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SUBMANIFOLDS WITH TOTALLY UMBILICAL GAUSS IMAGE IMMERSED IN A EUCLIDEAN SPACE

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1. Introduction

Since the classical Gauss map has been introduced by Gauss in order to define the Gauss curvature, it has become one of the fundamental tools in the study of submanifolds. For a submanifold M of dimension n in a Euclidean m-space E^m of higher codimension, the Gauss map is defined as the mapping $\Gamma : M \to G(n, m-n)$ which maps a point p in M into the n-dimensional linear subspace of E^m which is obtained by parallel displacement of the tangent space T_pM of M at p, where $G(n, m-n)$ denotes the Grassmann manifold consisting of n-dimensional linear subspaces of E^m. It is well known that the Grassmann manifold $G(n, m-n)$ admits a standard Riemannian metric $\tilde{g}$ which makes $G(n, m-n)$ into a symmetric space.

Throughout this paper we consider only submanifolds of E^m whose Gauss maps are regular.

Let G denote the metric on M induced from $\tilde{g}$ via the Gauss map. Then M is said to have *totally umbilical (resp. totally geodesic) Gauss image* if the image $\Gamma(M)$ of (M, G) under Γ is totally umbilical (resp. totally geodesic) in $(G(n, m-n), \tilde{g})$. One fundamental problem concerning the Gauss map is to classify submanifolds of E^m whose Gauss images are totally umbilical. Recently B. Y. Chen and S. Yamaguchi have solved this problem completely for surfaces in E^m and studied submanifolds of E^m with totally geodesic Gauss image [4], [5], [6].

In this paper we shall study submanifolds of E^m with totally umbilical Gauss image and especially isotropic immersions with totally geodesic Gauss image.

In §2, we first recall structure equations for submanifolds in a Euclidean m-space E^m. In §3, we also recall definitions of Grassmann manifolds and Gauss map, and derive some fundamental formulas for Gauss map. §4 is devoted to classify submanifolds of dimension higher than 2 in E^m whose Gauss maps are conformal.

In §5, we study submanifolds with totally geodesic Gauss image.

2. Preliminaries

Let M be a submanifold in a Euclidean m-space E^m with the induced metric g on M. Denote by $<,>$ the scalar product in E^m. Let ∇ and $\tilde{\nabla}$ be the connections on M and E^m, respectively. Then the second fundamental form h of the immersion in given by

$$(2.1) \qquad h(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y,$$

where X and Y are vector fields tangent to M. For a vector field ξ normal to M, we put

$$(2.2) \qquad \tilde{\nabla}_X\xi=A_\xi X+D_X\xi,$$

where $-A_\xi X$ and $D_X\xi$ denote the tangential and normal component of $\tilde{\nabla}_X\xi$, respectively. It is clear that $\langle A_\xi X, Y\rangle=\langle h(X, Y), \xi\rangle$.

For the second fundamental form h, the covariant differentiation $'\nabla$ with respect to the connection in $TM\oplus T^\perp M$ is defined by

$$(2.3) \qquad ('\nabla_X h)(Y,Z)=D_X(h(Y, Z))-h(\nabla_X Y, Z)-h(Y, \nabla_X Z).$$

Let K and K^D denote the curvature tensors associated with ∇ and D, respectively. Then the equations of Gauss, Codazzi, and Ricci are given respectively by

$$(2.4) \qquad K(X,Y:Z,W)=\langle h(X,W), h(Y,Z)\rangle-\langle h(X,Z), h(Y,W)\rangle,$$

$$(2.5) \qquad ('\nabla_X h)(Y,Z)=('\nabla_Y h)(X,Z),$$

$$(2.6) \qquad K^D(X,Y:\xi,\eta)=\langle [A_\xi, A_\eta]X,Y\rangle$$

for vector fields X,Y,Z,W tangent to M and ξ, η normal to M.

