) \in A_{k+1}(t)$, $t \in J_{k+1}$ if and only if the optimal policy remains unchanged.

5.3 Sensitivity Ranges for Holding Costs

Let $\triangle h^1$ be a change in the unit holding cost of period k such that

$$h_k^1 \rightarrow h_k^1 + \triangle h^1.$$

Since $\triangle h^1$ affects $C_k(s,\ t)$ only if there is positive inventory at the start of period $k+1$, it follows from equation (10) that

$$\triangle_k(s,\ t) = \begin{cases} \triangle h^1\{R(t) - R(k)\} & \text{if } t>k \\ 0 & \text{if } t\leq k. \end{cases}$$

Note that $\triangle_k(s,\ t)$ caused by $\triangle h^1$ does not depend on s. The LHS of inequality (18) now can be expressed as follows:

$$\triangle_k(s^*_k,\ s^*_{k+1}) - \triangle_k(s,\ t) = \begin{cases} \triangle h^1\{R(s^*_{k+1}) - R(t)\} & \text{if } s^*_{k+1}>k \text{ and } t>k \\ \triangle h^1\{R(s^*_{k+1}) - R(k)\} & \text{if } s^*_{k+1}>k \text{ and } t\leq k \\ \triangle h^1\{R(k) - R(t)\} & \text{if } s^*_{k+1}\leq k \text{ and } t>k \\ 0 & \text{if } s^*_{k+1}\leq k \text{ and } t\leq k \end{cases}$$

Hence, depending on whether $s^*_{k+1}>k$ or $s^*_{k+1}\leq k$, Proposition 2 concludes

Proposition 6. *If* $s^*_{k+1}>k$, $\triangle h^1$ *for* h^1_k satisfies $\triangle h^1>-h^1_k$,

$$\triangle h^1\{R(s^*_{k+1})-R(t)\}\leq e_k(s,\ t)+V^*_{k+1}(t)$$

for every transition $((k,\ s),\ (k+1,\ t))\in A_{k+1}(t),\ t\in J_{k+1}$ *with* $t>k$, *and*

$$\triangle h^1\{R(s^*_{k+1})-R(k)\}\leq e_k(s,\ t)+V^*_{k+1}(t)$$

for every transition $((k,\ s),\ (k+1,\ t))\in A_{k+1}(t),\ t\in J_{k+1}$ with $t\leq k$ *if and only if the optimal policy remains unchanged.*

Proposition 7. *If* $s^*_{k+1}\leq k$, $\triangle h^1$ *for* h^1_k *satisfies* $\triangle h^1\geq -h^1_k$ *and*

$$\triangle h^1\ \{R(k)-R(t)\}\leq e_k(s,\ t)+V^*_{k+1}(t)$$

for every transition $((k,\ s),\ (k+1,\ t))\in A_{k+1}(t),\ t\in J_{k+1}$ *with* $t>k$ *if and only if the optimal policy remains unchanged.*

Obviously, the inventory holding cost for period n can be increased arbitrarily since $I_n=0$.

5.4 Sensitivity Ranges for Backlogging Costs

Let $\triangle h^2$ be a change in the unit backlogging cost of period k such that

$$h^2_k\rightarrow h^2_k+\triangle h^2$$

Since $\triangle h^2$ affects $C_k(s,\ t)$ only if requirement r_k is backlogged, the resulting change $\triangle_k(s, t)$ is given by

$$\triangle_k(s,\ t)=\begin{cases}0 & \text{if } t\geq k\\ \triangle h^2\{R(k)-R(t)\} & \text{if } t<k.\end{cases}$$

Thus, as in the case of holding costs, the LHS of inequality (18) can be expressed as:

$$\triangle_k(s^*_k,\ s^*_{k+1})-\triangle_k(s,\ t)=\begin{cases}0 & \text{if } s^*_{k+1}\geq k \text{ and } t>k\\ \triangle h^2\{R(t)-R(k)\} & \text{if } s^*_{k+1}\geq k \text{ and } t<k\\ \triangle h^2\{R(k)-R(s^*_{k+1})\} & \text{if } s_{k+1}<k \text{ and } t\geq k\\ \triangle h^2\{R(t)-R(s^*_{k+1})\} & \text{if } s_{k+1}<k \text{ and } t<k\end{cases}$$

Hence, depending on whether $s^*_{k+1}\geq k$, *or* $s^*_{k+1}<k$, Proposition 2 concludes

Proposition 8. *If* $s^*_{k+1}\geq k$, *then* $\triangle h^2$ *for* h^2_k *satisfies* $\triangle h^2\geq -h^2_k$ *and*

$$\triangle h^2\{R(t)-R(k)\}\leq e_k(s,\ t)+V^*_{k+1}(t)$$

for every transition $((k,\ s),\ (k+1,\ t))$, $t\in J_{k+1}$ *with* $t<k$ *if and only if the optimal policy remains unchanged.*

