ASYMPTOTIC BEHAVIOR OF SEMIGROUPS OF
ASYMPTOTICALLY NONEXPANSIVE
TYPE ON BANACH SPACES

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1. Introduction

Let $G$ be a semitopological semigroup. $G$ is called right reversible if any two closed left ideals of $G$ has non-void intersection. In this case, $(G, \succeq)$ is a directed system when the binary relation "$\succeq$" on $G$ is defined by $t \succeq s$ if and only if $[s] \cup \overline{G_s} \supseteq [t] \cup \overline{G_t}$, $s, t \in G$. Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups (see [9]). Left reversibility of $G$ is defined similarly. $G$ is called reversible if it is both left and right reversible.

In 1976, Kirk [12] introduced any non-Lipschitzian self-mapping which extends, in a sense, an asymptotically nonexpansive mapping inherited by Goebel and Kirk [4]; a continuous mapping $T : K \to K$, $K$ a nonempty closed subset of a real Banach space $X$, is said to be of asymptotically nonexpansive type if for each $x \in K$,

$$\limsup_{n \to \infty} \{ \sup\{||T^nx - T^ny|| - ||x - y|| : y \in K\} : x \in K\} \leq 0.$$ 

Now, we introduce a semigroup of non-Lipschitzian self-mappings; let $C$ be a nonempty closed convex subset of a real Banach space $X$ with norm $|| \cdot ||$. A family $\mathcal{G} = \{T_s : s \in G\}$ of continuous mappings of $C$ into $C$ is said to be a right reversible semigroup of asymptotically nonexpansive type on $C$ if the following conditions are satisfied:

(a) the index set $G$ is a right reversible semitopological semigroup with the above order $\succeq$;
(b) $T_sT_t x = T_{st} x$ for all $s, t \in G$ and $x \in C$;
(c) for each $x \in C$,

$$\limsup_{s \in G} \{ \sup\{||T_s x - T_s y|| - ||x - y|| : y \in C\} : y \in C\} \leq 0.$$
(d) $T$ is continuous with respect to the strong operator topology: $T_s x \to T_t x$ for each $x \in C$ as $s \to t$ in $G$.

Left reversible semigroup of asymptotically nonexpansive type is defined similarly. For semigroups of another non-Lipschitzian self-mappings, see [3], [10], [11] etc.

For each $x \in C$, $\bar{\varnothing}(x) = \{ T_s x : s \in G \}$ is called the orbit of $x$ under $\varnothing$ and a point $z \in C$ such that $\varnothing(z) = \{ z \}$ is called a common fixed point of $\varnothing$. We denote by $F(\varnothing)$ the set of common fixed points of $\varnothing$ and by $\omega_w(x)$ the set of weak subnet limits of the net $\{ T_s x : s \in G \}$ and set $E(x) = \{ y \in C : \lim_{s \in G} \| T_s x - y \| \text{ exists} \}$.

It is the purpose of this paper that some of the weak convergence and fixed point theory of semigroups of nonexpansive mappings ([8], [13], [14]) carries over to the larger class of mappings defined above.

2. Weak convergence

Unless otherwise specified, let $G$, $X$, $C$, $\varnothing = \{ T_s : s \in G \}$ be as before. We begin with the following

**Lemma 2.1.** For each $x \in C$, $F(\varnothing) \subseteq E(x)$.

**Proof.** Let $y \in F(\varnothing)$ and $r = \inf_{s \in G} \| T_s x - y \|$. Given $\varepsilon > 0$, there is $s_0 \in G$ such that $\| T_s x - y \| < r + \frac{\varepsilon}{2}$. Since $\varnothing$ is of asymptotically nonexpansive type, there also exists $t_0 \in G$ such that

$$\| T_t T_s x - y \| \leq \| T_s x - y \| + \frac{\varepsilon}{2}$$

for all $t \geq t_0$. Let $b \geq a_0 = t_0 s_0$. Since $G$ is right reversible, we may assume $b \in \overline{G a_0}$. Let $\{ s_a \}$ be a net in $G$ such that $s_a a_0 \to b$. Then, for each $\alpha$,

$$\| T_t a_\alpha T_s x - y \| \leq \| T_s x - y \| + \frac{\varepsilon}{2}$$

Hence $\| T_b x - y \| \leq \| T_s x - y \| + \frac{\varepsilon}{2}$. So, we have

$$\inf_{s} \sup_{T_s x - y} \sup_{T_b x - y} \| T_b x - y \| \leq \| T_s x - y \| + \frac{\varepsilon}{2} < r + \varepsilon.$$ 