If we define the second covariant derivative of h by

$$(2.7) \qquad ('\nabla_W{}'\nabla_X h)(Y,Z)=D_W(('\nabla_X h)(Y,Z))-('\nabla_X h)(\nabla_W Y,Z)-('\nabla_X h)(Y,\nabla_W Z) -('\nabla_{\nabla_W X}h)(Y,Z),$$

then we have

$$(2.8) \qquad ('\nabla_W{}'\nabla_X h)(Y,Z)-('\nabla_X{}'\nabla_W h)(Y,Z)=K^D(W,X)(h(Y,Z)) -h(K(W,X)Y,Z)-h(Y,K(W,X)Z).$$

For the second fundamental form h the vector $h(X, X)$ is called a ***normal curvature vector*** in the direction of a unit vector X. If every normal curvature vector has the same length for any unit vector X at P, then the immersion is said to be ***isotropic*** at P. If the immersion is isotropic at any point on M, namely, the length of a normal curvature vector depends only on the initial point, then the immersion is said to be ***isotropic***. The immersion is isotropic at a point P if and only if the second fundamental form h satisfies

(2.9) $$C_3\langle h(X_1,\ X_2),\ h(X_3,\ Y)\rangle=\lambda^2 C_3\langle X_1,\ X_2\rangle\langle X_3,\ Y\rangle$$

for a scalar λ, where $X_i (i=1,2,3)$ and Y are unit tangent vectors at P and C_3, denotes the cyclic sum with respect to vectors X_1, X_2, X_3. The condition is equivalent to

(2.10) $$\langle h(X,X),\ h(X,Y)\rangle=0$$

for any orthonormal vectors X and Y at P [13].

3. Grassmann maniford $G(n, m-n)$ and the Gauss map

In this section we shall derive some fundamental formulars for the Gauss map. We follow closely that given by Chen and Yamaguchi [4].

Let $G(n,\ m-n)$ be the Grassmann manifold consisting of n-dimensional linear subspaces of E^m endowed with the standard metric $\tilde{g}$. Let Π_0 be any given point in $G(n, m-n)$. We fix an orthonormal basis $(f_i,\ f_x)$, $i=1,\cdots,n$; $x=n+1,\cdots,m$, of E^m such that f_i lie in Π_0. Let Π be a point in a suitable neighborhood U of Π_0 and $(e_i,\ e_x)$ be an orthonormal frame such that e_i lie in Π. We put

(3.1) $$e_i=\sum \xi_{ij}f_j+\sum \xi_{iy}f_y,\quad e_x=\sum \xi_{xj}f_j+\sum \xi_{xy}f_y,$$
$$h,i,j,k,l=1,\cdots,n\ ;\ x,y,z=n+1,\cdots,m.$$

Though the frame $(f_i,\ f_x)$ is fixed, we can take a frame $(e_i,\ e_x)$ in such a way that (see, e.g., [10])

(3.2) $$\langle e_i,\ f_j\rangle=\langle e_j,\ f_i\rangle,\ \langle e_x,\ f_y\rangle=\langle e_y,\ f_x\rangle,\ e_i(\Pi_0)=f_i,\ e_x(\Pi_0)=f_x.$$

If U is a sufficiently small neighborhood of Π_0, the $n(m-n)$ numbers ξ_{ix} can be used as local coordinates on U.

Let $\Pi(\xi)$ be a point of U whose local coordinates are ξ_{ix} and $\Pi(\xi+d\xi)$ a point whose local coordinates are $\xi_{ix}+d\xi_{ix}$. Then the distance ds between $\Pi(\xi)$ and $\Pi(\xi+d\xi)$ is given by [10]

(3.3) $$\tilde{g}=ds^2=\sum_{i,x}\langle de_i,\ e_x\rangle^2,$$

and the corresponding Christoffel symbols of $\tilde{g}$ satisfy

(3.4) $$\widetilde{\left\{ {kz \atop ix\ \ jy} \right\}}=0 \text{ at } \xi_{ix}=0.$$

Let M be an n-dimensional submanifold in E^m. For each point P in M, the tangent space T_PM can be taken, after a suitable parallel displacement, as a point $\Gamma(P)$ of $G(n, m-n)$. This mapping $\Gamma : M\to G(n, m-n)$ is called the *Gauss map* of M into $G(n, m-n)$.

