EXTENSIONS OF HIGHER ANTI-DERIVATIONS TO MODULES OF QUOTIENTS

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1. Introduction

Throughout the following, $R$ will denote an associative ring with unit element 1 and $R$-Mod will denote the category of all unitary left $R$-modules. And let $w : R \rightarrow R$ be an involution (i.e. $w$ is an endomorphism of $R$ whose square is identity map.) Then anti-derivation with respect to $w$ of $R$ is a mapping $d : R \rightarrow R$ such that $d(a + b) = d(a) + d(b)$ and $d(ab) = d(a)b + w(a)d(b)$ for all elements $a, b \in R$ ([4]). If $w$ is an identity map, then $d$ is called an ordinary derivation.

If $M$ is a unitary left $R$-module and if $d$ is a fixed anti-derivation (with respect to $w$) on $R$ then anti-$d$-derivation on $M$ is a mapping $\tilde{d} : M \rightarrow M$ satisfying the condition that $\tilde{d}(m + n) = \tilde{d}(m) + \tilde{d}(n)$ and $\tilde{d}(am) = d(a)m + w(a)d(m)$ for all elements $m, n \in M$ and $a \in R$. If $w$ is an identity map, then $\tilde{d}$ is called a $d$-derivation on $M$ ([3]).

Let $S$ be a segment of $\mathbb{N}$, i.e. $S = \{0, 1, 2, \ldots, s\}$ for some $s \geq 0$. A family $d = (d_n)_{n \in S}$ of mappings $d_n : R \rightarrow R$ is called anti-$d$-derivation of order $s$ of $R$ (where, $s = \sup S \leq \infty$) if the following properties are satisfied

(i) $d_n(a + b) = d_n(a) + d_n(b)$
(ii) $d_n(ab) = d_n(a)b + \sum_{i+j=n-1} d_i(a)d_j(b) + w(a)d_n(b)$ for all $a, b \in R$

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If $d$ is a fixed anti-$d$-derivation of order $s$ on $R$, then anti-$d$-derivation of order $s$ on $M$ is a family $\tilde{d} = (\tilde{d_n})_{n \in S}$ of mappings satisfying that

(i) $\tilde{d_n}(m + m') = \tilde{d_n}(m) + \tilde{d_n}(m')$
(ii) $\tilde{d_n}(am) = \tilde{d_n}(a)m + \sum_{i+j=n-1} d_i(a)\tilde{d_j}(m)$
$+ w(a)\tilde{d_n}(m)$ for all $a \in R$ and $m, m' \in M$
(iii) $\tilde{d_0} =$ identity map on $M$ ([3]).

**Lemma 1.** ([4, 5]) The set of ordinary derivations of $R$ corresponds bijectively to the set of derivations of order 1 of $R$. And the set of der-

Received June 20, 1986.

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ivations of order infinite corresponds bijectively to the inverse limit of the set of derivations of finite orders.

2. Preliminaries

Notations and terminology concerning (hereditary) torsion theories on $R$-$\text{Mod}$ will follow [2]. In particular, if $\tau$ is a torsion theory on $R$-$\text{Mod}$ then a left ideal $H$ of $R$ is said to be $\tau$-dense in $R$ if and only if the cyclic left $R$-module $R/H$ is $\tau$-torsion. If $M$ is a left $R$-module then we denote by $T_\tau(M)$ the unique largest submodule of $M$ which is $\tau$-torsion. If $E(M)$ is the injective hull of a left $R$-module $M$ then we define the submodule $E_\tau(M)$ of $E(M)$ by $E_\tau(M)/M = T_\tau(E(M)/M)$. The module of quotients of $M$ with respect to $\tau$, denoted by $Q_\tau(M)$, is then defined to be $E_\tau(M/T_\tau(M))$. Note that, in particular, if $M$ is $\tau$-torsionfree then $Q_\tau(M) = E_\tau(M)$, and this is a left $R$-module containing $M$ as a largest submodule. In general, we have a canonical $R$-homomorphism from $M$ to $Q_\tau(M)$ obtained by composing the canonical surjection from $M$ to $M/T_\tau(M)$ with the inclusion map into $Q_\tau(M)$.

