ON WC-CONTINUOUS FUNCTIONS (*)

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1. Introduction

In 1970, Gentry and Hoyle [5] defined a function $f: X \rightarrow Y$ to be $C$-continuous if for each $x \in X$ and each open set $V$ containing $f(x)$ and having the compact complement, there exists an open set $U$ containing $x$ such that $f(U) \subseteq V$. These functions have been investigated by Long and Hendrix [6] and Long and Herrington [8]. In 1980, Long and Hamlett [7] called a function $H$-continuous by replacing "compact" in the definition of $C$-continuous functions with "$H$-closed" (quasi $H$-closed relative to $Y$ [10]). The investigation of $H$-continuous functions has been continued by the second author [9] of the present paper.

Recently, Lo Faro and the first author [1,2] have introduced and investigated a new weak form of compactness in topological spaces, called weakly compact spaces. In this paper, we introduce and characterize sets called weakly compact relative to a topological space. Then we define a new class of functions called $WC$-continuous functions analogous to $H$-continuous functions and $C$-continuous functions. It will be shown that $WC$-continuity implies $H$-continuity and they are equivalent if the range of the function is almost-regular [11].

2. Definitions

Throughout this paper $X$ and $Y$ represent topological spaces on which no separation axioms are assumed unless explicitly stated. Let $S$ be a subset of a space $X$. The closure and the interior of $S$ in $X$ are deno-

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A subset \( S \) is said to be regular open (resp. regular closed) if \( \text{Int}(\text{Cl}(S)) = S \) (resp. \( \text{Cl}(\text{Int}(S)) = S \)). For definitions and notations used in this paper, readers can find them in [2] except for the following.

**Definition 2.1.** An open cover \( \{V_\alpha | \alpha \in \mathcal{P}\} \) of a space \( X \) is said to be regular [2] if for each \( \alpha \in \mathcal{P} \) there exists a nonempty regular closed set \( F_\alpha \) in \( X \) such that \( F_\alpha \subseteq V_\alpha \) and \( X = \bigcup \{\text{Int}(F_\alpha) | \alpha \in \mathcal{P}\} \).

**Definition 2.2.** A space \( X \) is said to be weakly compact [2] (resp. quasi \( H \)-closed [10]) if every regular (resp. open) cover of \( X \) has a finite subfamily whose closure is a cover of \( X \).

In [12], Singal and Singal called quasi \( H \)-closed spaces almost compact. A quasi \( H \)-closed Hausdorff space is usually called \( H \)-closed. Urysohn-closed spaces are characterized by weakly compact Urysohn spaces [3]. It has been shown in [2] that almost compactness is strictly stronger than weak compactness.

**Definition 2.3.** A space \( X \) is said to be almost-regular [11] if for each regular closed set \( F \) of \( X \) and each point \( x \in X - F \), there exist disjoint open sets \( U \) and \( V \) such that \( F \subseteq U \) and \( x \in V \).

**Definition 2.4.** A subset \( K \) of a space \( X \) is said to be weakly compact relative to \( X \) if for each cover \( \{V_\alpha | \alpha \in \mathcal{P}\} \) of \( K \) by open sets of \( X \) satisfying the following property (P), there exists a finite subset \( \mathcal{P}_0 \) of \( \mathcal{P} \) such that \( K \subseteq \bigcup \{\text{Cl}_X(V_\alpha) | \alpha \in \mathcal{P}_0\} \).

(P) For each \( \alpha \in \mathcal{P} \), \( V_\alpha \) contains a nonempty regular closed set \( F_\alpha \) of \( X \) and \( K \subseteq \bigcup \{\text{Int}_X(F_\alpha) | \alpha \in \mathcal{P}\} \).

**Definition 2.5.** Let \( \mathcal{F} \) be a filter on a space \( X \). A point \( x \in X \) is called a \( \gamma \)-adherence point of \( \mathcal{F} \) [2] if \( \mathcal{F} \cap \mathcal{U}(\overline{\{x\}}) \neq \emptyset \).

