A GENERALIZATION OF PRIME IDEALS IN SEMIGROUPS

HYEKYUNG KIM

In [3], Murata and his coauthors defined f-prime ideals in rings and obtained analogous results of Van der Walt [4]. In this paper, f-prime ideals in semigroups are defined and obtained results similar to those in [3]. One found that the f-radical of an ideal $A$ of a semigroup defined by the author is the intersection of all f-prime ideals containing $A$. Under the left regularity assumption, the radical of an ideal $A$ turns out to be the f-radical of $A$. Moreover, the properties of primary ideals in semigroups [1] such as the uniqueness of decomposition theorem by Laske-Noether could be extended for f-primary ideals.

1. f-prime ideals and the f-radical of an ideal

Throughout, $S$ will denote a semigroup and $F$ will denote the set of all functions $f$ from $S$ into the set of all ideals in $S$ such that, for each $s$ in $S$,

1. $s \in f(s)$,
2. $x \in f(s)$ implies $f(x) \subseteq f(s)$,
3. $x \in f(s) \cup A$ implies $f(x) \subseteq f(s) \cup A$ for each ideal $A$ of $S$.

It is clear that the function $f$ defined by $f(s) = (s)$, the principal ideal generated by $s$, is in $F$. For a fixed ideal $B$ of $S$, the function defined by $f(s) = (s) \cup B$ is also in $F$.

Definition. A subset $Q$ of $S$ is called a $p$-system iff $(a)(b) \cap Q \neq \phi$ for any $a, b$ in $Q$. $Q$ is said to be an $sp$-system iff $(a)^2 \cap Q \neq \phi$ for each $a$ in $Q$.

It is evident that every subsemigroup of $S$ is a $p$-system and every $p$-system is an $sp$-system. Let $S = \{a, b, c, d\}$ be the semigroup with the
following multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>a</td>
<td>c</td>
<td>a</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>d</td>
</tr>
</tbody>
</table>

As is easily seen, \{a, b\} is a \(p\)-system and \{b, c, d\} is an \(sp\)-system which is not a \(p\)-system.

**Definition.** For \(f \in F\), a subset \(Q\) of \(S\) is called an \(f\)-system \([sf\text{-}system]\) iff it contains a \(p\)-system \([sp\text{-}system]\) \(Q^*\) such that \(Q^* \cap f(q) = \phi\) for each \(q\) in \(Q\). In each case, \(Q^*\) will be called a kernel of \(Q\).

A proper ideal \(P\) in \(S\) is called \(f\)-prime \([f\text{-}semiprime]\) iff its complement \(P^c\) is an \(f\)-system \([sf\text{-}system]\).

It is clear that every \(f\)-prime ideal is \(f\)-semiprime.

A proper ideal \(P\) of \(S\) is completely prime iff \(xy \in P\) for some \(x, y\) in \(S\) implies \(x \in P\) or \(y \in P\). A proper ideal \(P\) of \(S\) is prime if \(XY \subseteq P\) where \(X\) and \(Y\) are ideals of \(S\) implies \(X \subseteq P\) or \(Y \subseteq P\).

In a commutative semigroup with identity, every prime ideal is completely prime. Every completely prime ideal in \(S\) is \(f\)-prime, but the converse is not true.

**Example (1)** Let \(N\) be the semigroup of positive integers with the usual product. Consider a function \(f\) from \(N\) into the set of all ideals in \(N\) which is defined by \(f(n) = 3n \cup nN\). It is clear that \(f\) is contained in \(F\). Let \(P = 4N\) and \(Q^* = 3N - 6N\). Then \(Q^* \subseteq P^c\) and for any \(q_1, q_2\) in \(Q^*\), \((q_1) \cap Q^* \neq \phi\) which proves that \(Q^*\) is a \(p\)-system. Since \(f(q) \cap Q^* \neq \phi\) for any \(q \in P^c\), the ideal \(P\) is \(f\)-prime. But \(P\) is not prime. In this case, every prime ideal is \(f\)-prime.

