SASAKIAN SUBMANIFOLDS OF CODIMENSION 2 IN A 
SASAKIAN MANIFOLD WITH HARMONIC CURVATURE

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0. Introduction

Let $N$ be a Sasakian manifold with the structure tensor $(F, G, \nu)$. If $M$ is a submanifold immersed in $N$, $M$ is said to be an invariant submanifold provided that the tangent space to $M$ at each point of the submanifold is invariant under the action $F$. It is well known that an invariant submanifold tangent to the structure vector field $\nu$ of a Sasakian manifold is also a Sasakian manifold. The study of invariant $C$-Einstein submanifolds of codimension 2 in a Sasakian manifold, which is called a problem of Nomizu-Smyth, were made by Endo [3], Kon [8], Pak and Oh [10], Yano and Ishihara [13] and so on. One of which, done by Yano and Ishihara [13], asserts that any invariant Einstein submanifold of codimension 2 immersed in a Sasakian manifold of constant curvature is totally geodesic. However, Pak and Oh [10] obtained the same result replacing the Sasakian manifold of constant curvature by that with vanishing $C$-Bochner curvature tensor.

The main purpose of the present paper is to investigate invariant submanifolds of codimension 2 immersed in an Einstein Sasakian manifold.

1. Invariant submanifolds of codimension 2 in a Sasakian manifold

Let $N$ be a $(2n+1)$-dimensional Sasakian manifold with Sasakian structure $(F, G, \nu)$ covered by a system of coordinate neighborhoods $\{U : x^A\}$, where here and in the sequel the indices $A, B, C...$ run over the range $\{1, 2, ..., 2n+1\}$. We denote by $F_B^A$, $G_B^A$ and $\nu^A$ components of the $(1,1)$-tensor $F$, of the Riemannian metric tensor $G$ and of the
structure vector field $v$ respectively. We then have

$$
\begin{align*}
F^B_C F^A_B &= -\delta^A_C + v^C A, & F^A_C v^E = 0, & v^A F^A_C = 0, \\
G_{BA} v^A &= 1, & F^B_D F^C_A G_{BA} = G_{DC} - v^D v_C,
\end{align*}
$$

where $v_C = G_{CE} v^E$. Denoting by $\nabla_A$ the operator of the covariant differentiation with respect to the fundamental tensor $G_{BA}$, we have

$$
\nabla_C v_B = F_{CB}, \quad \nabla_C F_{EB} = -G_{CE} v_B + G_{CB} v_E.
$$

From the last equation of (1.2) and the Ricci formula for $F_{CB}$, we find

$$
R_{DCA} F^A_E = -R_{DCE} F^A_B - G_{CB} F_{DE} + G_{DB} F_{CE} + G_{CE} F_{DB} - G_{DE} F_{CB},
$$

where $R_{DCA}$ denotes components of the Riemannian curvature tensor of $N$. Moreover, we have

$$
\begin{align*}
R_{DA} v^A &= 2nv^D, \\
R_{DE} F^A_B + R_{BE} F^A_E &= 0,
\end{align*}
$$

$R_{DA}$ being components of the Ricci tensor of $N$.

Let $M$ be an invariant submanifold of codimension 2 in $N$ covered by a system of coordinate neighborhoods $\{ V; y^A \}$, where here and in the sequel the indices $h, i, j, \ldots$ run over the range $\{1, 2, \ldots, 2n-1\}$. And let $M$ be immersed isometrically in $N$ by the immersion $i: M \to N$. We represent the immersion $i$ locally by $x^A = x^A(y^k)$ and put $B_j = (B_j^A)$ are $(2n-1)$–linearly independent local tangent vector fields of $M$. We denote by $C^A$ and $D^A$ two mutually orthogonal unit normals to $M$. Then the induced Riemannian metric $g_{ji}$ on $M$ is given by

$$
g_{ji} = G_{CB} B_j^C B_i^B
$$

because the immersion is isometric.