Since $\varepsilon$ is arbitrary, we have $\inf_{s} \sup_{T_s x - y} \| T_s x - y \| = \inf_{s} \| T_s x - y \|$. Therefore, $\lim_{s} \| T_s x - y \|$ exists and so $y \in E(x)$. 
LEMMA 2.2. Let $X$ be uniformly convex and suppose that $F(\emptyset) \neq \emptyset$. Let $x \in C$, $f \in F(\emptyset)$ and $0 < \alpha \leq \beta < 1$. Then, for each $\varepsilon > 0$, there is $a_{\circ} \in G$ such that
\[ \| T_s(\lambda T_s x + (1-\lambda)f) - (\lambda T_s T_s x + (1-\lambda)f) \| < \varepsilon \]
for all $s, t \in G$ with $s, t \geq a_{\circ}$ and $\lambda : \alpha \leq \lambda \leq \beta$.

Proof. Let $\varepsilon > 0$, $c = \min \{ 2\lambda(1-\lambda) : \alpha \leq \lambda \leq \beta \}$, $c' = \max \{ 2\lambda(1-\lambda) : \alpha \leq \lambda \leq \beta \}$ and $r = \lim \| T_s x - f \|$. For $r = 0$, it is easy. Let $r > 0$. Then we can choose $d > 0$ so small that
\[(r+d)\left[ 1-c\delta\left( \frac{\varepsilon}{r+d} \right) \right] < r,\]
where $\delta$ is the modulus of convexity of the norm. Since $r = \inf \| T_s x - f \|$, and $\emptyset$ is of asymptotically nonexpansive type, there exists $a_{\circ} \in G$ such that
\[ \| T_s x - f \| < r + \frac{d}{2}, \quad \| f - T_s z \| < \frac{c}{4} d + \| f - z \| \]
and
\[ \| T_s T_s x - T_s z \| < \frac{c}{4} d + \| T_s x - z \| \]
for all $s \geq a_{\circ}$, $z \in C$ and each $t \in G$. Suppose that
\[ \| T_s(\lambda T_s x + (1-\lambda)f) - (\lambda T_s T_s x + (1-\lambda)f) \| \geq \varepsilon \]
for some $s, t \geq a_{\circ}$ and $\lambda : \alpha \leq \lambda \leq \beta$. Put $u = (1-\lambda)(T_s z - f)$ and $v = \lambda(T_s T_s x - T_s z)$, where $z = \lambda T_s x + (1-\lambda)f$. Then we have that $\| u \|$, $\| v \| < \lambda(1-\lambda)(r+d)$, $\| u - v \| = \| T_s x - (\lambda T_s T_s x + (1-\lambda)f) \| \geq \varepsilon$ and $\lambda u + (1-\lambda) v = \lambda(1-\lambda)(T_s T_s x - f)$. So, by using the Lemma in [7] we have
\[ \| T_s x - f \| \leq (r+d)\left[ 1-c\delta\left( \frac{\varepsilon}{r+d} \right) \right] < r. \]
This contradicts $r = \inf \| T_s x - f \|$ by Lemma 2.1.

Let $x$ and $y$ be elements of a Banach space $X$. Then we denote by $[x, y]$ the set $\{ x + (1-\lambda)y : 0 \leq \lambda \leq 1 \}$ and $\overline{co}(A)$ denotes the closure of the convex hull of $A$.

LEMMA 2.3. Let $X$ have a Fréchet differentiable norm and $\{ x_\alpha \}$ a bounded net in $C$. Let $z \in \bigcap \overline{co}\{ x_\alpha : \alpha \geq \beta \}$, $y \in C$ and $\{ y_\alpha \}$ a net of elements in $C$ with $y_\alpha \in [y, x_\alpha]$ and $\| y_\alpha - z \| = \min \{ \| u - z \| : u \in [y, x_\alpha] \}$.
If \( y_\alpha \to y \), then \( y = z \).