**Definition 2.6.** Let \( A \) be a subset of a space \( X \). A point \( x \in X \) is called a \( \gamma \)-adherence point of \( A \) if \( A \cap V \neq \emptyset \) for every \( V \in \mathcal{U}(\overline{\{x\}}) \). The set of all \( \gamma \)-adherence points of \( A \) is called the \( \gamma \)-closure of \( A \). If \( A \) contains the \( \gamma \)-closure of \( A \), then it is called \( \gamma \)-closed.

### 3. Sets weakly compact relative to a space

**Definition 3.1.** A filter \( \mathcal{F} \) on a space \( X \) is said to be quasi-regular [2] if there exists an open filter \( \mathcal{Q} \) on \( X \) such that \( \mathcal{F} = \mathcal{U}(\mathcal{Q}) \).
REMARK 3.2. It is obvious that for any subset $A$ of a space $X$ $\text{tr}_A F \neq \emptyset$ if $\text{tr}_A \emptyset \neq \emptyset$, where $\text{tr}_A F$ denotes the trace of $F$ on $A$. However, the converse is not true in general as the following example shows.

EXAMPLE 3.3. Let $X = \{x, y, z, t\}$ and $\mathcal{U} = \{\emptyset, X, \{x\}, \{z\}, \{x, z\}, \{x, z, t\}\}$. Let $A = \{z, t\}$. Then the filter $F = \mathcal{U}(\emptyset) = \{\{z, x, y\}, \emptyset\}$ is quasi-regular [2, Controesempio 4]. Moreover, $\text{tr}_A F = \{\emptyset, A\} \neq \emptyset$ but $\text{tr}_A \emptyset = \emptyset$ because $\{x\} \cap \{z, t\} = \emptyset$.

THEOREM 3.4. For a subset $A$ of a space $X$, the following are equivalent:

1. $A$ is weakly compact relative to $X$.
2. Every open filter $\emptyset$ with $\text{tr}_A \emptyset \neq \emptyset$ has a $\gamma$-adherence point in $A$.
3. Every filter $\emptyset$ such that $\emptyset$ is an open filter and $\text{tr}_A \emptyset \neq \emptyset$ has an $\gamma$-adherence point in $A$.
4. Every quasi-regular filter $F = \mathcal{U}(\emptyset)$ such that $\text{tr}_A \emptyset \neq \emptyset$ has an adherence ($\delta$-adherence or $\gamma$-adherence) point in $A$.
5. Every filter $\emptyset$ such that $\emptyset$ is a quasi-regular filter $\emptyset$ with $\text{tr}_A \emptyset \neq \emptyset$ has an adherence ($\delta$-adherence point in $A$.
6. Every filter $\emptyset$ such that $\emptyset$ is a quasi-regular filter $\emptyset$ with $\text{tr}_A \emptyset \neq \emptyset$ has an adherence ($\delta$-adherence or $\gamma$-adherence) point in $A$.
7. Every open ultra filter $\emptyset$ with $\text{tr}_A \emptyset \neq \emptyset$ $\gamma$-converges.
8. Let $\{C_\alpha | \alpha \in \mathcal{\mathcal{V}}\}$ be a family of closed sets of $X$ such that for each $\alpha \in \mathcal{\mathcal{V}}$ there exists an open set $A_\alpha$ of $X$ satisfying $C_\alpha \subset A_\alpha$ and $\cap \{\text{Cl}(A_\alpha) | \alpha \in \mathcal{\mathcal{V}}\} \subset X - A$. Then there exists a finite subset $V_0$ of $\mathcal{\mathcal{V}}$ such that $\cap \{\text{Int}(C_\alpha) | \alpha \in V_0\} \subset X - A$.