(2) Let \(T = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1\}\) be a triangle semigroup under \((x, y)(x', y') = (xx', xy' + y)\). Consider a function \(f\) from \(T\) into the set of all ideals in \(T\) defined by \(f((x, y)) = ((x, y)) \cup ((\frac{1}{2}, 0))\). Then \(f \in F\). Since \((1, 0)\) is a unit and \((x, y) T \subseteq T(x, y), ((x, y)) = \)
A generalization of prime ideals in semigroups 209

\[ T(x, y) \text{. Let } P = T \left( \frac{1}{4}, \frac{3}{4} \right) \text{. Take } Q^* = \{ (x, 0) \mid 0 < x \leq 1 \} \subseteq P \text{, it is clearly that } Q^* \text{ is a } p\text{-system. Since } f(a) \cap Q^* \neq \emptyset \text{ for any } a \in P, \text{ } P \text{ is } f\text{-prime but not prime. For, } (\left( \frac{1}{2}, \frac{1}{2} \right)) = T \left( \frac{1}{2}, \frac{1}{2} \right) \subseteq P \text{ and } \left( \frac{1}{3}, \frac{1}{2} \right) = T \left( \frac{1}{3}, \frac{1}{2} \right) \subseteq T T \left( \frac{1}{2}, \frac{1}{2} \right) \left( \frac{1}{3}, \frac{1}{2} \right) = T \left( \frac{1}{6}, \frac{3}{4} \right) \subseteq P. \]

**Proposition 1.1.** For any \( f\text{-prime } [f\text{-semiprime}] \) ideal \( P \) of \( S \), \( f(a_1) f(a_2) \subseteq P \) implies \( a_1 \in P \) or \( a_2 \in P \) \( [f(a)^2 \subseteq P \text{ implies } a \in P]. \)

**Proof.** Suppose \( a_i \in P^c (i=1,2) \). Since \( P^c \) is an \( f\text{-system} \), there exists a \( p\text{-system } Q^* \subseteq P^c \) such that \( f(a_i) \cap Q^* \neq \emptyset \) \( (i=1,2) \). Let \( x_1 \in f(a_1) \cap Q^* \) and \( x_2 \in f(a_2) \cap Q^* \). Then \( (x_1)(x_2) \cap Q^* \neq \emptyset \) and hence \( f(x_1)f(x_2) \cap Q^* \neq \emptyset \) which is a contradiction. The proof of the other half could be done similarly.

It is clear that the union of prime ideals in \( S \) is prime. However, the (finite) union of \( f\text{-prime ideals in } S \) need not be \( f\text{-prime. In Example (1), let } P_1=3N \text{ and } P_2=4N \cup 6N. \text{ Then } f(2)f(2) \subseteq P_1 \cup P_2 =3N \cup 4N \text{ and } 2 \notin P_1 \cup P_2. \text{ Then by Proposition 1.1, } P_1 \cup P_2 \text{ is not } f\text{-prime.} \)

Let \( A \) be any ideal of \( S \). Then the ideal \( \cup f(a) \) is denoted by \( f(A) \). Clearly \( A \subseteq f(A) \) and \( f(A) \subseteq f(B) \) if \( A \subseteq B \). Moreover, \( f(a) = f(\{a\}) \) since \( x \in (a) \subseteq f(\{a\}) \) implies \( \cup f(x) \subseteq f(a) \). In general, \( f(A) \cap A = A. \) But if \( f(a) = (a) \), then \( f(A) = A. \)

**Proposition 1.2.** Let \( P \) be an \( f\text{-prime } [f\text{-semiprime}] \) ideal of \( S \).

(i) \( f(a) f(b) \subseteq P \) implies \( a \in P \text{ or } b \in P \) \( [f(a)^2 \subseteq P \text{ implies } a \in P]. \)

(ii) \( f(A)f(B) \subseteq P \) implies \( f(A) \subseteq P \text{ or } f(B) \subseteq P, \text{ for any ideals } A, B \text{ of } S \) \( [f(A)^2 \subseteq P \text{ implies } f(A) \subseteq P]. \)

**Proof.** Obviously (ii) implies (i). Let \( a, b \in P^c \), then \( f(a) \cap P^c \neq \emptyset \) and \( f(b) \cap P^c \neq \emptyset \). Since \( f(a) = f(\{a\}), f(\{a\}) \cap P^c \neq \emptyset \) and \( f(\{b\}) \cap P^c \neq \emptyset \). Thus \( f(\{a\})f(\{b\}) \cap P^c \neq \emptyset \) implies \( f(a)f(b) \cap P^c \neq \emptyset \). The proof of the other half is similar.

**Definition.** A subset \( A \) of \( S \) is called \textit{semiprime} if for \( a \in S \), \( a^2 \in A \)
implies $a \in A$.

**Corollary 1.3.** If $f(a) = (a)$ for each $a$ in $S$, then prime and $f$-prime are synonyms. Moreover, under the same condition, semiprime and $f$-semiprime are synonyms whenever $S$ is commutative.

