OPERATORS HAVING ANALYTIC SPECTRAL RESOLVENTS

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1. Introduction

Throughout this paper, $X$ is an abstract Banach space over the field of complex numbers $\mathbb{C}$, $T$ is an element of $B(X)$, $T^*$ denotes the dual operator of $T$ on the dual space $X^*$. For a set $S \subseteq X$, $S'$ is the annihilator of $S$, $\overline{S}$ for the closure of $S$ in an appropriate topology and $\partial S$ for the boundary of $S$. If $T$ is endowed with the single valued extension property (SVEP), then $\sigma(x, T)$ denotes the local spectrum for $x \in X$, and $X_T(S) = \{x \in X : \sigma(x, T) \subseteq S\}$. If $M$ is a $T$-invariant subspace, we write $T, M$ for the restriction and $T, M$ for the operator induced by $T$ on the quotient space $X/M$. We use $\sigma(T)$ for the spectrum of $T$ and $\rho(T)$ for its resolvent set, the symbol Cov[$\sigma(T)$] stands for the class of all finite open coverings of $\sigma(T)$. We write $\text{AI}(T)$, $\text{Inv}(T)$ for the Analytic invariant subspaces, invariant subspaces of $X$ for $T$ respectively. And the symbol $\text{SM}(T)$ denotes the spectral maximal subspaces of $X$ for $T$.

A decomposable operator, analytic invariant subspaces, and analytic spectral resolvent appear in this paper frequently, we begin with their definitions.

**Definition 1.1.** An operator $T \in B(X)$ said to be decomposable if, for every finite system $\{G_1, G_2, \ldots, G_n\}$ of open subsets of $\mathbb{C}$ that cover $\sigma(T)$, there exist spectral maximal subspaces $Y_1, Y_2, \ldots, Y_n$ such that

\begin{align*}
X &= \sum_{i=1}^{n} Y_i, \\
\sigma(T|Y_i) &\subseteq G_i \quad (i = 1, 2, \ldots, n).
\end{align*}

**Definition 1.2.** A $T$-invariant subspace $Y$ of $X$ is said to be analytic invariant if, for every $X$-valued analytic function defined on a region
$D \subset \mathbb{C}$ such that

$$(\lambda - T)f(\lambda) \in Y \text{ for } \lambda \in D, \text{ then } f(\lambda) \in Y \text{ for } \lambda \in D.$$  

**Definition 1.3.** $E$ is said to be an analytic spectral resolvent (ASR) for $T$ if

(i) $E : \mathcal{U} \to \text{Al}(T)$, where $\mathcal{U}$ is the usual topology of $\mathbb{C}$,

(ii) $E(\emptyset) = \{0\},$

(iii) for $\{G_i\}_{i=1}^n \subseteq \text{Cov}[\sigma(T)]$, $X = \sum_{i=1}^n E(G_i)$, and

(iv) $\sigma(T|E(G)) \subseteq \overline{G}$ for any $G \subseteq \mathcal{U}$.

In the definition 1.3, if one replace $\text{Al}(T)$ by $\text{Inv}(T)$ then $E$ is called the spectral resolvent [13].

It is shown through lengthy computations that if $T \in B(X)$ has a spectral resolvent then $T$ is a decomposable operator ([13], p.77, Theorem 11). Thus it is true that if $T$ has an ASR then $T$ is decomposable. But the later case the decomposability follows from the following results:

**Theorem 1.4.** [10]. For an operator $T$, the following are equivalent.

(i) $T$ is decomposable.

(ii) For every open set $G$ in $\mathbb{C}$, there is a $T$--invariant subspace $M$ such that $\sigma(T/M) \subseteq \overline{G}$ and $\sigma(T/M) \subseteq \mathbb{C} \setminus G$.

**Theorem 15.** [15]. Let $E : \mathcal{U} \to \text{Inv}(T)$ be a spectral resolvent for $T$, then $\sigma(T/E(G)) \subseteq \mathbb{C} \setminus G$ if and only if $E(G)$ is analytic invariant under $T$.

