NONEMBEDDABILITY AND NONIMMERSIBILITY
OF A PRODUCT LENS SPACE

Moo Young Sohn

1. Introduction

In [1], M. F. Atiyah introduced a method to solve non-immersion and nonembedding problems of a smooth manifold $M$ by using Adams operators on the Grothendieck ring $\tilde{KO}(M)$, and he applied his method to the case of the real projective space $RP(n)$. Later, H. Suzuki [8J estimated a lower bound of $i, j$ such that $RP(n) \times RP(m)$ can be immersed in $R^{s+m+i}$ and embedded in $R^{s+m+j}$, and M. Yasuo [9J considered the case of product lens space $L^{2n+l}(p) \times L^{2m+1}(q)$, where $p, q$ are any odd prime numbers. In this paper, making use of the method initiated by [1], [8], [9], we estimate a upper bound of the number of linearly independent tangent vector fields over a product lens space $L^{2n+1}(p) \times L^{2m+1}(q)$, where $p, q$ are any odd prime numbers, and a lower bound of $k, l$ such that $L^{2n+1}(p) \times L^{2m+1}(q)$ can be immersed in $R^{2(n+m+1)+k}$ and embedded in $R^{2(n+m+1)+l}$. In what follows, $M$ will mean a smooth closed manifold. Immersion and embedding will mean $C^\infty$-differentiable ones.

2. $r$-operator over the Grothendieck ring

Let $F$ denote either the real field $R$ or the complex field $C$, and let $\text{Vect}_F(M)$ denote the set of equivalence classes of $F$-vector bundles over $M$. The Whitney sum of vector bundles makes $\text{Vect}_F(M)$ a semi-group and the Grothendieck group $K_F(M)$ is the associated abelian group. The tensor product of vector bundles defines a commutative ring structure in $K_F(M)$. As usual, we use the notation $KO(M)$ and $K(M)$ for $K_R(M)$ and $K_C(M)$ respectively. The trivial bundle of dimension $n$ will simply be denoted by $n$. Let $x_0$ be a base point of $M$, then clearly $KO(x_0) = \mathbb{Z}$ (the ring of integers).

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We define $\overline{KO}(M) = \text{Ker}\{ i^* : KO(M) \to \mathbb{Z} \}$, where $i^*$ is the homomorphism induced by the natural inclusion $\{x_0\} \to M$, then clearly $KO(M) \approx \mathbb{Z} \oplus \overline{KO}(M)$.

For $x \in \text{Vect}_R(M)$, the vector bundle $\lambda^i(x)$ is defined by the exterior power operation $\wedge^i(x)$, $i = 0, 1, 2, 3, \ldots$.

We get the following formal properties of the operation $\lambda^i$.

1. $\lambda^0(x) = 1$
2. $\lambda^1(x) = x$
3. $\lambda^i(x+y) = \sum_{j=0}^{i} \lambda^i(x) \lambda^{i-j}(x)$
4. $\lambda^i(x) = 0$ for $i > \text{dim}x$.

We define $\lambda_t(x) = \sum_{i=0}^{\infty} \lambda^i(x) t^i$, where $t$ is an indeterminate. Let $A_t(M)$ denote the multiplicative group of formal power series in $t$ with coefficient in $\overline{KO}(M)$ and with constant term 1. Then (1) and (3) assert that $\lambda_t$ defines a homomorphism $\text{Vect}_R(M) \to A_t(M)$. Hence we get a homomorphism $\lambda_t : KO(M) \to A_t(M)$ and operators $\lambda_t : KO(M) \to KO(M)$ with $\lambda_t(x) = \sum_{i=0}^{\infty} \lambda_t(x) t^i$. The $\gamma$-operation in $KO(M)$, $\gamma_t : KO(M) \to A_t(M)$, is defined by the requirement that $\gamma_t(x) = \lambda_t(x) / 1 - t(x)$ and $\gamma_t(x) = \sum_{i=0}^{\infty} \gamma_t(x) t^i$ for $x \in KO(M)$.

Now let $\tau(M)$ denote the tangent bundle over $M$ and put $\varpi(M) = \tau(M) - n \in \overline{KO}(M)$, then the operations $\gamma^i$ give us an information of the structure of tangent bundle over $M$ as follows.

**Theorem 2.1 ([8]).** If $\gamma^i(\varpi(M)) \neq 0$ for an $i$, $0 \leq i \leq n$, then the number of linearly independent tangent vector fields over $M$ does not exceed $n-i$.

The following Atiyah criterion for an immersion and an embedding will be used for our main result.

