1. Introduction

Throughout this paper, $A$ denotes a complex unital Banach algebra. An element $h$ of $A$ is Hermitian if its numerical range is real. Let $H$ be the set of all Hermitian elements of $A$. This paper deals with the following question; If $a, b, ab \in H$, does it then follow that $ab = ba$?

Berkson [1] has proved various partial positive results, one is that, if $a, b, ab, a^2$ and $b^2$ are all Hermitian, then $ab = ba$. Murphy [4] extended Berkson’s result, that is, if $a, b$ and $ab$ are Hermitian and also either $a^2$ or $b^2$ is Hermitian, then $ab = ba$.

2. Main results

The following three lemmas contain the elementary properties of the Hermitian elements, which can be found in [2].

**Lemma 2.1.** (1) $H$ is a real linear subspace of $A$. (2) $H \cap iH = \{0\}$.

**Lemma 2.2.** If $h, k \in H$, then $i(hk - kh) \in H$.

**Lemma 2.3.** (Sinclair’s Theorem) If $h \in H$, then $r(h) = ||h||$, where $r$ denotes spectral radius.

We use the following lemma, which was proved by Kleinecke [3].

**Lemma 2.4.** Let $B$ be a Banach algebra. Let $x, y \in B$. Let $x$ commute with $xy - yx$. Then $xy - yx$ is quasinilpotent, that is, $r(xy - yx) = 0$.

Now we have the main theorem.

**Theorem 2.5.** Let $a, b, ab \in H$. Suppose also that either $i(hxb - xb^2) \in H$ and $a^2 + xb \in H$ for some $x$ in $A$ or $i(aya - ya^2) \in H$ and $b^2 + ya \in H$ for some $y$ in $A$. Then $ab = ba$.

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Proof. Suppose first that $i(bxb-xb^2) \in H$ and $a^2+xb \in H$ for some $x$ in $A$.

Apply Lemma 2.2 with $h=a$, $k=b$. So

$$i(ab-ba) \in H.$$  \hspace{1cm} (1)

Apply Lemma 2.2 with $h=-a$, $k=i(ab-ba)$. So

$$a(ab-ba)-(ab-ba)a \in H.$$  \hspace{1cm} (2)

Apply Lemma 2.2 with $h=a$, $k=ab$. So

$$i(a^2b-aba) \in H.$$  \hspace{1cm} (3)

Apply Lemma 2.2 with $h=b$, $k=a^2+xb$. So

$$i(ba^2-a^2b+xb-xb^2) \in H.$$  \hspace{1cm} (4)

Since $i(bxb-xb^2) \in H$,

$$i(ba^2-a^2b) \in H.$$  \hspace{1cm} (5)

Taking twice (3) plus (4), we conclude that

$$i(a^2b-2aba+ba^2) \in H.$$

i.e. $i(a(ab-ba)-(ab-ba)a) \in H$.  \hspace{1cm} (5)

Apply Lemma 2.1(2) to (2) and (5) to deduce that

$$a(ab-ba)=(ab-ba)a.$$

Hence, by Lemma 2.4 $ab-ba$ is quasinilpotent. So, by (1), $i(ab-ba)$ is both Hermitian and quasinilpotent. Sinclair’s Theorem then applies to $i(ab-ba)$ to prove $ab=ba$.

The same conclusion follows when $i(aya-ya^2)$ and $b^2+ya$ for some $y$ in $A$ are Hermitian by considering $A$ with its multiplication reversed.

**Corollary 2.6.** Let $a, b, ab \in H$. Suppose also that either $a^2+rb^n \in H$ for some real number $r$ and positive integer $n$ or $b^2+sa^n \in H$ for some real number $s$ and positive integer $n$. Then $ab=ba$.

*Proof.* Apply Theorem 2.5 to $x=rb^{n-1}$ and $y=sa^{n-1}$.

We obtain Murphy’s Theorem as a corollary.

**Corollary 2.7.** (Murphy’s Theorem) Let $a, b, ab \in H$. Suppose also that either $a^2 \in H$ or $b^2 \in H$. Then $ab=ba$.

*Proof.* Apply Corollary 2.6 to $r=0$ and $s=0$.

**Corollary 2.8.** Let $a, b, ab \in H$. Suppose also that either $a^2+xb \in H$, $bx=xb$ for some $x$ in $A$ or $b^2+ya \in H$, $ay=ya$ for some $y$ in $A$. Then $ab=ba$.

*Proof.* It is trivial by Theorem 2.5.
References


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