If $R$ is the endomorphism ring of the left $R$-module $Q_\tau(RR)$ then $Q_\tau(M)$ is canonically a left $R$-module for every $R$-module and the canonical map $R \rightarrow R_\tau$ is a ring homomorphism, the ring $R_\tau$ is called as the ring of quotients or localization of $R$ at $\tau$. A torsion theory on $R$-$\text{Mod}$ is said to be faithful if and only if $R$, considered as a left module over itself, is $\tau$-torsionfree. In this case, $R$ is canonically subring of $R_\tau$.

Before entering our discussion, we assume that any anti-derivations are related with a fixed involution $w$.

**Lemma 2.** ([2]) Let $H$ be a $\tau$-dense ideal in $R$, and let $\alpha_{H,q}$ be $R$-module homomorphism defined on $H$ into $Q_\tau(M)$, then $R/H$ is $\tau$-torsion and there exist unique $R$-module homomorphism $\beta_{R,q} : R \rightarrow Q_\tau(M)$ which makes the diagram

$$
\begin{array}{ccc}
0 & \rightarrow & H \\
& \alpha_{H,q} \downarrow & \rightarrow R \\
& \nearrow \beta_{R,q} & \uparrow Q_\tau(M)
\end{array}
$$

commutes.
Let $H$ and $K$ be $\tau$–dense ideals of $R$ then we have the following results.

1. $H \cap K$ is $\tau$–dense ideal.
2. $(H : a) = \{r \in R \mid ra \in H\}$ is $\tau$–dense ideal.
3. Homomorphic image of $H$ is $\tau$–dense ideal.

Let $H$ and $K$ be $\tau$–dense ideals of $R$ and let $\alpha_{H,q} : H \rightarrow Q_\tau(M)$ and $\alpha_{K,q} : K \rightarrow Q_\tau(M)$ be defined as in the Lemma 2. Then $\alpha_{H,q}$ and $\alpha_{K,q}$ define the same element in $Q_\tau(M)$.

3. Extension theorems

In this section we consider extensions of higher anti-$d$–derivation to modules of quotients, in the case module $M$ is $\tau$–torsionfree left $R$–module, where $\tau$ is a torsion theory on $R$–mod. We begin with a Lemma.

For each $q$ in $Q_\tau(M)$, the map $\alpha_{H,q} : H \rightarrow Q_\tau(M)$ defined by $h \mapsto \bar{d}_n(w(h)q) - \sum_{i+j=n-1} d'_i(w(h))\bar{d}'_j(q) - d_n(w(h))q$ is an $R$–module homomorphism for every $h \in H$, where $d'_i$ is a derivation of order $i$ on $R$ and $\bar{d}'_j$ is derivation of order $j$ on $Q_\tau(M)$ which restricts to $M$ is $\bar{d}_j$. Moreover the map defined by $k \mapsto (k)\alpha_{K,q} = (kw(a))\alpha_{K,q}$ is an $R$–module homomorphism.

Proof. The proof is routine use the definition of higher anti-$d$–derivation and higher derivation.

Let $d$ be an anti-$d$–derivation of order $s$ on $R$ and let $\tau$ be a torsion theory on $R$–Mod and $M$ be $\tau$–torsionfree left $R$–module on which we have defined an anti-$d$–derivation $\bar{d}$ of order $s$. Then there exists an anti-$d$–derivation of order $s$, $\bar{d}$ defined on $Q_\tau(M)$, the restriction of which to $M$ is $\bar{d}$.