Denoting by $\nabla_j$ the operator of van der Waerden–Botolotti covariant differentiation formed with $g_{ji}$, The equations of Gauss and Weingarten for $M$ are respectively obtained:

$$
\begin{align*}
\nabla_j B_i^A &= h_j^i C^A + k_j^i D^A, \\
\nabla_j C^A &= h_j^r B_r^A + l_j D^A, \\
\nabla_j D^A &= -k_j^r B_r^A - l_j C^A,
\end{align*}
$$

where $h_j^r = h_j^r g^{ir}$, $k_j^r = k_j^r g^{ir}$ are components of the second fundamental tensors, $l_j$, those of the third fundamental tensor and $g^{ji}$ being contravariant components of $g_{ji}$. As to the transformations of $B_j^A$, $C^A$ and $D^A$ by $F_B^A$ we have respectively equations of the form
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\[ F_B^A B_i^B = f_i^j B_j^A, \]
\[ F_B^A C^B = D^A, \quad F_B^A D^B = - C^A, \]

where we have put \( f_{ji} = G(FB_j, B_i) \). The structure vector field \( v \) is also represented by

\[ v^A = p^i B_i^A, \]

where \( p_i = G(B_i, v) \), \( p^i \) being contravariant components of \( p_i \). From the last three equations, it follows that

\[ \begin{cases} f_j^r f_r^h = - \delta_j^h + p_j p^h, & p f_r^h = 0, \\ g_{ij} f_j^r f_i^t = g_{ji} - p_j p_i, & p f_r^p = 1. \end{cases} \]

Differentiating (1.8) covariantly along \( N \) and making use of (1.7), (1.8) and (1.9), we have

\[ \nabla_j p_i = f_{ji}, \quad \nabla_j f_r^h = - g_{ji} p^h + \delta_j^h p_i. \]

Then it is shown that the set \( (f, g, p) \) defines a Sasakian structure ([12, 14]). Similarly differentiating (1.10) covariantly and using (1.7) and (1.11), we have

\[ \begin{cases} h_{ji} - k_{ji} f_i^t, & k_{ji} = - h_{ji} f_i^t, \end{cases} \]

which imply

\[ \begin{cases} h_{ji} p^r = 0, & k_{ji} p^r = 0, \\ h_{ij} = k_{ij} = 0. \end{cases} \]

Thus, the submanifold of codimension 2 \( M \) is minimal.

From (1.13) we can easily see that

\[ \begin{cases} k_{ji} h^i_r + k_i h^i_r = 0, \\ k_{ji}^2 = h_{ji}^2, & k_2 = h_2, \\ k_3 = h_3 = 0, \end{cases} \]

where we have defined \( h_{ji}^2 = h_{ji} h^i, \quad k_{ji}^2 = k_{ji} k^i, \quad h_2 = h_{ji} h^i \) and \( h_3 = h_{ji} h^i \), \( h_{ji} \).

On the other hand, Gauss, Codazzi and Ricci equations for \( M \) are given respectively by

\[ \begin{align*}
R_{DCB} B_k^D B_j^C B_i^B B_h^A & = R_{kjih} - (h_{kk} h_{ji} - h_{kh} h_{ji} + k_{kh} k_{ji} - k_{kj} k_{jh}), \\
R_{DCBA} B_k^D B_i^C B^B C^A & = \nabla_j h_{ji} - \nabla_j h_{ki} - (l_k k_{ji} - l_k k_{kj}), \\
R_{DCBA} B_i^D B_j^C B^B D^A & = \nabla_k l_j - \nabla_j l_k + h_{kl} k_{jr} - h_{jr} k_{jr}, \\
R_{DCBA} B_k^D B_j^C B^B D^A & = \nabla_k l_j - \nabla_j l_k + h_{kr} k_{jr} - h_{jr} k_{jr}.
\end{align*} \]
where $R_{kjih}$ denote covariant components of the Riemannian curvature tensor of $M$.