**Definition.** Let $A$ be an ideal of $S$. Then $r_f(A) = \{x \mid Q \cap A \neq \phi \text{ for each } f\text{-system } Q \text{ containing } x\}$, $r_{sf}(A) = \{x \mid Q \cap A \neq \phi \text{ for each } sf\text{-system } Q \text{ containing } x\}$ will be called the $f$-radical and $sf$-radical of $A$ respectively.

**Theorem 1.4.** Let $A$ be an ideal of $S$. Then $r_f(A) \cup r_{sf}(A)$ is the intersection of all $f$-prime $[f$-semiprime$]$ ideals of $S$.

**Proof.** Let $C$ be the intersection of all $f$-prime ideals containing $A$. It is clear that $r_f(A) \subseteq C$. Conversely, if $x \notin r_f(A)$, then there exists an $f$-system $Q$ such that $x \in Q$ and $Q \cap A = \phi$. Let $P$ be the union of all ideals $B$ such that $A \subseteq B$ and $B \cap Q = \phi$ and let $Q^*$ be a kernel of $Q$. Then $Q^* \subseteq P^c$. For any element $a$ in $P^c$, $A \subseteq f(a) \cup P$ and $P$ is maximal with respect to the properties $A \subseteq P$ and $P \cap Q = \phi$. Since $P \subseteq f(a) \cup P$, $(f(a) \cup P) \cap Q = \phi$. Thus $f(a) \cap Q \neq \phi$ and there exists $q$ in $Q$ such that $q \in f(a)$. By a property of $f$, $f(q) \subseteq f(a)$. Since $Q$ is an $f$-system, $f(q) \cap Q^* \neq \phi$. It follows that $f(a) \cap Q^* \neq \phi$ and $P^c$ is an $f$-system with the kernel $Q^*$. Hence $P$ is $f$-prime and $x \notin P$, i.e., $C \subseteq r_f(A)$.

For any ideal $A$ of $S$, we denote

$\overline{A} = \{x \in S \mid f(x)^n \subseteq A \text{ for some positive integer } n\}$

$A' = \{x \in S \mid x^n \subseteq A \text{ for some positive integer } n\}$.

Let $x \in \overline{A}$. Then $f(x)^n \subseteq A \cap r_f(A)$ for some $n$. Hence $x \in r_f(A)$ by Proposition 1.1. Thus $\overline{A} \subseteq r_f(A)$. Let $x \in S$ and $x^n \notin A$ for all $n$. Then $\{x, x^2, \ldots, x^n, \ldots\}$ is an $f$-system of $S$ and $\{x, x^2, \ldots\} \cap A = \phi$. Hence $x \notin r_f(A)$ and $r_f(A) \subseteq A'$. Therefore, $\overline{A} \subseteq r_f(A) \subseteq A'$.

**Theorem 1.5.** Let $A$ be an ideal of a left regular semigroup $S$. Then $r_f(A) = A'$ for any $f \in F$.

**Proof.** Suppose $x \notin r_f(A)$. It is well known that $S$ is left regular iff every left ideal of $S$ is semiprime [5]. Hence $A$ is semiprime. It follows that for each positive integer $n$, $x^n \notin A$ implies $x \notin A$. Therefore $x \notin A$ implies $x^n \notin A$ for each $n$. Hence $x \notin A'$. 
Let $Q^*$ be a $p$-system such that $Q^* \cap A = \emptyset$. Let $C$ be the collection of all $p$-systems which contain $Q^*$ and do not meet $A$. Since $Q^* \subseteq C$, $C$ is nonempty. It is clear that the union of a chain in $C$ is in $C$, and hence $C$ has a maximal element $M^*$. Let $M = \{ x \in S | f(x) \cap M^* \neq \emptyset \} \cap A^c$. Then $M$ is an $f$-system with the kernel $M^*$ and $M \cap A = \emptyset$. As is seen in the proof of Theorem 1.4, there exists an $f$-prime ideal $P$ such that $A \subseteq P$ and $P \cap M = \emptyset$. Since $P^c$ is an $f$-system with the kernel $M^*$, $P^c = M$.

**Definition.** An $f$-prime ideal $P$ is called a minimal $f$-prime ideal belonging to an ideal $A$ iff $P$ contains $A$ and there exists a kernel $Q^*$ for the $f$-system $P^c$ such that $Q^*$ is a maximal $p$-system which does not meet $A$.