Proof. In the case of finite order, we use the mathematical induction on the order $s$. For $s=0$, the statement is trivial. For $s=1$, if $q \in Q_\tau(M)$, then there exists a $\tau$–dense left ideal $H$ of $R$ satisfying $Hq \subseteq M$. Define a function $\alpha_{H,q} : H \rightarrow Q_\tau(M)$ by setting $h \mapsto \bar{d}(w(h)q) - d(w(h))q$, by the Lemma 2 we see that $\alpha_{H,q}$ extends uniquely to $R$–homomorphism from $R$ to $Q_\tau(M)$ and so there exists unique element $\bar{q}$ of $Q_\tau(M)$ satisfying the condition that $\bar{d}(q) = \bar{q}$. This function is well–defined and becomes anti-$d$–derivation of order 1, moreover restricts to $M$ is $\bar{d}$.
Assume that for the case of \( s = n - 1 \), the statement is true. If \( q \) is an element of \( Q_\tau(M) \) then there exists \( \tau \)-dense left ideal \( H \) of \( R \) satisfying \( Hq \subseteq M \) and \( \omega(H)q \subseteq M \). Let \( \alpha_{H,q} \) be as in the Lemma 5, then by the Lemma 2, we see that \( \alpha_{H,q} \) extends uniquely to \( R \)-homomorphism from \( \mathcal{R} \) to \( Q_\tau(M) \) and so there exists unique element \( \tilde{q} \in Q_\tau(M) \) satisfying the condition that \( (h)\alpha_{H,q} = h\tilde{q} \) for all element \( h \in H \). We define a function \( \tilde{\alpha}_n : Q_\tau(M) \rightarrow Q_\tau(M) \) by setting \( \tilde{\alpha}_n(q) = \tilde{q} \). This function is well-defined. Indeed, suppose that \( q \) is an element of \( Q_\tau(M) \) and let \( H \) and \( K \) be \( \tau \)-dense left ideals of \( R \) satisfying \( Hq \subseteq M \) and \( Kq \subseteq M \). Then \( (H \cap K)q \subseteq M \) and \( H \cap K \) is \( \tau \)-dense left ideal of \( R \), by the Lemma 4 \( \alpha_{H,q} \) and \( \alpha_{K,q} \) define the same element \( \tilde{q} \).

Now we claim that such \( \tilde{\alpha}_n \) is anti-\( d \)-derivation of order \( n \) on \( Q_\tau(M) \). Indeed, let \( q \) and \( q' \) be elements of \( Q_\tau(M) \) and let \( a \) be an element of \( R \), then there exist \( \tau \)-dense left ideals \( H \) and \( H' \) of \( R \) satisfying \( Hq \subseteq M \) and \( H'q' \subseteq M \). Take \( K = H \cap H' \), then we have \( Kq \subseteq M \) and \( Kq' \subseteq M \), so \( K(q + q') \subseteq M \). Moreover, for each element \( k \in K \) we have

\[
(k)\alpha_{K,q+q'} = \tilde{\alpha}_n\left(\omega(k)(q + q')\right) = \sum_{i+j=n-1} \tilde{d}_i'(w(k))\tilde{d}_j'(q+q') - \sum_{i+j=n-1} \tilde{d}_i'(q+q')
\]

By the Lemma 2, the uniqueness of extension, this implies that \( \tilde{\alpha}_n(q + q') = \tilde{\alpha}_n(q) + \tilde{\alpha}_n(q') \). Similary there exists a \( \tau \)-dense left ideal \( H \) of \( R \) satisfying conditions that \( Hq \subseteq M \), \( Haq \subseteq M \), \( \omega(H)q \subseteq M \) and \( \omega(H)aq \subseteq M \), let \( K = H \cap w(H) \cap (H;a) \cap (w(H):a) \), by the Lemma 3, \( K \) is a \( \tau \)-dense left ideal of \( R \), we therefore have an \( R \)-homomorphism from \( \mathcal{R} \) to \( Q_\tau(M) \), which can be extended to from \( \mathcal{R} \) to \( Q_\tau(M) \). We see that

\[
(k)\alpha_{K,q} - (kw(a))\alpha_{K,q} = \tilde{\alpha}_n(\omega(k)aq) - \sum_{i+j=n-1} \tilde{d}_i'(w(k))\tilde{d}_j'(aq)
\]

By the Lemma 2, this implies that \( \tilde{\alpha}_n(\omega(k)aq) = \tilde{\alpha}_n(\omega(k)aq) + \sum_{i+j=n-1} \tilde{d}_i'(w(k))\tilde{d}_j'(aq) \). Thus \( \tilde{\alpha}_n \) is an anti-\( d \)-derivation of order \( n \) on \( Q_\tau(M) \).