For each point $P\in M$, we take an orthonormal frame $(e_i,\ e_x)$ of E^m such shat $e_i\in T_PM$. We put

(3.5) $$de_i = w_i^j e_j + w_i^x e_x,\ w_i^j = -w_j^i,$$

where, here and in the sequel, we use the Eienstein convention on summations. Let w^i be the dual frame of e^i. Then we have

(3.6) $$w_i^x = h_{ij}^x w^j,\ h_{ij}^x = \langle h(e_i,\ e_j),\ e_x\rangle.$$

From (3.3), (3.5) and (3.6), it follows that the metric G on M induced from $(G(n, m-n),\ \tilde{g})$ is given by

(3.7) $$G = \sum_{i,x} (w_i^x)^2 = \sum h_{ik}^x h_{kj}^x w^i w^j.$$

Let (x^i) be a local coordinate system on a suitable neighborhood V of a given point $0 \in M$ given by $x^i = 0$. We take a fixed orthonormal frame $(f_i,\ f_x)$ at 0 such that f_i lie in T_0M. For any point P in V with local coordinates (x^i) we choose $(e_i,\ e_x)$ such that e_i lie in T_PM satisfying the condition (3.2). Let $\xi_{ix} = \langle e_i,\ f_x\rangle$. Then $\xi_{ix}(x^j)$ are local coordinates of $\Gamma(P)$ in $G(n, m-n)$. We put $X_i = \partial/\partial x^i$ and $e_i = \sum \beta_i^j X_j$. Then we have

(3.8) $$\partial \xi_{ix}/\partial x^k = (\nabla_k \beta_i^j)\langle X_j,\ f_x\rangle + \beta_i^j \langle h(X_k,\ X_j),\ f_x\rangle$$

where $\nabla_k \beta_i^j = \partial \beta_i^j/\partial x^k + \left\{ {j \atop kl} \right\} \beta_i^l$ and $\nabla_{X_k} X_l = \left\{ {i \atop kl} \right\} X_i$. Let $\tilde{X}_j = \Gamma_* X_j$. We have

(3.9) $$\tilde{X}_j = \frac{\partial \xi_{ix}}{\partial x^j} \frac{\partial}{\partial \xi_{ix}}$$

Denote by $\tilde{h}$ the second fundamental form of the Gauss image. $\Gamma(M)$ in $G(n,\ m-n)$. If we denote by $\overline{\nabla}$ and ∇^G the connections of $(G(n,\ m-n),\ \tilde{g})$ and $(\Gamma(M),\ G)$, respectively, it is clear that

$$\tilde{h}(\tilde{X}_j,\ \tilde{X}_i) = \overline{\nabla}_{\tilde{X}_j} \tilde{X}_i - \Gamma_*(\nabla^G_{X_j} X_i).$$

Thus we have

(3.10) $$\tilde{h}(\tilde{X}_j,\ \tilde{X}_i) = \left[\frac{\partial^2 \xi_{kz}}{\partial x^j \partial x^i} + \left\{ {\widetilde{kz} \atop ly\ \ hx} \right\} \frac{\partial \xi_{ly}}{\partial x^j} \frac{\partial \xi_{hx}}{\partial x^i} - \left\{ {h \atop j\ \ i} \right\}^G \frac{\partial \xi_{kz}}{\partial x^h} \right] \frac{\partial}{\partial \xi_{kz}},$$

where $\nabla^G_{X_j} X_i = \left\{ {h \atop j\ \ i} \right\}^G X_h$. At 0 with $x^i = 0$, $\langle X_j,\ f_x\rangle = 0$ and consequently $(\xi_{jx}) = 0$.