### 2. Invariance of an analytic spectral resolvent

In the first part of this section, we will give an answer to the following question:

If $E$ is an ASR of $T_1 \in B(X)$, when does it also be an ASR for another operator $T_2 \in B(X)$?

Before stating the result, we need a Lemma and a definition.

**Definition 2.1.** [3]. We say that $T_1$ and $T_2$ are quasi-nilpotent equivalent ($T_1 \sim T_2$) if,

$$\lim_{n \to \infty} \| (T_1 - T_2)^n \|^\frac{1}{n} = 0 = \lim_{n \to \infty} \| (T_2 - T_1)^n \|^\frac{1}{n},$$

where
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\[ (T_1 - T_2)^{[x]} = \sum_{k=0}^{n} \binom{n}{k} T_1^k T_2^{n-k}. \]

**Lemma 2.2.** [3].

(i) If \( T_1 \sim T_2 \) then \( \sigma(T_1) = \sigma(T_2) \).

(ii) If \( T_1 \) has the SVEP and if \( T_1 \sim T_2 \), then \( T_2 \) has the SVEP.

(iii) If \( T_1 \) is decomposable and \( T_1 \sim T_2 \), then \( T_2 \) is also decomposable and \( \chi_{T_1}(F) = \chi_{T_2}(F) \) for every closed \( F \subseteq \mathbb{C} \).

(iv) If \( T \) has the SVEP and if \( T_1 \sim T_2 \), then \( \sigma(x, T_1) = \sigma(x, T_2) \) for every \( x \in X \).

**Theorem 2.3.** For \( T_1, T_2 \in \mathcal{B}(X) \), let \( E \) be an ASR for \( T_1 \). If \( T_1 \sim T_2 \) and if \( E(G) \in AI(T_1) \cap AI(T_2) \) for each \( G \in \mathcal{U} \), then \( E \) is also an ASR for \( T_2 \).

**Proof.** By Lemma 2.2, (i), we have \( \sigma(T_1) = \sigma(T_2) \). For any \( \{G_i\}_{i=1}^{n} \in \text{Cov}[\sigma(T_1)] \), \( \sum_{i=1}^{n} E(G_i) = X \) and \( \sigma(T_1 | E(G)) \subseteq \bar{G} \) for any \( G \in \mathcal{U} \).

It remains to prove that \( \sigma(T_1 | E(G)) = \sigma(T_2 | E(G)) \) for any \( G \in \mathcal{U} \). For any \( x \in E(G) \in AI(T_1) \cap AI(T_2) \), we have

\[ \sigma(x, T_1) = \sigma(x, T_1 | E(G)), \quad \sigma(x, T_2) = \sigma(x, T_2 | E(G)). \]

Since \( T_1 \) is decomposable so it has the SVEP, thus \( \sigma(x, T_1) = \sigma(x, T_2) \) by Lemma 2.2, (iv). Therefore we have

\[ \bigcup_{x \in E(G)} \sigma(x, T_1) = \bigcup_{x \in E(G)} \sigma(x, T_1 | E(G)) = \sigma(T_1 | E(G)), \quad \text{and} \]
\[ \bigcup_{x \in E(G)} \sigma(x, T_2) = \sigma(T_2 | E(G)). \]

It follows that \( \sigma(T_1 | E(G)) = \sigma(T_2 | E(G)) \), \( G \in \mathcal{U} \).

The condition \( E(G) \in AI(T_1) \cap AI(T_2) \) for every \( G \in \mathcal{U} \) in Theorem 2.3 seems to be crucial, but the following example shows it is not so unreasonable.

**Example.** Let \( T \) be a spectral operator with the spectral measure \( \mu \) in the sense of N. Dunford. Thus it can be represented by \( T = S + Q \), where \( S \) and \( Q \) are the scalar and radical part respectively, and \( \sigma(T) = \sigma(S) \). Putting \( T_1 = T \), \( T_2 = S \), we have

\[ \| (T_1 - T_2)^{[x]} \|^\frac{1}{2} = ||Q||^\frac{1}{2} = || (T_2 - T_1)^{[x]} ||^\frac{1}{2}, \]
whence $T_1 \sim T_2$.