**Theorem 2.2 ([1]).** If $M$ is immersed in the $(n+k)$-dimensional euclidean space $R^{n+k}$, then we have $\gamma^i(-\varpi(M)) = 0$ for $i \geq k$. If $M$ is embedded in $R^{n+l}$, then we have $\gamma^i(-\varpi(M)) = 0$ for $i \geq l$.

The $\gamma$-dimension and $\gamma$-codimension of an $n$-dimensional manifold $M$ are defined as follows;

$$\text{Dim}_\gamma(M) = \sup \{ i \in \mathbb{N} | \gamma^i(\tau(M) - n) \neq 0 \},$$

$$\text{Cocod}_\gamma(M) = \sup \{ i \in \mathbb{N} | \gamma^i(n - \tau(M)) \neq 0 \}.$$

Let $p, q$ be any odd prime numbers. $L^{2n+1}(p) = S^{2n+1}/Z_p$, $L^{2m+1}(q) = S^{2m+1}/Z_q$ the standard lens spaces, and let
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\[ \prod_1 : L^{2n+1}(p) \times L^{2m+1}(q) \to L^{2n+1}(p), \]
\[ \prod_2 : L^{2n+1}(p) \times L^{2m+1}(q) \to L^{2m+1}(q), \]
\[ \prod_\wedge : L^{2n+1}(p) \times L^{2m+1}(q) \to L^{2n+1}(p) \wedge L^{2m+1}(q) \]

be canonical projections.

The following is easily obtained by using cohomology properties of the Grothendieck ring, where \( L^{2n+1}(\cdot) \wedge L^{2m+1}(\cdot) \) is the smash product.

**Theorem 2.3.** (i) The induced homomorphisms
\[ \prod_1^* : KO(L^{2n+1}(p)) \to KO(L^{2n+1}(p) \times L^{2m+1}(q)), \]
\[ \prod_2^* : KO(L^{2n+1}(q)) \to KO(L^{2n+1}(p) \times L^{2m+1}(q)), \]
\[ \prod_\wedge^* : KO(L^{2n+1}(p) \wedge L^{2m+1}(q)) \to KO(L^{2n+1}(p) \times L^{2m+1}(q)) \]

are injective and we have a direct sum decomposition
\[ KO(L^{2n+1}(p) \times L^{2m+1}(q)) = \prod_1^*(KO(L^{2n+1}(p))) \]
\[ \oplus \prod_2^*(KO(L^{2m+1}(q))) \oplus \prod_\wedge^*(KO(L^{2n+1}(p) \wedge L^{2m+1}(q))). \]

(ii) If \( u \in KO(L^{2n+1}(p)) \) and \( v \in KO(L^{2m+1}(q)) \), then
\[ \prod_1^*(u) \prod_2^*(v) \in \prod_\wedge^*(KO(L^{2n+1}(p) \wedge L^{2m+1}(q))). \]

3. Applications to a product product lens space

Throughout this section, let \( p \) and \( q \) be any odd prime numbers, \( \tau \) the tangent bundle over \( L^{2n+1}(p) \times L^{2m+1}(q) \), and \( \tilde{\tau} = \tau - 2(n+m+1) \).

Let \( \xi, \eta \) be the canonical complex line bundles over \( L^{2n+1}(p) \) and \( L^{2m+1}(q) \) respectively and let \( a = \xi - 1_C, b = \eta - 1_C \), where \( 1_C \) denote the complex trivial bundle. Then we have the following relations (cf. \([3]\))
\[ \tau(L^{2n+1}(p)) - (2n+1) = (n+1) \text{re}(a), \]
\[ \tau(L^{2m+1}(q)) - (2m+1) = (m+1) \text{re}(b), \]
\[ \gamma_i(\text{re}(a)) = 1 + \text{re}(a)t - \text{re}(a)t^2, \]
\[ \gamma_i(\text{re}(b)) = 1 + \text{re}(b)t - \text{re}(b)t^2, \]

where \( \text{re} \) denote the realification of a vector bundle.

**Theorem 3.1.** \( \text{Dim}_\tau(L^{2n+1}(p) \times L^{2m+1}(q)) \)
\[ = 2 \sup \{ k \left( \binom{n+1}{k} (\prod_1^* \text{re}(a))^k + \binom{m+1}{k} (\prod_2^* \text{re}(b))^k \right) \}
\[ + \sum_{i+j+k \leq n+1 \atop i+j \leq m+1} \binom{n+1}{i} \binom{m+1}{j} (\prod_1^* \text{re}(a))^i (\prod_2^* \text{re}(b))^j, \]
where \( \binom{n}{i} \) is the binomial coefficient.