Therefore (3.4) and (3.8) imply

(3.11) $$\left\{ {\widetilde{kz} \atop ly\ \ hx} \right\}_0 = 0,\ \left(\frac{\partial \xi_{ix}}{\partial x^k} \right)_0 = (\beta_i^j)_0 \langle h(X_k,\ X_j),\ f_x\rangle_0.$$

On the other hand by a direct computation, we also have $(\nabla_k \beta_i^j)_0 = 0$ (see, e.g., [11]). Thus from (3.8), (3.10) and (3.11), we may obtain the following formula for the second fundamental form of the Gauss image:

LEMMA. 3.1 (Chen and Yamaguchi [4]). *The second fundamental form $\tilde{h}$ of* *the Gauss image at $O \in M$ satisfies*

$$\tilde{h}(\tilde{X}_j, \tilde{X}_i) = \sum (\beta_k^l)_0 \langle (\nabla_{X_i}^{h})(X_j, X_l) + h(\nabla_{X_i} X_j, X_l) - h(\nabla_{X_i}^{G} X_j, X_l), f_z \rangle_0 \frac{\partial}{\partial \xi_{kz}}. \tag{3.12}$$

4. Submanifolds with conformal Gauss maps

Let M be a submanifold of E^m. It is said that the Gauss map of M is *conformal* (respectively, *homothetic*) if $G = e^{2\rho} g$ for some function ρ (respectively, for some constant ρ) on M.

Assume that Γ is conformal with $G = e^{2\rho} g$. Denoting by G_{ji} and g_{ji} local components of G and g, respectively, with respect to the local coordinates (x^i) in M, we have from (3.7)

$$G_{ji} = \sum h_{jk}^{x} h_{ki}^{x} = e^{2\rho} g_{ji}, \tag{4.1}$$

where h_{ji}^{x} denote the second fundamental tensors of M in E^m. Applying the covariant derivative ∇_h to (4.1), we have

$$\sum \{(\nabla_h h_{jk}^{x}) h_{ki}^{x} + h_{jk}^{x} \nabla_h h_{ki}^{x}\} = 2e^{2\rho} \rho_h g_{ji}, \tag{4.2}$$

where, here and in the sequel we put $\rho_h = \partial\rho / \partial x^h$. Transvecting (4.2) with g^{ji}, we obtain

$$\sum (\nabla_h h_{ik}^{x}) h_{ki}^{x} = n e^{2\rho} \rho_h \tag{4.3}$$

where $(g^{ji}) = (g_{ji})^{-1}$. Also transvecting (4.2) with g^{hi} and taking account of the equation (2.5) of codazzi for M in E^m, we find

$$\sum \{(\nabla_h h^{x}) h_{ih}^{x} + h_{hk}^{x} \nabla_i h_{hk}^{x}\} = 2e^{2\rho} \rho_i, \tag{4.4}$$

where $h^x = g^{ji} h_{ji}^{x}$. Therefore (4.3) with (4.4) implies

$$\sum (\nabla_h h^{x}) h_{ih}^{x} = (2-n) e^{2\rho} \rho_i. \tag{4.5}$$

If the Gauss map Γ is harmonic, namely, $\nabla_h h^x = 0$ (see, Corollary 1 of [14]), (4.5) reduces to $(n-2)e^{2\rho}\rho_i = 0$. The last equation gives

LEMMA 4.1. *If the Gauss map Γ is conformal and harmonic, then $n=2$ or Γ is homothetic.*

From now on, we put

$$W_{ji}^{h} = \left\{ {h \atop j\ i} \right\}^{G} - \left\{ {h \atop j\ i} \right\}. \tag{4.6}$$

Then, the conformality of the Gauss map Γ implies (see. p23 of [1])

$$W_{ji}^{h} = \rho_j \delta_i^h + \rho_i \delta_j^h - g_{ji} \rho^h, \tag{4.7}$$

where $\rho^h = g^{hi} \rho_i$.