Proof. (1) $\Rightarrow$ (2): Let $\emptyset$ be an open filter on $X$ with $\text{tr}_A \emptyset \neq \emptyset$. We suppose that $\emptyset \cap \mathcal{U}(\emptyset) = \emptyset$ for every $x \in A$. Then, there exist open sets $G_x \subset \emptyset$, $U_x \subset U_x$ and $A_x \subset U(\emptyset)$ such that $G_x \cap A_x = \emptyset$ and $U_x \subset \text{Cl}(U_x) \subset A_x$. By $G_x \cap A_x = \emptyset$, we obtain $\text{Cl}(G_x) \cap A_x = \emptyset$ and hence $\text{Cl}(G_x) \cap \text{Cl}(U_x) = \emptyset$. Let us put $B_x = X - \text{Cl}(G_x)$, then $\text{Cl}(U_x) \subset B_x$ and $B_x \subset U(\emptyset)$.

The family $\{B_x | x \in A\}$ is a cover of $A$ by open sets of $X$ and $A \subset \bigcup \{\text{Int}(\text{Cl}(U_x)) | x \in A\}$. Therefore, there exists a finite number of points $x_1, x_2, \ldots, x_n$ in $A$ such that $A \subset \bigcup \{\text{Cl}(B_{x_i}) | i = 1, 2, \ldots, n\}$. Therefore, we have

(*) $\cap \{X - \text{Cl}(B_{x_i}) | i = 1, 2, \ldots, n\} \subset X - A$.

For each $i = 1, 2, \ldots, n$, $G_{x_i} \subset \text{Int}(\text{Cl}(G_{x_i}))$, hence we have $X - \text{Cl}(B_{x_i}) = \text{Int}(X - B_{x_i}) = \text{Int}(\text{Cl}(G_{x_i})) \in \emptyset$. 

Therefore, by (*) we obtain $X - A \in \mathcal{G}$. This is a contradiction.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (1): These implications are proved similarly to the proof of [2, Lemma 2.1].

(4) $\Rightarrow$ (8): Let $\mathcal{F}(\mathcal{V})$ be the family of all finite subsets of $\mathcal{V}$. We suppose that

\[ \cap \{ \text{Int}(C_a) \mid a \in \mathcal{A} \} \not\subset X - A \text{ for every } \mathcal{A} \in \mathcal{F}(\mathcal{V}). \]

Then, $\mathcal{F} = \cap_{a \in \mathcal{A}} \text{Int}(C_a)$ is an open filter base with $\text{tr}_a \mathcal{F} \neq 0$. Thus, $\mathcal{U}(\mathcal{F})$ is a quasi-regular filter on $X$ such that $\text{tr}_a \mathcal{F} \neq 0$. By (4), there exists a point $x \in A$ such that $\mathcal{U}(\mathcal{F}) \cap \mathcal{U}_x \neq 0$. Put

\[ \mathcal{L} = \{ \cap_{a \in \mathcal{A}} \mathcal{A}_a \mid \mathcal{A} \in \mathcal{F}(\mathcal{V}) \}, \]

then it is an open filter base such that $\mathcal{U}(\mathcal{F}) \subset \mathcal{L}$. Therefore, $\mathcal{L} \cap \mathcal{U}_x \neq 0$ and hence $x \in \text{Cl}(A_a)$ for every $a \in \mathcal{V}$. Thus, we obtain $x \in \cap \{ \text{Cl}(A_a) \mid a \in \mathcal{V} \}$. This is a contradiction because $\cap \{ \text{Cl}(A_a) \mid a \in \mathcal{V} \} \subset X - A$.

(8) $\Rightarrow$ (1): Let $\{ A_a \mid a \in \mathcal{V} \}$ be an open cover of $A$ with Property (P). For each $a \in \mathcal{V}$, there exists a nonempty regular closed set $C_a$ such that $C_a \subset A_a$ and $A \subset \cup \{ \text{Int}(C_a) \mid a \in \mathcal{V} \}$. We consider the family $\{ X - A_a \mid a \in \mathcal{V} \}$ of closed sets. For each $a \in \mathcal{V}$, $X - C_a$ is open in $X$, $X - C_a \subset X - A_a$ and

\[ \cap \{ \text{Cl}(X - A_a) \mid a \in \mathcal{V} \} = X - \cup \{ \text{Int}(C_a) \mid a \in \mathcal{V} \} \subset X - A. \]