It is clear that any $f$-prime ideal $P$ containing $A$ contains a minimal $f$-prime ideal belonging to $A$ and the $f$-radical of an ideal $A$ coincides with the intersection of all minimal $f$-prime ideals belonging to $A$.

In general, an arbitrary intersection of $f$-prime ideals of $S$ may not be $f$-prime. However, an arbitrary intersection of $f$-semiprime ideals of $S$ is $f$-semiprime. It follows that an arbitrary intersection of $f$-prime ideals of $S$ is $f$-semiprime, and an ideal $A$ in $S$ is $f$-semiprime iff $r_f(A) = A$.

### 2. $f$-primary ideals

**Definition.** An element $a$ is *(right)* $f$-related to an ideal $A$ of $S$ iff for each $b \in f(a)$, there exists an element $c \in A$ such that $cb \in A$. An ideal $B$ is *(right)* $f$-related to an ideal $A$ of $S$ iff every element of $B$ is $f$-related to $A$.

**Lemma 2.1.** Let $A$ be an ideal of $S$ and let $K$ be the set of all elements of $S$ which are not $f$-related to $A$. Then $K$ is an $f$-system.

**Proof.** Let $q$ be an element of $K$. Then there exists $b$ in $f(q)$ such that $cb \notin A$ for every element $c \in A$. Let $K^*$ be the set of all such $b$. Then $K^*$ is a $p$-system and $f(q) \cap K^* \neq \emptyset$. Hence $K$ is an $f$-system with the kernel $K^*$.

In Example (1), let $A = 4N$ and $f(a) = aN \cup 3N$ for any $a \in S$. Then $3 \in f(a)$ and $3(4n+i) \in A$ for $i = 1, 2, 3$. It follows that for any $c \in A$, $3c \notin A$. Hence $A$ is not $f$-related to $A$. However, each element of a proper ideal $A$ is $f$-related to $A$ if $f$ is defined to be $f(a) = (a)$ for each
a in S.

For the rest of this section, we assume that

(\alpha) Every ideal \(A\) of \(S\) is \(f\)-related to \(A\)

**Proposition 2.2.** The \(f\)-radical \(r_f(A)\) of an ideal of \(S\) is \(f\)-related to \(A\).

**Proof.** Let \(K\) be the set of all elements of \(S\) which are not \(f\)-related to \(A\). Suppose \(x \in r_f(A)\) and \(x\) is not \(f\)-related to \(A\). Then by Lemma 2.1, \(K\) is an \(f\)-system containing \(x\). It follows that \(K \cap A \neq \emptyset\), which contradicts the assumption (\(\alpha\)).

Let \(K\) be the set of all elements of \(S\) which are not \(f\)-related to \(A\). Then \(K\) is an \(f\)-system and \(K \cap A = \emptyset\) by Lemma 2.1 and the assumption (\(\alpha\)). Let \(P\) be the union of all ideals which are \(f\)-related to \(A\) and do not meet \(K\). As the proof of Theorem 1.4, \(P\) becomes \(f\)-prime. This unique maximal ideal \(P\) will be called the maximal \(f\)-prime ideal belonging to \(A\). By the assumption (\(\alpha\)), \(P\) contains \(A\). Since an element \(x\) is \(f\)-related to an ideal \(A\) iff \(f(x)\) is \(f\)-related to \(A\), every element \(f\)-related to \(A\) is contained in \(P\).

For ideals \(A\) and \(B\) of \(S\) and \(x \in S\), we adopt the notation \(A : x = \{y \in S \mid f(y)f(x) \subset A\}\) and \(A : B = \cap \{A : x \mid x \in B\}\)

**Proposition 2.3.** Let \(A\) be an ideal of \(S\) and \(b \in S\). If \(A : b \neq \emptyset\), then \(A : b\) is an ideal containing \(A\).

**Proof.** Let \(x \in A : b\) and \(s \in S\). Then \(x \in f(x)\) and \(xs \in f(x)\). It follows that \(f(xs) \subset f(x)\) and \(f(xs)f(b) \subset f(x)f(b) \subset A\). Thus \(xs \in A : b\). Similarly, \(sx \in A : b\). Let \(a \in A\) and \(x \in A : b\). Then \(xa \in A : b \cap A\), and \(f(xa)f(b) \subset A\). For any \(a' \in A\), \(f(a') \subset f(xa) \cup A\) since \(a' \in f(xa) \cup A\). Then \(f(a')f(a) \subset (f(xa) \cup A)f(b) = f(xa)f(b) \cup A f(b) \subset A\), and hence \(a' \in A : b\).