Now we prove that \( \tilde{\alpha} \) restricts to \( \tilde{\alpha} \) on \( M \). Indeed, for every \( m \in M \), then we take \( H \) equal to \( R \) itself and so we see that for any \( a \in R \) we
have $\bar{d}_n(am) = \sum_{i+j=n-1} d_i(a)\bar{d}_{j}(m) - d_n(a)m = w(a)d_n(am)$, which implies that $\bar{d}_n(am) = \bar{d}_n(am)$ for each $n \in \mathbb{S}$.

In the case of infinite order, we use the Lemma 1 not only ring $R$, but also module $M$ and $Q_\tau(M)$, i.e. for any infinite order (anti-$d^-$) derivation $d_\infty(\bar{d}_\infty$ or $\bar{d}_\infty$) on $R(M$ or $Q_{\tau}(M))$, then there exists unique sequence (anti-$d^-$) derivations $d_n(\bar{d}_n$ or $\bar{d}_n$) on $R(M$ or $Q_{\tau}(M))$ such that we can write $d_\infty = \lim d_n(\bar{d}_n = \lim \bar{d}_n$ or $\bar{d}_\infty = \lim \bar{d}_n$). For the given $d_\infty$, there is unique sequence $\{d_n\}_{n \in \mathbb{N}}$ on $M$ which we can write $\bar{d}_\infty = \lim d_n(\bar{d}_n$ or $\bar{d}_n$), by the finite order case we can extend each $\bar{d}_n$ to $\bar{d}_n$ on $Q_{\tau}(M)$ which restricts to $\bar{d}_n$ to $M$. Now take $\bar{d}_\infty$ as an inverse limit of such $\{d_n\}_{n \in \mathbb{N}}$ on $Q_{\tau}(M)$, then $\bar{d}_\infty$ satisfies all results.

For the anti-$d$-derivations (of order 1) $d$ on a ring $R$, then there exists a unique anti-$d$-derivation $\bar{d}$ defined on $R$, the restriction of which to $R$ is $d$, in the case $\tau$ is a faithful torsion theory on $R$-Mod ([6]). Now we generalize this result to the higher order case.

**Theorem 7.** Let $d$ be an anti-$d$-derivation of order $s$ on $R$ and let $\tau$ be a faithful torsion theory on $R$-Mod. Then there exists a unique anti-$d$-derivation $\bar{d}$ of order $s$ defined on $R$, the restriction of which to $R$ is $d$.

**Proof.** The existence of $\bar{d}$ follows from the Theorem 6 and the fact that $Q_{\tau}(R)$ and $R$ are isomorphic, as left $R$-modules. To show uniqueness assume that $d'$ and $d''$ be anti-$d$-derivations of order $s$ defined on $R$ and $d' = d''$ on $R$. For any non zero element $q \in R$, there is a $\tau$-dense left ideal $H$ of $R$ satisfying conditions $Hq \subseteq R$ and $w(H)q \subseteq R$, take $K = H \cap w(H)$ as $\tau$-dense ideal of $R$, then for any element $k \in K$ we have $0 = (d'_n - d''_n)(kq) = w(k)(d'_n - d''_n)(q)$, for each $n \in \mathbb{S}$. Thus we have $w(K)(d'_n - d''_n)(q) = 0$, for each $n \in \mathbb{S}$. Since $w(K)$ is a $\tau$-dense ideal of $R$, this implies that $d'_n(q) = d''_n(q)$ for all $q \in R$.