Putting $A_{kj} = \nabla_k l_j - \nabla_j l_k$ and utilizing (1.15), the equation (1.21) reduces to

$$(1.22) \quad R_{DCBA}B^DB^DCDB^A = A_{kj} + 2h_{kr}k_j^r.$$ 

2. Sasakian submanifolds of codimension 2

Let $M$ be an invariant submanifold of codimension 2 of a Sasakian manifold $N$. Transvecting (1.18) by $g^{ij}$ and using (1.16), we find

$$R_{DA}B^DB^A = R_{DCBA}B^DCDB^A = R_{kk} + 2h_{kr}k_j^r,$$

where $R_{jj}$ denote components of the Ricci tensor of $M$. Transvecting $f_{jk}$, we get

$$(2.1) \quad R_{kr}f_j^r - 2h_{kr}k_j^r = R_{DA}B^DF^jB^A - R_{DCBA}F^DB^FDB^A (CCCB + DCDB)^B$$

On the other hand, from (1.3) we have

$$R_{DCBA}F^DB^E(EEE + DCD) = R_{DCEA}F^DF^DB^E(CCCE + DCDB)^B + 2F^{DE}B^DB^E.$$

Thus, using (1.8), (1.9) and (1.22), it follows that

$$R_{DCBA}F^DB^E(CCCE + DCDB)^B = A_{kj} + 2h_{kr}k_j^r - 2f_{kj}.$$ 

Therefore (2.1) turns out to be

$$R_{DA}B^DE^F = R_{DA}B^DB^A + A_{kj} - 2f_{kj}.$$ 

Transvecting $f_{ij}$ and taking account of (1.11), we get

$$R_{DA}B^D(-\delta_i^r + p_{i}p_{r})B^A = R_{kr}(-\delta_i^r + p_{i}p_{r}) + A_{kj}f_j^r - 2(g_{ki} - p_{k}p_{i}),$$

which together with (1.4) and (1.10) yields

$$R_{DA}B^DB^A = R_{jk} + 2g_{ji} - A_{ij}f_j^r.$$ 

**LEMMA 1.** Let $N$ be a $(2n+1)$-dimensional Sasakian manifold with harmonic curvature.

**Proof.** Differentiating (1.4) covariantly along $N$ and using (1.1), we find

$$(\nabla CR_{DA})v^A + R_{DA}F^A = 2nF^E,$$

Since $N$ has harmonic curvature, namely, $\nabla CR_{DA} - \nabla D R_{CA} = 0$, it follows that

$$R_{DA}F^A = R_{CA}F^A = 4nF^E.$$
which together with (1.5) yields

\[(2.3) \quad R_{DA}F^A = 2nF_{CD}.\]

Transvecting (2.3) by $F_B^C$ and making use of (1.1) and (1.4), we see that $N$ is Einstein. This completes the proof.

Since $M$ is also Sasakian manifold, we see that

\[(2.4) \quad f_j^* R_{ir} = f^{ik} R_{ijk} + (2n-3)f_{ji}.\]

Thus, if $M$ is of harmonic curvature, then, by Lemma 1, $M$ is Einstein, i.e. $R_{ji} = 2(n-1)g_{ji}$. Therefore (2.4) turns out to be

\[f^{ik} R_{ijk} = f_{ji}.\]

Transvecting (1.22) with $\rho^j$ and using (1.10) and (1.14), we find

\[R_{DCBA}B_k^Dv^C G_B^A = A_{kj} \rho^j.\]

However, we have $R_{DCBA}v^A = v_D G_{CB} - v_C G_{DB}$, which is a direct consequence of (1.2), we get

\[R_{BADCB}B_k^Dv^C G_B^A = 0.\]

Therefore we have

\[(2.6) \quad A_{ji} \rho^r = 0.\]

**Lemma 2.** Let $M$ be an invariant submanifold of codimension 2 in a Sasakian manifold. Then we have

\[(2.7) \quad \bar{V}^l \bar{V}_j h_{il} = (\bar{V} \bar{V}_j k_{ik}) f^{ik} + (2n-3) h_{ji}.\]