Let \(P\) be the maximal \(f\)-prime ideal belonging to an ideal \(A\) of \(S\) and let

\[
A_p = \begin{cases} 
\bigcup_{s \in P} (A : s) & \text{if } P \neq S \\
A & \text{if } P = S.
\end{cases}
\]

If \(f(a) = (a)\), for any \(a\) of \(S\), then \(A_p \neq \emptyset\) since \(A \subset A : s\) for any
A generalization of prime ideals in semigroups

$s
\not\in A$. In Example (1), let $A=4N$ and $P=2N$. Then for any $s\in S$, $9N \subset f(x)f(s)$. It follows that $A : s = \{x \in S | f(x)f(s) \subset 4N\} = \phi$, and hence $A_p = \phi$ whenever $P \neq S$.

For the rest of this section, we will also assume that

(β) For any ideals $A$ and $B$ with $B \subseteq r_f(A)$, $A : B \neq \phi$.

**Proposition 2.4.** Let $P$ be the maximal $f$-prime ideal belonging to an ideal $A$ of $S$. Then $A = A_p$.

**Proof.** By the assumption (β), $A_p \neq \phi$. For any element $x$ in $A_p$, there exists $s \in P^c$ such that $f(x)f(s) \subset A$. Since $s$ is not $f$-related to $A$, there exists $s' \in f(s)$ such that $cs' \in A$ implies $c \in A$. Then $xs' \in A$, and hence $x \in A$. Therefore $A = A_p$.

**Definition.** Let $K$ be an $f$-system in $S$. A kernel $K^*$ of $K$ is said to be dense in $K$ iff $K^* \cap A \neq \phi$ for any ideal $A$ in $S$ with $K \cap A \neq \phi$.

If $f(a) = (a)$ for any $a$ in $S$, then every kernel $K^*$ of an $f$-system $K$ is dense in $K$. However, in Example (1), since $P = 4N$ is $f$-prime, $P^c$ is an $f$-system with the kernel $K^* = 3N - 6N$. Then $K^* \cap 6N = \phi$ while $P^c \cap 6N \neq \phi$, and hence $K^*$ is not dense in $P^c$.

**Definition.** An ideal $A$ of $S$ is (right) $f$-primary iff $f(a)f(b) \subset A$ implies $a \in A$ or $b \in r_f(A)$.

Every $f$-prime ideal must be $f$-primary by Proposition 1.1.

**Proposition 2.5.** Let $A$ and $B$ be ideals of $S$. Then

1. $A \subseteq B$ implies $r_f(A) \subseteq r_f(B)$
2. $r_f(r_f(A)) = r_f(A)$
3. $r_f(AB) = r_f(A \cap B) = r_f(A) \cap r_f(B)$ if every $f$-system in $S$ has a dense kernel.

**Proof.** Clearly (1) and (2) hold. Now $r_f(AB) \subseteq r_f(A \cap B) \subseteq r_f(A) \cap r_f(B)$ by (1). Let $x \in r_f(A) \cap r_f(B)$ and let $K$ be any $f$-system containing $x$. Then $K \cap A \neq \phi$ and $K \cap B \neq \phi$. Since $K$ has the dense kernel $K^*$, $K^* \cap A \neq \phi$ and $K^* \cap B \neq \phi$. Let $a \in K^* \cap A$, $b \in K^* \cap B$. Then $(a)(b) \cap K^* \neq \phi$. Since $(a)(b) \subset AB$, $AB \cap K^* \neq \phi$ and hence $AB \cap K \neq \phi$, which means $x \in r_f(AB)$.

**Corollary 2.6.** Assume that every $f$-system in $S$ has a dense kernel.
Let $Q$ and $T$ be $f$-primary ideals such that $rf(Q) = rf(T)$. Then $Q \cap T$ is an $f$-primary ideal and $rf(Q \cap T) = rf(Q) = rf(T)$.

**Proposition 2.7.** An ideal $A$ is $f$-primary iff $A : B = A$ for every ideal $B \subset rf(A)$.

**Proof.** Suppose $A$ is $f$-primary and $B$ is an ideal such that $B \subset rf(A)$. By the assumption (β), $A : B \neq \emptyset$ implies $A \subset A : B$. Let $b \in B$ and $b \notin rf(A)$. For each element $x \in A : B$, $x \in A$ since $A$ is $f$-primary. Hence $A : B \subset A$ implies $A : B = A$. Conversely, suppose $f(a)f(b) \subset A$ and $b \notin rf(A)$. Then $f(b) \notin rf(A)$. Hence $A : f(b) = A$ implies $f(a)f(b') \subset f(a)f(b) \subset A$, for every $b' \in f(b)$. Therefore $a \in \cap \{A : b'|b' \in f(a)\} = A : f(b) = A$.