By (8), there exists a finite subset $\mathcal{V}_0$ of $\mathcal{V}$ such that

\[ \cap \{ \text{Int}(X - A_a) \mid a \in \mathcal{V}_0 \} \subset X - A. \]

Therefore, we obtain $A \subset \cup \{ \text{Cl}(A_a) \mid a \in \mathcal{V}_0 \}$. This shows that $A$ is weakly compact relative to $X$.

4. WC-continuous functions

DEFINITION 4.1. A function $f : X \rightarrow Y$ is said to be WC-continuous if for each $x \in X$ and each open neighborhood $V$ of $f(x)$ having the complement weakly compact relative to $Y$, there exists an open neighborhood $U$ of $x$ such that $f(U) \subset V$.

THEOREM 4.2. For a function $f : X \rightarrow Y$ the following are equivalent:

(1) $f$ is WC-continuous.

(2) If $V$ is open in $Y$ and $Y - V$ is weakly compact relative to $Y$, then $f^{-1}(V)$ is open in $X$.

(3) If $F$ is closed in $Y$ and weakly compact relative to $Y$, then $f^{-1}(F)$ is closed in $X$. 

Proof. (1) $\Rightarrow$ (2): Let $V$ be an open set of $Y$ such that $Y - V$ is weakly compact relative to $Y$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists an open neighborhood $U$ of $x$ such that $f(U) \subseteq V$. Therefore, we have $x \in U \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in $X$.

(2) $\iff$ (3): This is obvious.

(3) $\Rightarrow$ (1): Let $x \in X$ and $V$ an open neighborhood of $f(x)$ such that $Y - V$ is weakly compact relative to $Y$. By (3), $f^{-1}(Y - V)$ is closed in $X$ and hence $U = f^{-1}(V)$ is an open set containing $x$ such that $f(U) \subseteq V$.

**Lemma 4.3.** If $A_1$ and $A_2$ are weakly compact relative to a space $X$, then $A_1 \cup A_2$ is weakly compact relative to $X$.

**Proof.** Let $\mathcal{O} = \{V_\alpha | \alpha \in \mathcal{A}\}$ be a cover of $A_1 \cup A_2$ by open sets of $X$ satisfying Property (P). Then $\mathcal{O}$ is a cover of $A_1, A_2$ satisfying (P) and hence for each $i = 1, 2$ there exists a finite subset $V_i$ of $\mathcal{A}$ such that $A_i \subseteq \cup \{\text{Cl}(V_\alpha) | \alpha \in V_i\}$. Therefore, we have

$$A_1 \cup A_2 \subseteq \cup \{\text{Cl}(V_\alpha) | \alpha \in V_1 \cup V_2\}.$$  

This shows that $A_1 \cup A_2$ is weakly compact relative to $X$.

Let $(X, \tau)$ be a topological space. It follows from Lemma 4.3 that the family of open sets having the complement weakly compact relative to $(X, \tau)$ may be used as a base for a topology $\tau_{WC}$. It has been shown that the family of open sets having the compact (resp. quasi $H$-closed) complement may be used as a base to generate a topology $\tau_C$ (resp. $\tau_H$) on $X [5,7]$.

**Remark 4.4.** For a topological space $(X, \tau)$, we have $\tau_C \subseteq \tau_H \subseteq \tau_{WC} \subseteq \tau$.

**Theorem 4.5.** A function $f : X \to (Y, \sigma)$ is WC-continuous if and only if $f : (X, \sigma_{WC}) \to (Y, \sigma)$ is continuous.

**Proof.** This is obvious from the definition of $\sigma_{WC}$.

**Remark 4.6.** It is obvious that continuity implies WC-continuity and WC-continuity implies $H$-continuity. The following example shows that WC-continuity does not necessarily imply continuity.