For the proof of Lemma 2.3, see Lemma 3 in [14]. Now we can prove the following:

**Proposition 2.4.** Let \( C \) be a closed convex subset of a uniformly convex Banach space \( X \) with a Fréchet differentiable norm. Suppose that \( F(\overline{0}) \) is nonempty. Then, for each \( x \in C \), the set \( \cap_{s \in G} \{ T_s x : s \geq s \} \cap F(\overline{0}) \) consists of at most one point.

**Proof.** For each \( x \in C \), let \( W(x) = \cap t \co \{ T_t x : t \geq s \} \). Suppose that \( f, g \in W(x) \cap F(\overline{0}) \) and \( f \neq g \). Put \( h = \frac{1}{2}(f + g) \) and \( r = \lim \| T_s x - g \| \). Since \( h \in W(x) \), \( \| h - g \| \leq r \). For each \( s \in G \), choose \( p_s \in [T_s x, h] \) such that \( \| p_s - g \| = \min \{ \| y - g \| : y \in [T_s x, h] \} \). If \( \liminf \| p_s - g \| = \| h - g \| \), then obviously \( p_s \to h \). Hence, by Lemma 2.3, \( h = g \). This contradicts \( f \neq g \). Now we suppose that \( \liminf \| p_s - g \| < \| h - g \| \). Then there exists \( x \geq 0 \) and \( s_a \in G \) such that \( s_a \geq \alpha \) and

\[
\| p_{s_a} - g \| + c < \| h - g \|
\]

for every \( \alpha \in G \). Put \( p_{s_a} = a_a T_s x + (1 - a_a) h \) for every \( \alpha \). Then there is \( \beta > 0 \) and \( \gamma < 1 \) such that \( \beta \leq a \leq \gamma \) for all \( \alpha \). By Lemma 2.2, and since \( \overline{0} \) is of asymptotically nonexpansive type, there exists \( \alpha_0 \in G \) such that

\[
\| T_s (\lambda T_s x + (1 - \lambda) h) - (\lambda T_s T_s x + (1 - \lambda) h) \| < \frac{c}{2}
\]

and

\[
\| g - T_s z \| < \frac{c}{2} + \| g - z \|
\]

for all \( s, t \geq \alpha_0 \), \( z \in C \) and \( \lambda : \beta \leq \lambda \leq \gamma \).

For \( s_\alpha \geq \alpha_0 \), let \( s \geq \beta_\alpha = \alpha_0 s_\alpha \). Then, since \( G \) is right reversible, \( s \in \{ \beta_\alpha \} \cup G_{\beta_\alpha} \), we may assume \( s \in G_{\beta_\alpha} \). Let \( \{ t_\beta \} \) be a net in \( G \) such that \( t_\beta \gamma_\alpha \to s \). Then, for each \( \beta \),

\[
\| p_{t_\beta - g} \| \leq \| a_{t_\beta} T_{t_\beta} x + (1 - a_{t_\beta}) h - g \|
\leq \| T_{t_\beta} p_{t_\beta} - (a_{t_\beta} T_{t_\beta} T_{t_\beta} x + (1 - a_{t_\beta}) h) \|
\|
+ \| g - T_{t_\beta} p_{t_\beta} \| \leq c + \| g - p_{t_\beta} \| < \| h - g \|.
\]

Hence, \( \| p_s - g \| < \| h - g \| \) for all \( s \geq \beta_\alpha \). Therefore we have \( p_s \neq h \) for all \( s \geq \beta_\alpha \). Let \( s \geq \beta_\alpha \) and \( u_k = k(h - T_s x) + T_s x \) for all \( k \geq 1 \). Then
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\[ ||u_k - g|| \geq ||h - g|| \] for all \( k \geq 1 \) and hence, by Theorem 2.5 of [2], we have

\[ \langle h - u_k, J(g - h) \rangle = \langle (1 - k) (h - T_k x), J(g - h) \rangle \geq 0 \]

for all \( k \geq 1 \), where \( J \) is the duality mapping of \( X \). Therefore, it follows that \( \langle h - y, J(g - h) \rangle \leq 0 \) for all \( y \in x_\beta \{ T_x : t \geq \beta_0 \} \). Put \( y = f = h + (h - g) \), then \( h = g \). This contradicts \( f \neq g \). The proof is completed.