We define $E(G) = \mu(G) X$ for $G \in \mathcal{U}$, then we have $\mu(G) X = X_{T_1}(G)$, $X_{T_1}(G) = X_{T_2}(G)$ ([3], p. 40, Theorem 2.1). Moreover, $X_{T_1}(G)$ is a spectral maximal space for $T_1$ since $T_1$ is decomposable, thus $E(G) = X_{T_1}(G)$ is analytically invariant under $T_1$. i.e. $E(G) \in \text{SM}(T_1) \subset \text{AI}(T_1)$, $\text{AI}(T_1) \cap \text{AI}(T_2) \neq \emptyset$. Furthermore, $T_1$ and $T_2$ are decomposable, thus $E : \mathcal{U} \rightarrow \text{AI}(T_1) \cap \text{AI}(T_2)$ is an ASR for $T_1$ and $T_2$.

A decomposable operator is not a strongly decomposable operator, thus if $T$ is decomposable, the restriction operator $T|E(G)$ and the quotient operator $T/E(G)$ are not decomposable in general even if $E(G)$ is a spectral maximal space for $T$. We will give the conditions under which these operators are decomposable for some fixed $G \in \mathcal{U}$. To do this we need a Lemma.

**Lemma 2.4.** [14]. Let $T$ be decomposable on a reflexive Banach space $X$. If $Y$ reduces $T$, then $T|Y$ is decomposable.

For an operator $T$ with the disconnected spectrum, $G \in \mathcal{U}$ is said to be disconnect the spectrum $\sigma(T)$ if.

$G \cap \sigma(T) \neq \emptyset, \sigma(T) \not\subset G$ and $\partial G \subset \sigma(T)$.

**Proposition 2.5.** Let $X$ be a reflexive Banach space, let $E$ be an ASR for $T$. If $G \in \mathcal{U}$ disconnects the spectrum $\sigma(T)$, then both $T|E(G)$ and $T/E(G)$ are decomposable.

**Proof.** By the assumption on $G$, $X = E(G) \oplus E(G^C)$ ([5], p. 62, Lemma 12). Thus $E(G)$ reduces $T$ so by Lemma 2.4, $T|E(G)$ is decomposable. $T^*$ is decomposable since $T$ is, $X^*$ is also reflexive and

$$X^* = E(G)^* \oplus E(G^C)^* = (X/E(G^C))^* \oplus (X/E(G))^*$$

$$\equiv E(G^C)^{\perp} \oplus E(G)^{\perp}.$$

A simple computation shows that both $E(G)^{\perp}$ and $E(G^C)^{\perp}$ are invariant under $T^*$. Using again the Lemma 2.4, $T^*|E(G)^{\perp}$ is decomposable.

Now, by the identification $T^*|E(G)^{\perp} \equiv (T/E(G))^*$, $T/E(G)$ is decomposable. The last conclusion follows from the fact that $T$ is decomposable if and only if $T^*$ is decomposable. ([10], p. 95, Corollary 1).
3. Constructions of new ASR from given ASR.

In this section, we will formulate an ASR for the functional calculus, the direct sum of two ASR and a transformation of an ASR by an invertible operator. In what follows, we use the symbol $\mathcal{F}(T)$ for the set of all non-constant complex valued analytic functions on some open neighborhood of the spectrum $\sigma(T)$, and $f(T)$ for the functional calculus for $T$ if $f \in \mathcal{F}(T)$.

A $T$-invariant subspace $Y$ of $X$ is said to be a $\nu$-space if $\sigma(T|Y) \subseteq \sigma(T)$.

**Lemma 3.1.** [9]. Let $T \in B(X)$, let $g$ be analytic on some open neighborhood of $\sigma(T)$. If $Y$ is analytically invariant under $T$, then $Y$ is analytically invariant under $g(T)$.