**Example 4.7.** Let $X$ be the set of real numbers with the usual topology and $f : X \to X$ a function defined as follows: $f(x) = \frac{1}{x}$ if $x \neq 0$; $f(0) = 1/2$. Then $f$ is $C$-continuous [5, Example 2] and by Theorem
4.17 (below) \( f \) is \( WC \)-continuous. However, \( f \) is not continuous.

For a function \( f : X \to Y \), the set \( \{(x, f(x)) \mid x \in X\} \) is called the graph of \( f \) and denoted by \( G(f) \).

**Theorem 4.8.** If \( f : X \to Y \) is an open function and \( G(f) \) is \( \gamma \)-closed in the product space \( X \times Y \), then \( f \) is \( WC \)-continuous.

**Proof.** We suppose that \( f \) is not \( WC \)-continuous at some point \( x \in X \). Then there exists an open set \( V \) containing \( f(x) \) and having the complement weakly compact relative to \( Y \) such that \( f(U) \cap (Y - V) \neq \emptyset \) for every open set \( U \) containing \( x \). Since \( f \) is open,

\[ \mathcal{Q} = \{f(U) \mid x \in U \text{ and } U \text{ is open in } X\} \]

is an open filter base with \( \text{tr}_Y \cdot \mathcal{Q} \neq \emptyset \). Since \( Y - V \) is weakly compact relative to \( Y \), by (2) of Theorem 3.4 \( \mathcal{Q} \) has a \( \gamma \)-adherence point \( y \in Y - V \). Therefore, \( y \neq f(x) \) and \((x, y)\) is a \( \gamma \)-adherence point of \( G(f) \). However, we have \((x, y) \notin G(f)\). This is a contradiction.

The following three theorems are immediate consequences of Theorem 4.5 and the proofs are omitted.

**Theorem 4.9.** If \( f : X \to Y \) is \( WC \)-continuous and \( A \) is a subset of \( X \), then the restriction \( f|A : A \to Y \) is \( WC \)-continuous.

**Theorem 4.10.** If \( f : X \to Y \) continuous and \( g : Y \to Z \) is \( WC \)-continuous then the composition \( g \circ f : X \to Z \) is \( WC \)-continuous.

**Theorem 4.11.** Let \( X \) be a space and let \( \{A_\alpha \mid \alpha \in \mathcal{V}\} \) be a cover of \( X \) such that

(a) each \( \alpha \in \mathcal{V} \), \( A_\alpha \) is open in \( X \)

(b) each \( \alpha \in \mathcal{V} \), \( A_\alpha \) is closed in \( X \) and the family \( \{A_\alpha \mid \alpha \in \mathcal{V}\} \) forms a neighborhood finite family.

If \( f : X \to Y \) is a function such that \( f|A_\alpha : A_\alpha \to Y \) is \( WC \)-continuous for each \( \alpha \in \mathcal{V} \), then \( f \) is \( WC \)-continuous.

**Theorem 4.12.** If \( X \) is Urysohn and \( A \) is weakly compact relative to \( X \), then \( A \) is closed.

**Proof.** Let \( x_0 \) be a point of \( X - A \). For each \( x \in A \), there exist open sets \( U_x \) and \( V_x \) containing \( x_0 \) and \( x \), respectively, such that \( \text{Cl}(U_x) \cap \text{Cl}(V_x) = \emptyset \). For each \( x \in A \), we have

\( x \in \text{Int}(\text{Cl}(V_x)) \subseteq \text{Cl}(V_x) \subseteq X - \text{Cl}(U_x) \) and \( A \subseteq \bigcup \{\text{Int}(\text{Cl}(V_x)) \mid x \in A\} \).
Therefore, the family \( \{ X - \text{Cl}(U_x) \mid x \in A \} \) is a cover of \( A \) by open sets of \( X \) satisfying Property (P). Since \( A \) is weakly compact relative to \( X \), there exist a finite number of points \( x_1, x_2, \ldots, x_n \) in \( A \) such that

\[
A \subseteq \bigcup_{i=1}^{n} \text{Cl}(X - \text{Cl}(U_{x_i})) = X - \bigcap_{i=1}^{n} \text{Int}(\text{Cl}(U_{x_i})).
\]