**Proof.** Let \( \tau_1, \tau_2 \) be the tangent bundles over \( L^{2n+1}(p), \ L^{2m+1}(q) \) respectively, then

\[
\tilde{r} = \prod_1^* \tau_1 + \prod_2^* \tau_2 - 2(n+m+1) \\
= \prod_1^* (\tau_1 - (2n+1)) + \prod_2^* (\tau_2 - (2m+1)) \\
= (n+1) \prod_1^* (\text{re}(a)) + (m+1) \prod_2^* (\text{re}(b)).
\]

By using the property \( \gamma_t(x+y) = \gamma_t(x)\gamma_t(y) \) and the naturality of the operator \( \gamma_t \) we have

\[
\gamma_t(\tilde{r}) = \left[ \prod_1^* \{ \gamma_t(\text{re}(a)) \} \right]^{n+1} \left[ \prod_2^* \{ \gamma_t(\text{re}(b)) \} \right]^{m+1} \\
= \left[ 1 + \prod_1^* \text{re}(a) - \prod_1^* \text{re}(a) t^2 \right]^{n+1} \left[ 1 + \prod_2^* \text{re}(b) - \prod_2^* \text{re}(b) t^2 \right]^{m+1} \\
= \sum_{0 \leq i \leq n+1} \sum_{0 \leq j \leq m+1} \left( \begin{array}{c} n+1 \noalign{\hline} i \noalign{\hline} \end{array} \right) \left( \begin{array}{c} m+1 \noalign{\hline} j \noalign{\hline} \end{array} \right) \left( \prod_1^* \text{re}(a) \right)^i \left( \prod_2^* \text{re}(b) \right)^j (t-t^2)^{i+j}.
\]

If we set

\[
A_k = \left( \begin{array}{c} n+1 \noalign{\hline} k \noalign{\hline} \end{array} \right) \left( \prod_1^* \text{re}(a) \right)^k + \left( \begin{array}{c} m+1 \noalign{\hline} k \noalign{\hline} \end{array} \right) \left( \prod_2^* \text{re}(b) \right)^k \\
+ \sum_{i+j=k} \sum_{0 \leq i \leq n+1} \sum_{0 \leq j \leq m+1} \left( \begin{array}{c} n+1 \noalign{\hline} i \noalign{\hline} \end{array} \right) \left( \begin{array}{c} m+1 \noalign{\hline} j \noalign{\hline} \end{array} \right) \left( \prod_1^* \text{re}(a) \right)^i \left( \prod_2^* \text{re}(b) \right)^j (t-t^2)^{i+j},
\]

then, by taking the coefficient of \( t^i \), we have

\[
\gamma^0(\tilde{r}) = 1, \quad \gamma^1(\tilde{r}) = A_1, \\
\gamma^2(\tilde{r}) = A_2 - A_1, \quad \gamma^3(\tilde{r}) = A_3 - 2A_2, \\
\gamma^4(\tilde{r}) = A_4 - 3A_3 + A_2, \quad \gamma^5(\tilde{r}) = A_5 - 4A_4 + 3A_3, \\
\gamma^6(\tilde{r}) = A_6 - 5A_5 + 6A_4 - A_3, \quad \gamma^7(\tilde{r}) = A_7 - 6A_6 + 10A_5 - 4A_4, \\
\gamma^8(\tilde{r}) = A_8 - 7A_7 + 15A_6 - 10A_5 + A_4,
\]

etc.

Therefore

\[
\text{Dim}_r(L^{2n+1}(p) \times L^{2m+1}(q)) = 2 \sup \{k | A_k \neq 0\}.
\]

**Corollary 3.2 ([9]).** \( \text{Dim}_r(L^{2n+1}(p)) \)

\[
= 2 \sup \{i \in \mathbb{N} | i \leq n+1, \left( \begin{array}{c} n+1 \noalign{\hline} i \noalign{\hline} \end{array} \right) \text{re}(a)^i \neq 0\}.
\]