**Proof.** From (1.14), we have respectively

\[(\bar{V} \bar{V}_j k_{ji}) \rho^r + k_{ji} f^r = 0, \quad (\bar{V} \bar{V}_j h_{ji}) \rho^r + h_{ji} f^r = 0\]

because of (1.12). By using (1.13), it follows that

\[(2.8) \quad (\bar{V} \bar{V}_j k_{ji}) \rho^r = -h_{kj}, \quad (\bar{V} \bar{V}_j h_{ji}) \rho^r = -k_{ji}.\]

Differentiating (1.13) covariantly along $M$ and taking account of (1.12), we find respectively

\[(2.9) \quad \bar{V} \bar{V}_j h_{ji} = (\bar{V} \bar{V}_j k_{ji}) f^i + k_{ji} p_i, \quad \bar{V} \bar{V}_j k_{ji} = -(\bar{V} \bar{V}_j h_{ji}) f^i - h_{kj} p_i.\]

Differentiating (2.9) covariantly and using (1.12) we obtain

\[\bar{V} \bar{V} \bar{V} \bar{V}_j h_{ji} = (\bar{V} \bar{V} \bar{V} \bar{V}_j k_{ji}) f^i + (\bar{V} \bar{V}_j k_{ji}) (-g_{li} f^r + g_{lr} p_i) + (\bar{V} \bar{V}_j k_{ik}) p_i + k_{kj} f_{li},\]

or, making use of (2.8),

\[\bar{V} \bar{V} \bar{V} \bar{V}_j h_{ji} = (\bar{V} \bar{V} \bar{V} \bar{V}_j k_{ji}) f^i + g_{li} h_{kj} + (\bar{V} \bar{V}_j k_{ji}) p_i + (\bar{V} \bar{V}_j k_{ik}) p_i + k_{kj} f_{li}.\]

If we transvect $g^{ti}$ and use (2.8), then we obtain the equation (2.7) which completes the proof.
3. Invariant submanifolds of an Einstein manifold

Suppose that $N$ is a $(2n+1)$-dimensional Einstein manifold. Then we have $R_{DA} = 2nG_{DA}$. It means that

$$R_{DA}B_j^PB_i^A = 2ng_{ji}.$$  

Thus (2.2) reduces to

$$(3.1) \quad R_{ji} = 2(n-1)g_{ji} + A_{ji}f_{ji}^r.$$  

Substituting (3.1) into (2.4) and using (2.6), we find

$$f^{il}R_{lji} = f_{ji} + A_{ji}.$$  

By the properties of the Sasakian structure, it follows that

$$f^{il}R_{ljk} = -2(f_{ji} + A_{ji}).$$  

On the other hand, from the Ricci identity for $k_{ji}$, we obtain

$$V_lV_{kji} - V_kV_{lji} = -R_{tkji}k^t_j - R_{tkij}k^t_j.$$  

Transvecting the above equation with $f^{ik}$ and taking account of (1.13) and (3.2), it reduces to

$$(3.3) \quad f^{ik}e_{kji} = 2h_{ji} + A_j, k^t_j + A_i, k^t_j.$$  

Thus, it follows that

$$(3.4) \quad h^{ji}f^{ik}e_{kji} = 2h_j^2 + A_j, k^t_j + A_i, k^t_j.$$  

Proposition 3. Let $N$ be a Sasakian with harmonic curvature. If $M$ is an invariant submanifold of codimension 2 in $N$, then the following assertions are true:

1. $M$ is an Einstein manifold if and only if the normal connection of $M$ is flat.
2. $M$ is C-Einstein manifold if and only if $A_{ji} = -bf_{ji}$ for some constant $b$.

Proof. (1) From Lemma 1 we have (3.1). Thus it is easily seen that $R_{ji} = 2(n-1)g_{ji}$ is equivalent to $A_{ji} = 0$ by taking account of (2.6).