**Lemma 3.2.** [6]. (1) Given $T \in B(X)$, let $f$ be an analytic injective function on some neighborhood of $\sigma(T)$. Then $Y$ is a $\nu$-space for $f(T)$ then $Y$ is a $\nu$-space for $T$.

(2) Given $T \in B(X)$, let $f$ be an analytic function on an open neighborhood of $\sigma(T)$. If $Y$ is $\nu$-space for $T$ then $Y$ is $\nu$-space for $f(T)$. Furthermore, we have

$$f(T)|Y = f(T|Y), \quad f(T)/Y = f(T/Y).$$

(3) Given $T \in B(X)$, let $f: D \to \mathbb{C}$ be analytic on an open neighborhood $D$ of $\sigma(T)$ and nonconstant on every component of $D$. If $Y \in AI[f(T)]$ and $Y \in Inv(T)$, then $Y \in AI(T)$.

**Theorem 3.3.** Let $E$ be an ASR for $T$, let $f \in \mathcal{F}(T)$ and if $f$ is continuous on $\mathbb{C}$ then the map $\xi$ defined by $\xi = E \circ f^{-1}$ is an ASR for $f(T)$.

**Proof.** For any $\{H_i\}_{i=1}^n \in Cov[\sigma(f(T))]$, $f(\sigma(T)) = \sigma(f(T)) \subseteq \bigcup_{i=1}^n H_i$, whence

$$\sigma(T) \subseteq f^{-1}\left(\bigcup_{i=1}^n H_i\right) = \bigcup_{i=1}^n f^{-1}(H_i)$$

We put $f^{-1}(H_i) = G_i$ $(i = 1, 2, \ldots, n)$, then $\{G_i\}_{i=1}^n \in Cov[\sigma(T)]$. Since $E$ is an ASR for $T$, we have
For each $H \in \mathcal{U}$, putting $f^{-1}(H) = G$, $\mathcal{E}(H) = E(G)$ is an analytically invariant subspace under $f(T)$ by Lemma 3.1. Thus $\mathcal{E}(H)$ is a $\nu$–space for $f(T)$. Therefore, by Lemma 3.2, (2), we have

$$\sigma(f(T)|\mathcal{E}(H)) = \sigma(f(T|\mathcal{E}(H))) = \sigma(T|\mathcal{E}(H)) = \sigma(T|E(G)).$$

And since $\sigma(T|E(G)) \subset \overline{\mathcal{G}} \cap \sigma(T)$,

$$\sigma(f(T)|\mathcal{E}(H)) \subset f(\mathcal{G}) \cap \sigma(f(T)) \subset \overline{H} \cap \sigma(f(T))$$

hold by the continuity of $f$ and the spectral mapping theorem. Therefore, $\mathcal{E}: \mathcal{U} \rightarrow \mathcal{A}[f(T)]$ is an ASR for $f(T)$.

The converse of the Theorem 3.3 is following.

**Theorem 3.4.** Let $f \in \mathcal{F}(T)$, let $f$ be an injective open function on $\mathcal{C}$. If $\mathcal{E}$ is an ASR for $f(T)$, then the map $E$ defined by $E = \mathcal{E} \circ f$ is an ASR for $T$.

**Proof.** Let $\{G_i\}_{i=1}^\infty \in \text{Cov} \{\sigma(T)\}$. Then $\{f(G_i)\}_{i=1}^\infty \in \text{Cov} \{\sigma(f(T))\}$ by the spectral mapping theorem, $\mathcal{E}(f(G_i)) \subset \mathcal{A}[f(T)]$ for each $i$. We put $f(G_i) = H_i$. Since $E(G_i) = \mathcal{E}(H_i)$, we have

$$E(\phi) = \mathcal{E}(\phi) = \{0\}, \quad \sum_i E(G_i) = \sum_i \mathcal{E}(H_i) = X.$$