**Theorem 2.5.** Let \( X, C, \) and \( F(\emptyset) \) as in Proposition 2.4. Let \( x \in C \). If \( \omega_w(x) \subseteq F(\emptyset) \), then the net \( \{ T_x : s \in G \} \) converges weakly to some \( y \in F(\emptyset) \).

**Proof.** Since \( F(\emptyset) \neq \emptyset \), by Lemma 2.1, \( \{ T_x : s \in G \} \) is bounded. So, there exists a subnet \( \{ T_{x_s} \} \) of the net \( \{ T_x : s \in G \} \) which converges weakly to some \( y \in C \). Since \( \omega_w(x) \subseteq F(\emptyset) \) and \( y \in \bigcap \{ T_x : t \geq s \} \), we have \( y \in \bigcap \{ T_x : t \geq s \} \cap F(\emptyset) \). Therefore, it follows from Proposition 2.4 that \( \{ T_x : s \in G \} \) converges weakly to \( y \in F(\emptyset) \).

Let \( K \) be a subset of a Banach space \( X \). A mapping \( T : K \to K \) is called asymptotically nonexpansive [4] if for each \( x, y \in K \),

\[ ||T^k x - T^k y|| \leq a_k ||x - y||, \quad k = 1, 2, \ldots, \]

where \( \{ a_k \} \) is a fixed sequence of real numbers such that \( \lim_{k \to \infty} a_k = 1 \). It is proved in [4] that if \( K \) is a bounded closed and convex subset of a uniformly convex space \( X \) then the set \( F(T) \) of fixed points of \( T \) is nonempty closed and convex. Taking \( G = N \) in Theorem 2.5, we have

**Corollary 2.6.** Let \( C \) be a closed convex and bounded subset of a uniformly convex Banach space \( X \) with a Fréchet differentiable norm. Let \( x \in C \). If \( T : C \to C \) is asymptotically nonexpansive mapping and \( \omega_w(x) \subseteq F(T) \), then the sequence \( \{ T^n x : n \in N \} \) converges weakly to a fixed point of \( T \), where \( \omega_w(x) \) denotes the set of subsequent limit points of \( \{ T^n x \} \).

### 3. Strong convergence and fixed point

Throughout this section, \( G \) denotes a commutative semitopological semigroup with the identity, directed by an order relation defined by \( t \geq s \) if and only if \( t = a s \) for some \( a \in G \). Let \( C \) be a closed convex subset of a uniformly convex Banach space \( X \). We suppose that the semigroup \( \emptyset = \{ T_s : s \in G \} \) is of asymptotically nonexpansive type and, for each \( x \in C \), its orbit \( \emptyset(x) = \{ T_s x : s \in G \} \) is bounded.

By slight modification of Theorem 1 in [11], we have the following:
Theorem 3.1. For each \( x \in C \), the asymptotic center \( c(x) \) of the orbit \( \mathcal{O}(x) \) with respect to \( C \) is a common fixed point of \( \mathcal{O} \).

Lemma 3.2. For each \( x \in C \), \( \lim P T_s x \) exists, where \( P \) is the metric projection of \( X \) onto \( F(\mathcal{O}) \).

Proof. Let \( r_s = \| T_s x - P T_s x \| \). With a proof as in Lemma 2.1, we have \( r = \inf \| T_s x - P T_s x \| = \limsup \| T_s x - P T_s x \| \). If \( r = 0 \), then \( \{ P T_s x : s \in G \} \) is clearly a Cauchy net. For \( r > 0 \), suppose that \( \{ P T_s x \} \) is not a Cauchy net. Then, there exists \( \varepsilon > 0 \) and \( \{ s_\alpha, t_\alpha \} \subseteq G \) such that

\[
\| P T_{s_\alpha} x - P T_{t_\alpha} x \| \geq \varepsilon
\]

for every \( \alpha \). Now choose a \( \sigma > 0 \) so small that

\[
(r+\sigma) \left[ 1 - \delta \left( \frac{\varepsilon}{r+\sigma} \right) \right] < r,
\]

where \( \delta \) is the modulus of convexity of the norm.