**Corollary 8.** Let $d$ be an anti-$d$-derivation of order $s$ on $R$ and $\bar{d}$ be anti-$d$-derivation of order $s$ on a left $R$-module $M$. Suppose that $\tau$ is a torsion theory on $R$-Mod satisfying the condition, for each $n \in \mathbb{S}$, $\bar{d}_n(T_{\tau}(M)) \subseteq T_{\tau}(M)$. Then there exist an anti-$d$-derivation $\bar{d}$ of order $s$ on $Q_{\tau}(M)$ in such manner that the diagram

\[
\begin{array}{ccc}
M & \longrightarrow & Q_{\tau}(M) \\
\bar{d} \downarrow & & \downarrow \bar{d} \\
M & \longrightarrow & Q_{\tau}(M)
\end{array}
\]
commutes.

Proof. Define $d'$ on $M/T_\tau(M)$ by denoting for each $n \in S$, $d'_n : m + T_\tau(M) \rightarrow \mathcal{A}_n(m) + T_\tau(M)$, by the condition $\mathcal{A}_n(T_\tau(M)) \subseteq T_\tau(M)$, such a map is well-defined. And $M/T_\tau(M)$ is $\tau$-torsionfree left $R$-module, by the Theorem 6, this derivation $d'$ can be extended to anti-$d$-derivation $\mathcal{A}$ on $Q_\tau(M)$ making the diagram commutes.

Now we consider inner derivation of order $s$ on $R$, if there exists an element $\alpha = (a_n)_{n \in S} \in R \times R \times \cdots \times R (s+1 \text{-times})$ such that $d = \Delta(\alpha)$, where

$$d_1(x) = \Delta(\alpha)_1(x) = a_1 x - x a_1,$$

$$d_2(x) = \Delta(\alpha)_2 = a_1^2 x - a_1 x a_1 + a_2 x - x a_2,$$

$$d_3(x) = \Delta(\alpha)_3(x) = a_1^2 x - a_1^2 x a_1 + a_1 a_2 x + x a_2 a_1 - a_1 x a_2 - a_2 x a_1 + a_3 x - x a_3, \ldots$$

we call $d$ as an inner derivation of order $s$ of $R$. ([1,4])

Corollary 9. The extension of any inner derivation $d$ of order $s$ of $R$ to a derivation $\mathcal{A}$ on $R$, is again inner. In particular, if $\tau$ is torsion-free, such extension $\mathcal{A}$ is unique and which restricts to $d$ on $R$.

Proof. Let $d$ be any inner derivation of order $s$ on $R$, then there exists a sequence $\alpha = (a_n)_{n \in S}$ such that $d = \Delta(\alpha)$. Since $R$ is $\tau$-torsionfree $T_\tau(R) = 0$, so for each $n \in S$ $d_n(T_\tau(R)) = 0 \subseteq T_\tau(R)$. Take $w =$ identity map on $R$ in the Corollary 8, there exists an extension $\mathcal{A}$ on $Q_\tau(R)$, so we can define a derivation $\mathcal{A}$ on $Q_\tau(R)$ for the element $\alpha = (a_n)_{n \in S}$ as follows $\mathcal{A}(q) = \Delta(\alpha)(q)$, then $\mathcal{A}$ is an inner derivation of order $s$. On the other hand $\tau$ is faithful, by the Theorem 7, such extension is unique and which restricts $d$ on $R$.

If we take $S = \{0, 1\}$, by the Lemma 1 we have following Corollary.

Corollary 10. If $\alpha : R \rightarrow R$ is the inner derivation of $R$ defined by an element $\alpha$ and if $\tau$ is a faithful torsion theory on $R-\text{Mod}$ then $\alpha$ defines an inner derivation $\mathcal{A}_\alpha$ on $R$, which restricts to $\alpha$ on $R$. ([3]).

References

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