For an admissible contour $C$ which surrounds $\sigma(T)$ and is contained in $D \cap \rho(T)$, where $D$ is a some open neighborhood of $\sigma(T)$. Applying Dunford's theorem on composite operator–valued function to the composite $f^{-1} \circ f$, we have

$$f^{-1}[f(T)] = \frac{1}{2\pi i} \int_C f^{-1}[f(\lambda)] R(\lambda, T) d\lambda = \frac{1}{2\pi i} \int_C R(\lambda, T) d\lambda = T.$$

By the same arguments as the proof of Theorem 3.3,

$$\sigma(T|E(G)) = \sigma(f^{-1}[f(T)]|E(G)) = \sigma(f^{-1}[f(T)]|\mathcal{E}(H))$$

$$= \sigma(f^{-1}[f(T|\mathcal{E}(H))]) = \sigma(f(T|\mathcal{E}(H)))$$

$$\subset f^{-1}(\overline{H} \cap \sigma(f(T))) = f^{-1}(\overline{H}) \cap \sigma(T).$$

And since $f$ is an injective open function on $\mathcal{C}$,

$$f^{-1}(\overline{H}) = f^{-1}(f(G)) \subset f^{-1}(f(G)) = \overline{G}.$$

Thus we have $\sigma(T|E(G)) \subset \overline{G} \cap \sigma(T)$ for $G \in \mathcal{U}$, therefore $E: \mathcal{U} \rightarrow \text{Inv}(T)$ is a spectral resolvent for $T$.

It remains to show that $E(G) = \mathcal{E}(H)$ is analytically invariant under $T$ for each $G \in \mathcal{U}$, where $H = f(G)$.
Since $E(G) = \mathcal{E}(H)$ is analytically invariant under $f(T)$,

(3. a) $\sigma(f(T)/\mathcal{E}(H)) \subset \mathcal{H} \cap \sigma(f(T)) = f(G)^C \cap f(\sigma(T)) = f(G^C \cap \sigma(T))$,

the last equality holds since $f$ is injective.

Now, by Lemma 3.2, $\mathcal{E}(H) = E(G)$ is a $\nu$-space for $f(T)$ if and only if it is a $\nu$-space for $T$. Thus

(3. b) $[f(T)/\mathcal{E}(H)] = \sigma[f(T/E(G))] = f(\sigma(T/E(G)))$.

It follows from (3. a) and (3. b) that

$$\sigma(T/E(G)) \subset G^C \cap \sigma(T) \subset \mathcal{C}\setminus G.$$ 

Therefore, by Theorem 1.5, $E(G)$ is analytically invariant under $T$ for each $G \in \mathcal{U}$. This completes the proof.

From Theorem 3.3 and Theorem 3.4, we have the following corollary.

**Corollary 3.5.** Let $f \in \mathcal{F}(T)$, if $f: \mathcal{C} \rightarrow \mathcal{C}$ is a homeomorphism then $E$ is an ASR for $T$ if and only if $E = Eof^{-1}$ is an ASR for $f(T)$.

**Theorem 3.6.** Let $X_k$ ($k=1,2$) be Banach spaces, let $T_k \in \mathcal{B}(X_k)$ ($k=1,2$). If $E_k$ is an ASR for $T_k$ for $k=1,2$. Then the map $E$ defined by $E(G) = E_1(G) \oplus E_2(G)$ ($G \in \mathcal{U}$) is an ASR for $T = T_1 \oplus T_2$.

**Proof.** For $\{G_i\}_{i=1}^n \in \text{Cov} [\sigma(T)]$, $\{G_i\}_{i=1}^n \in \text{Cov} [\sigma(T_k)] (k=1,2)$ since $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$. Thus $\sum_i E_k(G_i) = X_k (k=1,2)$ and $\sum_i E(G_i) = X$.

Clearly $E(\phi) = \{0\}$.

For any $G \in \mathcal{U}$, $E(G) \in \text{AI}(T)$ ([6], p. 20, proposition 2.18). Furthermore,

$$\sigma(T|E(G)) = \sigma(T_1 \oplus T_2|E_1(G) \oplus E_2(G)) = \sigma(T_1|E_1(G)) \cup \sigma(T_2|E_2(G)) \subset \overline{G}$$

Therefore, $E: \mathcal{U} \rightarrow \text{AI}(T)$ is an ASR for $T$.