For the \( \sigma > 0 \), there is \( s_\alpha, t_\alpha \in G \) such that \( r_{s_\alpha}, r_{t_\alpha} < r + \frac{\sigma}{2} \). Since \( \mathcal{O} \) is of asymptotically nonexpansive type, there is \( t_0 \in G \) such that

\[
\| T_{t_0} T_{s_\alpha} x - P T_{t_0} x \| \leq \| T_{s_\alpha} x - P T_{t_\alpha} x \| + \sigma \leq r + \sigma
\]

and also

\[
\| T_{t_0} T_{t_\alpha} x - P T_{t_\alpha} x \| < r + \sigma,
\]

for all \( t \geq t_0 \). Taking \( b = t_0 s_\alpha t_0 \), by commutativity of \( G \), we have that \( \| T_b x - P T_{s_\alpha} x \|, \| T_b x - P T_{t_\alpha} x \| < r + \sigma \) and \( \| P T_{s_\alpha} x - P T_{t_\alpha} x \| \geq \varepsilon \). So, by uniform convexity of \( X \), we have

\[
\| T_b x - (P T_{s_\alpha} x + P T_{t_\alpha} x)/2 \| \leq (r + \sigma) \left[ 1 - \delta \left( \frac{\varepsilon}{r + \sigma} \right) \right].
\]

Thus,

\[
\| T_b x - P T_b x \| \leq \| T_b x - (P T_{s_\alpha} x + P T_{t_\alpha} x)/2 \|
\]

\[
\leq (r + \sigma) \left[ 1 - \delta \left( \frac{\varepsilon}{r + \sigma} \right) \right] < r.
\]

This contradicts \( r = \inf \| T_s x - P T_s x \| \). The proof is completed.

Theorem 3.3. For each \( x \in C \), \( \lim P T_s x = c(x) \), where \( c(x) \) is the asymptotic center of \( \mathcal{O}(x) \) with respect to \( C \).
Proof. By Lemma 3.2, there exists $z \in C$ such that $PT_s x \to z$. It suffices to show that $z = c(x)$. Indeed,

$$
\limsup_i \| T_i x - z \| \leq \limsup_i (\| T_i x - PT_s x \| + \| PT_s x - z \|)
$$

$$
\leq \limsup_i \| T_i x - PT_s x \|
$$

$$
\leq \limsup_i \| T_i x - c(x) \|.
$$

Hence the uniqueness of asymptotic center implies that $z = c(x)$ (see[6]).

REMARK. With a proof as in Theorem 3.3, it is clear that if $X$ satisfies Opial's condition ([15], [13; Lemma 2.1]), and $\{T_s x : s \in G\} \text{ converges weakly to a } y \in F(\emptyset)$, then the net $\{PT_i x : s \in G\}$ converges strongly to the same fixed point $y$. It is easy that if $C$ is a closed convex subset of a uniformly convex space space $X$ and if $\emptyset$ is a right reversible semigroup of asymptotically nonexpansive type on $C$, then the set $F(\emptyset)$ of common fixed points is closed convex.

Finally, employing the method of the proof due to Goebel-Kirk-Thele [5; Theorem 3.1]. For each $s \in G$ and $x \in C$, we denote by $\omega^t(x)$ the set of subnet limits of the net $\{T_s (x) : t \in G\}$.

Lemma 3.4. Let $C$ be a compact convex subset of a Banach space $X$ and let $G, \emptyset = \{T_s : s \in G\}$ be as before. Then there exists two subsets $M$ and $H$ of $C$ satisfying the following properties:

(a) $H \subseteq C$ is minimal with respect to being nonempty, closed, convex and satisfying that

$$(\ast) \text{ for each } x \in H \text{ and } s \in G, \omega^t(x) \subseteq H;$$

(b) $M \subseteq H$ is minimal with respect to being nonempty, closed and satisfying that

$$(\ast\ast) \text{ for each } x \in M \text{ and } s \in G, \omega^t(x) \subseteq M;$$

(c) $M \subseteq \bigcap_{s \in G} \{T_s (M)\}$.