**Theorem 3.3.** If \( \gamma^k(\tilde{r}_1) \neq 0 \) or \( \gamma^k(\tilde{r}_2) \neq 0 \) then \( \gamma^k(\tilde{r}) \neq 0 \),

where \( \tilde{r}_1 = r(L^{2n+1}(p)) - (2n+1) \in KO(L^{2n+1}(p)) \) and

\( \tilde{r}_2 = r(L^{2m+1}(q)) - (2m+1) \in KO(L^{2m+1}(q)) \).
Proof. Since \( \gamma^k(\tilde{z}) = \gamma^k(\tilde{z}_1) + \gamma^k(\tilde{z}_2) + \text{terms of the form} \)
\[
\left(\begin{array}{c} n+1 \\ i \end{array}\right) \left(\begin{array}{c} m+1 \\ j \end{array}\right) \{\prod_1^* \text{re}(a)\}^i \{\prod_2^* \text{re}(b)\}^j
\]
\[
\prod_1^* \text{re}(a) \in \prod_1^* \text{KO}(L^{2n+1}(p)), \quad \prod_2^* \text{re}(b) \in \prod_2^* \text{KO}(L^{2m+1}(q)),
\]
\[
\{\prod_1^* \text{re}(a)\}^i \{\prod_2^* \text{re}(b)\}^j \in \prod_\wedge \text{KO}(L^{2n+1}(p) \wedge L^{2m+1}(q)),
\]
this theorem comes from theorem (2.3).

Next we compute the \( \gamma \)-codimension of \( L^{2n+1}(p) \times L^{2m+1}(q) \).

**Theorem 3.4.** \( \text{Cod}_\gamma(L^{2n+1}(p) \times L^{2m+1}(q)) \)
\[
= 2 \sup \{k | \left(\begin{array}{c} n+k \\ k \end{array}\right) (\prod_1^* \text{re}(a))^k + \left(\begin{array}{c} m+k \\ k \end{array}\right) (\prod_2^* \text{re}(b))^k
\]
\[
+ \sum_{i+j=k} \left(\begin{array}{c} n+i \\ i \end{array}\right) \left(\begin{array}{c} m+j \\ j \end{array}\right) (\prod_1^* \text{re}(a))^i (\prod_2^* \text{re}(b))^j, i \neq j \neq 0 \}.
\]

Proof. From the first part of the proof of theorem (3.1), we have
\[
\gamma_t(\tilde{z}) = \{1 + \prod_1^* \text{re}(a) t - \prod_1^* \text{re}(a) t^2\} \{1 + \prod_2^* \text{re}(b) t - \prod_2^* \text{re}(b) t^2\}^{-1}
\]
\[
= \sum_{i+j=0}^{\infty} \{1 + \prod_1^* \text{re}(a) t - \prod_1^* \text{re}(a) t^2\} \{1 + \prod_2^* \text{re}(b) t - \prod_2^* \text{re}(b) t^2\}^{-1} (t-t^2)^{i+j}.
\]

If we also set
\[
B_k = \left(\begin{array}{c} n+k \\ k \end{array}\right) (\prod_1^* \text{re}(a))^k + \left(\begin{array}{c} m+k \\ k \end{array}\right) (\prod_2^* \text{re}(b))^k
\]
\[
+ \sum_{i+j=k} \left(\begin{array}{c} n+i \\ i \end{array}\right) \left(\begin{array}{c} m+j \\ j \end{array}\right) (\prod_1^* \text{re}(a))^i (\prod_2^* \text{re}(b))^j,
\]
then, by taking the coefficient of \( t^i \), we have
\[
\gamma^0(-\tilde{z}) = 1, \quad \gamma^1(-\tilde{z}) = -B_1, \quad \gamma^2(-\tilde{z}) = B_2 + B_1,
\]
\[
\gamma^3(-\tilde{z}) = -B_3 - 2B_2, \quad \gamma^4(-\tilde{z}) = B_4 + \left(\begin{array}{c} 3 \\ 1 \end{array}\right) B_3 + B_2,
\]
\[
\gamma^5(-\tilde{z}) = -B_5 - \left(\begin{array}{c} 4 \\ 1 \end{array}\right) B_4 - \left(\begin{array}{c} 3 \\ 2 \end{array}\right) B_3,
\]
\[
\gamma^6(-\tilde{z}) = B_6 + \left(\begin{array}{c} 5 \\ 1 \end{array}\right) B_5 + \left(\begin{array}{c} 4 \\ 2 \end{array}\right) B_4 - B_3,
\]
\[
\gamma^7(-\tilde{z}) = -B_7 - \left(\begin{array}{c} 6 \\ 1 \end{array}\right) B_6 - \left(\begin{array}{c} 5 \\ 2 \end{array}\right) B_5 - \left(\begin{array}{c} 4 \\ 3 \end{array}\right) B_4,
\]
\[
\gamma^8(-\tilde{z}) = -B_8 + \left(\begin{array}{c} 7 \\ 1 \end{array}\right) B_7 + \left(\begin{array}{c} 6 \\ 2 \end{array}\right) B_6 + \left(\begin{array}{c} 5 \\ 3 \end{array}\right) B_5 + B_4,
\]

etc.
Therefore
\[ \text{Cod}_r(L^{2n+1}(p) \times L^{2m+1}(q)) = 2 \sup \{ k | B_k \neq 0 \}. \]