For the converse of the Theorem 3.6, we need some basic results for analytically invariant subspaces.

**Lemma 3.7.** [9]. (1) Let $T_i \in \mathcal{B}(X_i)$ ($i=1,2$), let $Y_i$ be an $T_i$-invariant subspace ($i=1,2$). Then $Y_1 \oplus Y_2$ is analytically invariant under $T_1 \oplus T_2$ if and only if $Y_i$ is analytically invariant under $T_i$.

(2) Let $Y, Z$ be $T$-invariant subspaces with $Y \subset Z$. Then the following hold.

(i) If $Y \in \text{AI}(T)$, then $Y \in \text{AI}(T|Z)$. 

(ii) If $Y \notin \mathcal{U}$. 

(iii) $Y \in \mathcal{U}$.

(iv) $Y \notin \mathcal{U}$. 

(v) $Y \notin \mathcal{U}$. 

(vi) $Y \in \mathcal{U}$.
(ii) If $Y \in AI(T|Z)$ and $Z \in AI(T')$, then $Y \in AI(T)$.

(iii) $Z \in AI(T)$ if and only if $Z/Y \in AI(T/Y)$.

(iv) If $T$ have the SVEP and let $P$ be a bounded projection operator on $X$ commuting with $T$, then $PX \in AI(T)$.

Lemma 3.8. [6]. Let $T$ have the SVEP and let $Y \in Inv(T)$. Then $T|Y$ has the SVEP and $\sigma(y, T) \subset \sigma(y, T|Y)$ for every $y \in Y$.

Theorem 3.9. Let $T \in B(X)$, $X = X_1 \oplus X_2$ and let $P_1$ be a bounded projection operator of $X$ onto $X_1$ commuting with $T$.

If $E$ is an ASR for $T$, then the map $E_k = P_k \circ E$ $(k=1, 2)$ are ASR for $T_k$ $(k=1, 2)$, where $P_2 = I - P_1$, $T_k = T_{X_k}$ for $k=1, 2$.

Proof. We note that the condition $TP_1 = P_1 T$ is equivalent to $TX_1 \subset X_1$, $TX_2 \subset X_2$. Obviously, $E_k(G) = P_k E(G)$ is a closed subspace for $k=1, 2$.

Since $TE(G) \subset E(G)$ $(G \in \mathcal{U})$, $P_k TE(G) \subset P_k E(G) = E_k(G)$, we have $TE_k(G) \subset E_k(G)$. Hence $T_k E_k(G) = (T_{X_k}) E_k(G) = TE_k(G) \subset E_k(G)$.

Now we will show that $E_k(G) \in AI(T|E(G))$, $k=1, 2$. Putting $A = T|E(G)$, $A : E(G) \to E(G)$ have the SVEP by Lemma 3.8. $P_k T = TP_k$ implies that $P_k A = AP_k$. Therefore $P_k E(G) = E_k(G)$ $(k=1, 2)$ are considered as subspaces of $E(G)$ invariant under $A$, and $E_k(G) \in AI(A) = AI(T|E(G))$ by Lemma 3.7, (2), (ii) and (iv).

It follows that
\[
\sigma(T_1|E_1(G)) = \sigma(T_1|E_1(G)) = \sigma([T|E(G)]|E_1(G)) \subset \sigma(T|E(G)),
\]
the last inclusion relation follows from Lemma 3.7, (i) and every analytically invariant subspace is a $\nu$-space. Thus we have
\[
\sigma(T_1|E_1(G)) \subset \overline{G}, \text{ and similorly } \sigma(T_2|E_2(G)) \subset \overline{G}.
\]

Obviously $E_k(\phi) = \{0\}$, $k=1, 2$. It remains to prove that $\sum_{i=1}^{n} E_k(G_i) = X_k$ for every $\{G_i\}_{i=1}^{n} \in \text{Cov}[\sigma(T_k)]$ $(k=1, 2)$.