(2) "If part" is evidently true because of (3.1).

Suppose that $M$ is of C-Einstein, i.e. $R_{ji} = ag_{ji} + bp_jp_i$. Then from (3.1) we have

$$-b(g_{ji} - p_jp_i) = A_{ji}f_{ji}^r.$$  

Differentiating the above equation covariantly and (2.6), we get

$$b(f_{kj} + f_{ki}p_j) = (\nabla_kA_{jr})f_{ji}^r + A_{jk}p_i.$$
Transvecting with \( p^i \), we find \( h f_{kj} = A_{jk} \). Thus we complete the proof.

**Theorem 4.** Let \( N \) be a \( (2n+1) \)-dimensional Sasakian manifold with harmonic curvature and \( M \) be a compact \( C \)-Einstein submanifold of codimension 2 in \( N \). If the second fundamental form is of Codazzi type and the scalar curvature of \( M \) is non-negative, then \( M \) is totally geodesic.

**Proof.** Since \( h_{ji} \) is of Codazzi type, Lemma 2 tells us that

\[
\Delta h_{ji} = (\mathcal{V}_j \mathcal{V}_i k_{ji}) f^{jk} + (2n-3) h_{ji},
\]

where \( \Delta = \mathcal{V}_i \mathcal{V}_i \) denotes the operator of Laplacian, which and (3.4) imply

\[
h^{ii} \Delta h_{ji} = (2n-1) h_2 + A_{jr} k^i h^{ji}.\]

Since \( M \) is \( C \)-Einstein manifold, by Proposition 3, we see that \( A_{jr} = -bf_{jr} \). Thus (3.6) reduces to

\[
h^{ii} \Delta h_{ji} = (2n-1-b) h_2
\]

because of (1.13). From (3.1) we have

\[
R = 2 (n-1) (2n-1-b) \geq 0
\]

because the scalar curvature of \( M \) is non-negative is assumed. Therefore (3.7) means

\[
h^{ii} \Delta h_{ji} = \frac{R}{2(n-1)} h_2.
\]

Therefore we have the following identity:

\[
\frac{1}{2} \Delta h_2 = \frac{R}{2(n-1)} h_2 + ||\mathcal{V}_k h_{ji}||^2 \geq 0.
\]

Since \( M \) is compact, we see, according to the Green's Theorem, that \( h_2 = k_2 = 0 \) because of (3.5). Thus \( M \) is totally geodesic because of (1.16).

Now, we prove the following theorem:

**Theorem 5.** Let \( M \) be a compact invariant submanifold of codimension 2 with harmonic curvature in a Sasakian manifold. If the second fundamental form is of Codazzi type, then \( M \) is totally geodesic.

**Proof.** Since \( M \) is of also a Sasakian manifold with harmonic curvature, by Lemma 1, it follows that \( M \) is Einstein, that is,

\[
R_{ji} = 2 (n-1) g_{ji}.
\]

Thus (2.5) is valid. Hence (3.3) becomes

\[
f^{ik} \mathcal{V}_k k_{ji} = 2 h_{ji}.
\]
Therefore (3.5) reduces to

\[ (3.9) \quad \Delta h_{ji} = (2n-1)h_{ji} \]

because \( h_{ji} \) is of Codazzi type. \( M \) being compact, combining (3.8) and (3.9), it is clear that \( h_{ji} = k_{ji} = 0 \). Thus \( M \) is totally geodesic.

**Remark 1.** If we replace the compactness condition in Theorem 4 and Theorem 5 by \( h_2 = \text{constant} \). Then we obtain the same result that \( M \) is totally geodesic.

**Remark 2.** Let \( N \) be a Sasakian space form, then it is clear that \( h_2 = \text{constant} \) and \( h_{ji} \) is of Codazzi type. Thus it follows that if \( M \) is an invariant submanifold of codimension 2 in a Sasakian space form \( N(c) \), then \( M \) is totally geodesic (cf. [10]).

**Bibliography**

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