**Definition.** If an ideal $A$ can be written as $A = A_1 \cap A_2 \cap \ldots \cap A_n$, where $A_i$ is an $f$-primary ideal for each $i$, it is called an $f$-primary decomposition of $A$. Every $A_i$ is called an $f$-primary component of $A$.

A decomposition is called irredundant iff $\bigcap_i A_i \subset A_i$ for each $i$.

An irredundant $f$-primary decomposition is said to be reduced iff $rf(A_i) \neq rrf(A_j)$ for $i \neq j$.

If an ideal $A$ of $S$ has an $f$-primary decomposition and if every $f$-system in $S$ has a dense kernel, then $A$ has a reduced $f$-primary decomposition by Corollary 2.6.

In the rest of this section, we assume the following:

(γ) $A : A = S$ for any $f$-primary ideal $A$.

In Example (1), let $A = 4N$. Since $9N \subset f(x)f(a) \subset A$ for $a \in A$ and $x \in S$, $A : A = \emptyset$. Thus the assumption (γ) is essential. However, (γ) holds if $f(a) = (a)$ for every $a$ in $S$.

**Theorem 2.8.** Let $A = A_1 \cap A_2 \cap \ldots \cap A_n = A'_1 \cap A'_2 \cap \ldots \cap A'_m$ be two reduced $f$-primary decompositions of $A$. Then $n = m$ and it is possible to renumber the $f$-primary components in such a way that $rf(A_i) = rf(A'_i)$ for $1 \leq i \leq n = m$.

**Proof.** Using Proposition 2.5, Proposition 2.7 and Corollary 2.6, the proof follows as in Theorem 3.7 of [3].

**3. $f$-primary semigroups**

**Proposition 3.1.** Let $A$ be an ideal of a semigroup $S$ with identity $1$. 

If \( r_f(A) = S - H(1) \), then \( A \) is \( f \)-primary. Where \( H(1) \) is the maximal subgroup containing 1.

Proof. Let \( f(x)f(y) \subseteq A \) and \( x \not\in A \). Suppose \( y \not\in r_f(A) \). Then \( f(y) \not\subseteq r_f(A) = S - H(1) \), and hence \( f(y) = S \). Then \( f(x)f(y) = f(x)S = f(x) \subseteq A \) and \( x \in A \) which is a contradiction. Thus \( A \) is \( f \)-primary.

**Proposition 3.2.** Let \( S \) be a semigroup with identity 1 and let every \( f \)-system in \( S \) has a dense kernel. Then for any \( n \in \mathbb{N} \), \( M^n \) is \( f \)-primary, where \( M = S - H(1) \).

Proof. By Proposition 2.5 (3), \( r_f(M^n) = r_f(M) \cap \ldots \cap r_f(M) = M \cap \ldots \cap M \). Hence \( M^n \) is \( f \)-primary by Proposition 3.1.

**Definition.** A semigroup \( S \) is called \( f \)-primary iff every ideal of \( S \) is \( f \)-primary.

**Theorem 3.3.** Let \( S \) be a semigroup with identity 1. If \( S \) has no \( f \)-prime ideal except \( S - H(1) \), then \( S \) is an \( f \)-primary semigroup. The converse is not true as in shown in [2].

Proof. Let \( A \) be a proper (nonzero) ideal. Then \( r_f(A) = S - H(1) \). By Proposition 3.1, \( A \) is \( f \)-primary.

**Theorem 3.4.** Let \( S \) be a left regular semigroup. If the set of all \( f \)-prime ideals of \( S \) is linearly ordered, then \( S \) is \( f \)-primary.

Proof. Let \( A \) be an ideal of \( S \) and let \( f(x)f(y) \subseteq A \). If \( x \not\in A \), \( x^n \not\in A \) for each positive integer \( n \) by the left regularity of \( S \). Then \( x \not\in r_f(A) \) by Theorem 1.5. Since \( f \)-prime ideals are linearly ordered, \( r_f(A) \) is \( f \)-prime. Now, since \( f(x)f(y) \subseteq r_f(A) \), \( y \in r_f(A) \) by Proposition 1.1.

**References**


Kyungpook National University
Daegu 635, Korea