Thus, we obtain \( A \cap \bigcap \{ \text{Int}(\text{Cl}(U_{x_i})) \mid i = 1, 2, \ldots, n \} = \phi \), where \( \bigcap \{ \text{Int}(\text{Cl}(U_{x_i})) \mid i = 1, 2, \ldots, n \} \) is a regular open set containing \( x_0 \). This shows that \( A \) is closed.

**Remark 4.13.** The proof of Theorem 4.12 shows that \( A \) is a \( \delta \)-closed set due to Veličko [13].

**Theorem 4.14.** Let \( Y \) be a Urysohn space. Then, a function \( f : X \rightarrow Y \) is WC-continuous if and only if \( f^{-1}(K) \) is closed in \( X \) for each set \( K \) of \( Y \) weakly compact relative to \( Y \).

**Proof.** This is an immediate consequence of Theorems 4.2 and 4.12.

A subset \( S \) of a space \( X \) is said to be \( N \)-closed relative to \( X \) [4] if every cover of \( S \) by regular open sets of \( X \) has a finite subcover.

**Theorem 4.15.** Let \( X \) be an almost-regular space and \( A \) a subset of \( X \). If \( A \) is weakly compact relative to \( X \), then it is \( N \)-closed relative to \( X \).

**Proof.** Let \( \{ V_{a} \mid a \in \mathcal{V} \} \) be a cover of \( A \) by regular open sets of \( X \). For each \( x \in A \), there exists an \( \alpha(x) \in \mathcal{V} \) such that \( x \in V_{\alpha(x)} \). Since \( X \) is almost-regular, there exist regular open sets \( G_{\alpha(x)} \) and \( W_{\alpha(x)} \) such that

\[
x \in G_{\alpha(x)} \subseteq \text{Cl}(G_{\alpha(x)}) \subseteq W_{\alpha(x)} \subseteq \text{Cl}(W_{\alpha(x)}) \subseteq V_{\alpha(x)}.
\]

The family \( \{ W_{\alpha(x)} \mid x \in A \} \) is a cover of \( A \) by open sets of \( X \) satisfying Property (P). There exists a finite subset \( A_0 \) of \( A \) such that

\[
A \subseteq \bigcup \{ \text{Cl}(W_{\alpha(x)}) \mid x \in A_0 \}.
\]

Therefore, we have \( A \subseteq \bigcup \{ V_{\alpha(x)} \mid x \in A_0 \} \). This shows that \( A \) is \( N \)-closed relative to \( X \).

**Theorem 4.16.** Let \( Y \) be an almost-regular space. Then, a function \( f : X \rightarrow Y \) is WC-continuous if and only if \( f \) is \( H \)-continuous.

**Proof.** This is an immediate consequence of Theorem 4.15 and the fact that \( N \)-closed relative to \( Y \) implies quasi \( H \)-closed relative to \( Y \).

**Theorem 4.17.** Let \( Y \) be a regular space. Then, for a function
f : X → Y the following are equivalent:
(a) WC-continuous.
(b) H-continuous.
(c) C-continuous.

Proof. Since Y is regular, Y is almost-regular and hence by Theorem 4.15 every set weakly compact relative to Y is N-closed relative to Y. Moreover, every subset of a regular space is compact if it is N-closed relative to X [4, Theorem 4.1].

Theorem 4.18. Let Y be a compact space. Then, for a function f : X → Y the following are equivalent:
(a) continuous.
(b) WC-continuous.
(c) H-continuous.
(d) C-continuous.

Proof. By Remark 4.6, it is only necessary to show that (d) implies (a). Let F be a closed set of Y. Since Y is compact, F is compact and hence f⁻¹(F) is closed in X [5, Theorem 1]. Therefore, f is continuous.

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