We choose $G \in \mathcal{U}$ such that $\sigma(T_1) \cap \overline{G} = \phi$ and satisfying $\bigcup_{i=1}^{n} G_i \cup G \supset \sigma(T)$.

By Lemma 3.7, (1), $E_1(G) \oplus E_2(G) = E(G) \in AI(T)$ if and only if $E_k(G) \in AI(T_k)$ for $k=1, 2$. Thus $\sigma(T_1|E_1(G)) \subset \overline{G} \cap \sigma(T_1) = \phi$. Therefore,
\[
\sigma(T_1) \cap \overline{G} = \phi \text{ implies that } E_1(G) = \{0\}.
\]

Since $\{G_1, G_2, \cdots, G_n, G\} \in \text{Cov}[\sigma(T)]$, $\sum_{i=1}^{n} E(G_i) + E(G) = X$. So we
have \( \sum_{i=1}^{n} E_{1}(G_{i}) = X_{1} \). Similarly \( \sum_{i=1}^{n} E_{2}(G_{i}) = X_{2} \) for any \( \{G_{i}\}_{i=1}^{n} \in \text{Cov}[\sigma(T_{2})] \). We have proved the theorem.

**Theorem 3.10.** Let \( E \) be an ASR for \( T \in B(X) \), let \( Y \) be another Banach space. If \( T \) is similar to \( S \in B(Y) \), then the map \( E = VE \) defined by \( E(G) = VE(G) \) (\( G \in \mathcal{U} \)) is an ASR for \( S \), where \( V \in B(X, Y) \) is an invertible operator such that \( VT = SV \).

**Proof.** We propose to show that \( E(G) = VE(G) \in AI(S) \). Clearly, \( VE(G) \) is a closed subspace of \( X \) for each \( G \in \mathcal{U} \), and invariant under \( S \).

Let \( f : D \to X \) be analytic and satisfy

\[
(\lambda I - S)f(\lambda) \in VE(G) \text{ on } D.
\]

Then

\[
V^{-1}(\lambda I - S)f(\lambda) \in E(G) \text{ and } (\lambda I - T) V^{-1}f(\lambda) \in E(G) \text{ on } D.
\]

And since \( V^{-1}f(\lambda) \) is analytic on \( D \) and \( E(G) \in AI(T) \), we have

\[
V^{-1}f(\lambda) \in E(G) \text{, thus } f(\lambda) \in VE(G), \lambda \in D.
\]

Now, for any \( \{G_{i}\}_{i=1}^{n} \in \text{Cov}[\sigma(T)] = \text{Cov}[\sigma(S)] \),

\[
\sum_{i=1}^{n} VE(G_{i}) = V \sum_{i=1}^{n} E(G_{i}) = VX = Y, \text{ i.e. } \sum_{i=1}^{n} E(G_{i}) = Y.
\]

Furthermore, \( S|VE(G) \) is similar to \( T|E(G) \); this follows from the facts that

\[
[V|E(G)][T|E(G)] = [S|VE(G)][V|E(G)], \text{ and } \]

\( V|E(G) \) is invertible. Therefore,

\[
\sigma(S|VE(G)) = \sigma(T|E(G)) \subset G \cap \sigma(T) \text{ i.e. } \sigma(S|E(G)) \subset G \cap \sigma(S).
\]

Hence \( E : \mathcal{U} \to AI(S) \) is an ASR for \( S \).

In definition of the ASR \( E \) for \( T \) if \( n=2 \) then \( E \) is said to be two-ASR for \( T \). Author previously proved that if \( T \) has a two-ASR \( E \), then the dual operator \( T^{*} \) has also a two-ASR \( E^{*} \), where \( E^{*} : \mathcal{U} \to AI(T^{*}) \) is defined by \( E^{*}(G) = E(C \setminus G)^{\perp} \) ([12], p. 77, Theorem 6).

**References**


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