Proof. Use Zorn's lemma to obtain the subset $H$ of $C$ which is minimal with respect to being nonempty, closed, convex and satisfying the property $(\ast)$. Again, we use Zorn's lemma to obtain the subset $M$ of $H$ which is minimal with respect to being nonempty, closed and satisfying the property $(\ast\ast)$. To prove (c), we note first that if $x \in M$
and \( w \in \omega^t(x) \) for some \( t \in G \), say, \( \lim_{\alpha} T_{t_{\alpha}} x = w \) for some subnet \( \{t_{\alpha}\} \) of \( G \), then \( \lim_{\alpha} T_{\mu_{\alpha}} x = T_z w \in M \) by (**) Therefore, for each \( s \in G \),
\[
H_s = M \cap T_s(M) \neq \emptyset.
\]
Obviously \( T_z(M) \) is nonempty and closed. By minimality of \( M \), to prove that \( H_s = M \), it suffices to show that for each \( x \in H_s \) and \( t \in G \), \( \omega^t(x) \subseteq H_s \). Indeed, let \( z \in \omega^t(x) \), say \( z = \lim_{\alpha} T_{t_{\alpha}} x \) for some subnet \( \{t_{\alpha}\} \) of \( G \). Then, since \( x \in M \), \( z \in M \) by (**). Also \( x \in T_z(M) \) implies that \( x = T_{t} y \) for some \( y \in M \). By continuity and commutativity of members of \( \mathcal{G} \), we get
\[
\lim_{\alpha} T_{t_{\alpha}} x = \lim_{\alpha} T_{t_{\alpha}} T_{t} y = \lim_{\alpha} T_{t_{\alpha} t} y = T_{t} z.
\]
This implies \( z = T_{t} v \) for some \( v \in M \); hence \( z \in M \cap T_z(M) = H_s \). Thus, \( T_z(M) \supseteq M \) and since \( s \in G \) is arbitrary, (c) is proved.

**Theorem 3.5.** Let \( C \) be a compact convex subset of a Banach space \( X \) and let \( G, \mathcal{G} \) as before. Then \( \mathcal{G} \) has a common fixed point in \( C \).

**Proof.** Let \( M, H \) be given as in Lemma 3.4. Then it suffices to show that \( \text{diam}(M) = 0 \). Now suppose that
\[
\delta = \text{diam}(M) > 0.
\]
Since \( \text{co}(M) = H \), by Lemma 1 in [1], there exists \( r < \delta \) such that for some \( u \in H \),
\[
\sup \{ \|u - x\| : x \in M \} \leq r.
\]
Set
\[
D = \{ x \in H : M \subseteq B(x, r) \}, \text{ where } B(x, r) = \{ u \in X : \|u - x\| \leq r \}.
\]
Since \( u \in D \), \( D \) is nonempty, closed and convex subset of \( H \). Moreover, because \( \delta > r \) and \( D \) can not contain points of \( M \) whose distance exceeds \( r \), it follows that \( D \) is a proper subset of \( H \).

Next, let \( z \in D \) and suppose \( \lim_{\alpha} T_{t_{\alpha}} z = w \) for some \( s \in G \) and some subnet \( \{t_{\alpha}\} \) of \( G \). To show that \( w \in D \), let \( y \in M \). Since \( T_{t_{\alpha}}(M) \supseteq M \) for every \( t_{\alpha} \in G \) by (c) of Lemma 3.4, there exists \( u_{\alpha} \in M \) such that \( y = T_{t_{\alpha}} u_{\alpha} \). Therefore, we have
\[
\|w - y\| \leq \|w - T_{t_{\alpha}} z\| + \|T_{t_{\alpha}} z - y\|
\]
\[
= \|w - T_{t_{\alpha}} z\| + \|T_{t_{\alpha}} z - T_{t_{\alpha}} u_{\alpha}\|
\]
\[
- \|z - u_{\alpha}\| + \|z - u_{\alpha}\|
\]
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Taking both sides by limsup, we obtain

$$\|w-y\| \leq r.$$ 

Since $w \in H$ by (*), $w \in D$. This contradicts the minimality of $H$; hence $\delta = 0$. The proof is completed.

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