**COROLLARY 3.5** ([9]). \[ \text{Cod}_r(L^{2n+1}(p)) = 2 \sup \{ i \in N | \binom{n+1}{i} (\text{re}(a))^i \neq 0 \}. \]

**THEOREM 3.6.** If \( \gamma^k(-\bar{z}_1) \neq 0 \) or \( \gamma^k(-\bar{z}_2) \neq 0 \) then \( \gamma^k(-\bar{z}) \neq 0. \)

**Proof.** Since \( \gamma^k(-\bar{z}) = \gamma^k(-\bar{z}_1) + \gamma^k(-\bar{z}_2) + \text{terms of the form} \)
\[ \binom{n+1}{i} \binom{m+j}{i} \{ \prod_1 \text{re}(a) \}^i \{ \prod_2 \text{re}(b) \}^j \]
\[ \prod_1 \text{re}(a) \in \prod_1 \text{KO}(L^{2n+1}(p)), \prod_2 \text{re}(b) \in \prod_2 \text{KO}(L^{2m+1}(q)), \]
\[ \{ \prod_1 \text{re}(a) \}^i \{ \prod_2 \text{re}(b) \}^j \in \prod \text{KO}(L^{2n+1}(p) \wedge L^{2m+1}(q)), \]
this theorem comes from theorem (2.3).

The order of \( \text{re}(a)^i \) in \( \text{KO}(L^{2n+1}(p)) \) was computed by Kawaguchi–Sugawara.

**THEOREM 3.7** ([5]). For \( 1 \leq i \leq \left[ \frac{n}{2} \right] \), the element \( (\text{re}(a))^i \in \text{KO}(L^{2n+1}(p)) \) is of order \( p^{1+\left[ \frac{n-2i}{p-1} \right]} \) and \( (\text{re}(a))^i \left[ \frac{n}{2} \right] = 0 \), where \( \left[ y \right] \) is the integral part of a real number \( y \).

For the next theorem, we set
\[ k(n, p) = \text{max} \left\{ k | k \leq \left[ \frac{n}{2} \right], V_p \left( \frac{n+1}{k} \right) < 1 + \left[ \frac{n-2k}{p-1} \right] \right\}, \]
where \( V_p(m) \) denote the \( p \)-adic valuation of \( m \).

Let \( \text{Span}(M) \) denote the maximal number of linearly independent tangent vector fields over \( M \).

**THEOREM 3.8.** \( \text{Span}(L^{2n+1}(p) \times L^{2m+1}(q)) \leq 2(n+m+1) - 2 \max \{ k(n, p), k(m, q) \} \).

**Proof.** Let \( k_0 = 2 \max \{ k(n, p), k(m, q) \} \). From the definition of \( k(n, p) \), we have \( 0 \leq k_0 \leq 2(n+m+1) \). By theorem (3.3), (3.7) and corollary (3.2), we obtain \( \gamma^{k_0}(\bar{z}) \neq 0. \) Applying theorem (2.1), we have \( \text{Span} (L^{2n+1}(p) \times L^{2m+1}(q)) \leq 2(n+m+1) - k_0. \)

For the next theorem, we set
\[ l(n, p) = \text{max} \left\{ k | k \leq \left[ \frac{n}{2} \right], V_p \left( \frac{n+k}{k} \right) < 1 + \left[ \frac{n-2k}{p-1} \right] \right\}. \]
As geometrical interpretation of theorem (3.6), we have non-embeddability and nonimmersibility of $L^{2n+1}(p) \times L^{2m+1}(q)$ into Euclidean spaces.

**Theorem 3.9.** (i) $L^{2n+1}(p) \times L^{2m+1}(q)$ cannot be immersible in $\mathbb{R}^{2(n+m+1)+2 \max \{l(n, p), l(m, q)\}}$.

(ii) $L^{2n+1}(p) \times L^{2m+1}(q)$ cannot be embeddable in $\mathbb{R}^{2(n+m-1)+2 \max \{l(n, p), l(m, q)\}}$.

**Proof.** Let $l_0=2\max \{l(n, p), l(m, q)\}$. Using the definition of $l(n, p)$ and theorem (3.6), (3.7) corollary (3.5), we have $r^{l_0}(-\pi) \neq 0$. Applying Atiyah Criterion theorem (2.2), we can get the desired results.

**References**


Kyungpook University
Taegu 635, Korea