Proposition 9. *If* $s^*_{k+1}<k$, *then* $\triangle h^2$ *for* h^2_k *satisfies* $\triangle h^2\geq -h^2_k$

$$\triangle h^2\{R(k)-R(s^*_{k+1})\}\leq e_k(s,\ t)+V^*_{k+1}(t)$$

for every transition $((k,\ s),\ (k+1,\ t))$, $t\in J_{k+1}$ *with* $t\geq k$, *and*

$$\triangle h^2\{R(t)-R(s^*_{k+1})\}\leq e_k(s,\ t)+V^*_{k+1}(t)$$

for every transition $((k,\ s),(k+1,\ t))$, $t\in J_{k+1}$ *with* $t<k$ *if and only if the optimal policy remains unchanged.*

Note that Propositions 8 and 9 are relevant only when $\alpha\geq 1$. When $\alpha=0$, no backlogging is permitted, and the backlogging cost is of no consequence.

6. Sensitivity Ranges for Market Requirements

Let $\triangle r$ be change in requirement r_k such that

$$r_k\rightarrow r_k+\triangle r.$$

We wish to find sensitivity ranges for requirements in individual periods.

$\triangle r$ for r_k affects "transition" costs whenever they involve the activity of producing, storing, or backlogging r_k. Specifically,

Case 1: When $r_k+\triangle r$ is produced and stored in period i (*i.e.*, $i<k$ and $s<k\leq t$), we have

$$\triangle_i(s,\ t)=p_i^1\{\delta(R(t)-R(s)+\triangle r)-\delta(R(t)-R(s))\}+\triangle r(p_i^2+h_{i1}).$$

Case 2: When $r_k+\triangle r$, which was produced earlier than in period i, is carried over to period $i+1$ (*i.e.*, $i<k$ and $k\leq s=t$), we have $\triangle_i(s,\ t)=\triangle r\ h_{i1}$

Case 3: When $r_k+\triangle r$, which has been backlogged until period i, is produced in period i (*i.e.*, $k\leq i\leq k+\alpha$ and $s<k\leq t$), we have $\triangle_i(s,\ t)-\triangle r\ p_i^2$.

Case 4: When $r_k+\triangle r$ is backlogged to period $i+1$ (*i.e.*, $k \leq i \leq k+\alpha$ and $s \leq t < k$), we have

$$\triangle_i(s,\ t) = p_i^1\{\delta(R(t)-R(s)+\triangle r)-\delta(R(t)-R(s))\}+\triangle r\, h^2$$

Case 5: Otherwise, we have $\triangle_i(s,\ t)=0$.

For notational convenience, we define

$$\delta_k(s,\ t,\ \triangle r) \equiv \delta(R(t)-R(s)+\triangle r)-\delta(R(t)-R(s))$$

where $r_k+\triangle r \geq 0$. Possible values of this function are 1, -1, and 0 as given below:

$$\delta_k(s,\ t,\ \triangle r) = \begin{cases} 1 \text{ if } R(t)-R(s)=0 \text{ and } \triangle r>0 \\ -1 \text{ if } R(t)-R(s)=r_k>0 \text{ and } \triangle r=-r_k \\ 0 \text{ otherwise} \end{cases}$$

by the definition of δ-function. For computational purposes, we may assume that (1) $\delta_k(s, t, \triangle r)=1$ when $R(t)-R(s)=0$ since $\triangle r=0$ is a trivial case, and (2) $\delta_k(s,\ t,\ \triangle r)=0$ when $R(t)-R(s)=r_k>0$ since we may check later on if $\triangle r=-r_k$ violates the inequalities that yield the sensitivity range for r_k

Using the transition diagram in Figure 2, we illustrate the behavior of $\triangle_i(s,\ t)$; for example, due to $\triangle r$ for r_2, $C_2(1,\ 4)$ is increased by $\triangle_2(1,\ 4)=p_2^1\delta_2(1,\ 4,\ \triangle r)+\triangle r\ (p_2^2+h_2^1)$ and $C_2(3,\ 3)$ by $\triangle_2(3,\ 3)=\triangle r\ h_2^1$.

We shall calculate how much the total penalty for a given policy is affected by $\triangle r$ *for* r_k. It is sufficient to keep track of the transition that involves the activity of producing $r_k+\triangle r$.

Let $U(k,\ i)$ be the marginal cost of supplying one unit of r_k from period i (in addition to the setup cost p_i^1) such that

$$U(k,\ i) = \begin{cases} p_i^2+h_i^1+\cdots+h_{k-1}^1 & \text{if } i<k \\ p_i^2 & \text{if } i=k \\ p_i^2+h_k^2+\cdots+h_{i-1}^2 & \text{if } k \leq i \leq k+\alpha. \end{cases}$$

If $i<k$ and $s<k \leq t$, the total pseudo-penalty for any policy that includes transition $((i,\ s),\ (i+1,\ t))$ would be no less than

$$\{e_i(s,\ t)+\triangle_i(s,\ t)\}+\{e_{i+1}(t,\ t)+\triangle_{i+1}(t,\ t)\}+\cdots+\{e_{k-1}(t,\ t)+\triangle_{k-1}(t,\ t)\}+V^*_k(t) \qquad (19)$$

since the minimum penalty from state (1, 0) to state $(i,\ s)$ is zero, and if there is positive inventory on hand we will produce nothing. The expression (19) is simplified to

$$p_i^1\delta_k(s,\ t,\ \triangle r)+\triangle r U(k,\ i)+e_i(s,\ t)+V^*_{i+1}(t) \qquad (20)$$

using the definition of $U(k, i)$ and the fact that

$$V^*_{i+1}(t) = e_{i+1}(t, t) + \cdots + e_{k-1}(t, t) + V^*_k(t).$$

Similarly, if $k \leq i \leq k+\alpha$ and $s < k \leq t$, the total pseudo-penalty for any policy that includes transition $((i, s), (i+1, t))$ would be no less than

$$\{e_k(\cdot,\cdot) + \triangle_k(\cdot,\cdot)\} + \{e_{k+1}(\cdot,\cdot) + \triangle_{k+1}(\cdot,\cdot)\} + \cdots + \{e_i(s, t) + \triangle_i(s, t)\} + V^*_{i+1}(t) \qquad (21)$$

where the dots$(\cdot)$ signify inventory levels. The expression (21) is simplified to

$$p_i^1 \delta_k(s, t, \triangle r) + \triangle r U(k, i) + e_i(s, t) + V^*_{i+1}(t) \qquad (22)$$

using the definition of $U(k, i)$ and the fact that the minimum penalty from state (1, 0) to state (i, s) is zero.

Suppose the optimal production policy supplies r_k from period j. Then $s^*_j < k \leq s^*_{j+1}$. Since the total penalty for the optimal policy is zero, its total pseudo-penalty can be given by

$$p_j^1 \delta_k(s^*_j, s^*_{j+1}, \triangle r) + \triangle r U \triangle U(k, j)$$

which must be smaller than the total pseudo-penalty for any other policy according to Theorem 1. Hence the sensitivity range for the requirement in period k can be obtained as follows:

Proposition 10. *Suppose* $s^*_j < k \leq s^*_{j+1}$ .Then $\triangle r$ *for* r_k *satisfies* $\triangle r \geq -r_k$ *and*

$$\triangle r\{U(k, j) - U(k, i)\} \leq e_i(s, t) + V^*_{i+1}(t) - p_j^1 \delta_k(s^*_j, s^*_{j+1}, \triangle r) + p_i^1 \delta_k(s, t, \triangle r)$$

for every transition $((i, s), (i+1, t)) \in A_{i+1}(t)$ *such that* $i \leq k+\alpha$, $i \neq j$, *and* $s < k \leq t$ *if and only if the optimal policy remains unchanged.*

Proof: The proof follows from expressions (20) and (22); the definitions of $\delta_k(s, t, \triangle r)$ and $U(k, i)$; finally, Theorem 1.

7. Conclusion

Based on the forward version of Zangwill's algorithm and the penalty network approach, we developed sensitivity analysis for deterministic production planning problems with infinite capacities and backlogging. For a complete implementation with PASCAL program, refer to the author's Ph.D. dissertation which is available upon request.

A next step would be to extend the sensitivity analysis to the case of 'constant capacities' for which Florian and Klein [1971] devised an $O(n^4)$ algorithm. Another interesting direction would be to extend the sensitivity analysis to the case of 'upper bounds on inventory' for which Love [1973] developed $O(n^3)$ algorithm. These two algorithms essentially arc shortest path algorithms once are costs are evaluated.

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References

1. Florian, M. and M. Klein. 1971. "Deterministic Production Planning with Concave Costs and Capacity Constraints," *Management Science,* Vol. 18, pp. 12-20.
2. Garey, M.R. and D.S. Johnson. 1979. *Computers and Intractability: A Guide to the Theory of NP-Completeness,* W.H. Freeman and Company, San Francisco.
3. Lee, In-Soo. 1986. "On Sensitivity Analysis for Shortest-path Dynamic Programming Models," Working Paper No. 337, Western Management Science Institute, UCLA.
4. Love, S.F. 1973. "Bounded Production and Inventory Models with Piecewise Concave Costs," *Management Science,* Vol. 20, pp. 313-319.
5. Zangwill, W.L. 1966. "A Deterministic Multiperiod Production Scheduling Model with Backlogging," *Management Science,* Vol. 13, pp. 105-119.
6. _______. 1967. "The Piecewise Concave Function," *Management Science,* Vol. 13, pp. 900-912.