Denoting by $\tilde{h}_{ji}^{ax}$ the local components of the second fundamental tensor of $\Gamma(M)$ in $G(n, m-n)$, we have from (3.12)

(4.8) $$\tilde{h}_{ji}^{ax}=\beta_a^r(\nabla_j h_{ir}^x - W_{ji}^s h_{sr}^x),$$

which together with (4.7) implies

$$\tilde{h}_{ji}^{ax}=\beta_a^r(\nabla_j h_{ir}^x-\rho_j h_{ir}^x-\rho_i h_{jr}^x+g_{ji}\rho^s h_{sr}^x).$$

From now on we assume that the Gauss image $\Gamma(M)$ is totally geodesic in $G(n, m-n)$, i.e., that the second fundamental from $\tilde{h}$ of $\Gamma(M)$ in $G(n, m-n)$ vanishes identically. Then the last equation reduces to

(4.9) $$\nabla_k h_{ji}^x=\rho_k h_{ji}^x+\rho_j h_{ki}^x-g_{kj}\rho^s h_{is}^x$$

since the matrix (β_a^r) is non-singular. Combining this with the equation (2.5) of codazzi, we find

(4.10) $$\rho_k h_{ji}^x-\rho_i h_{jk}^x-g_{kj}\rho^s h_{is}^x+g_{ji}\rho^s h_{ks}^x=0.$$

Transvecting (4.10) with g^{ji} we have

(4.11) $$\rho_k h^x=(2-n)\rho^s h_{ks}^x.$$

If $n>2$, then we have

(4.12) $$\rho^s h_{ks}^x=\rho_k h^x/(2-n)$$

Transvecting (4.10) with ρ^i gives

$$\|d\rho\|^2 h_{jk}^x=\rho_k\rho^s h_{js}^x+\rho_j\rho^s h_{ks}^x-g_{kj}\rho^s\rho^r h_{sr}^x,$$

which and (4.12) imply

(4.13) $$\|d\rho\|^2 h_{jk}^x=\frac{1}{2-n}[2\rho_k\rho_j-\|d\rho\|^2 g_{kj}]h^x.$$

Let $N_1=\{P\in M\,|\,d\rho\neq 0 \text{ at } P\}$. Then (4.13) shows that $h(X, Y)$ is contained in the linear subspace Span $\{H\}$ spanned by the mean curvature vector H with local components h^x on N_1. Since the Gauss map is assumed to be regular, $h\neq 0$ at any point $P\in M$. Thus we have the following.

LEMMA 4.2. *Let M be an n-dimensional $(n>2)$ submanifold of E^m. If the Gauss map is conformal and the Gauss image is totally geodesic, then $M=N_1\cup N_2$ such that* (1) *N_1 is open in M,* (2) *$d_\rho=0$ on N_2, and* (3) $\dim(\mathrm{Im}h)=1$ *on N_1.*

By using Lemma 4.2, we shall prove the following.

THEOREM 4.3. *Let M be an n-dimensional $(n>2)$ submanifold of E^m with totally geodesic Gauss image. If the Gauss map is conformal, then either*

(1) *the Gauss map is homothetic and the second fundamental form of M in E^m is parallel or*

(2) *M is a hypersurface in an affine $(n+1)$-subspace E^{n+1} of E^m and M is conformally flat.*

PROOF. Let N be a component of $N_1=\{P\in M \mid d_\rho\neq 0 \text{ at } P\}$. Then by means of Lemma 4.2 there exists an orthonormal local frame $\{E_1,\cdots,E_n,E_{n+1},\cdots,E_m\}$ such that $E_1,\cdots,E_n$ are tangent to M and the second fundamental tensors $A_x=A_{E_x}$, $x=n+1,\cdots,m$ take the following forms;

$$(4.14)\qquad A_{n+1}=\begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix},\quad A_{n+2}=\cdots=A_m=0.$$

For each fixed $i, i=1,2,\cdots,n,$ equations (2.3) and (4.14) imply

$$D_{E_i}(a_jE_{n+1})=D_{E_i}h(E_j,\ E_j)=('\nabla_{E_i}h)(E_j,\ E_j)+2h(\nabla_{E_i}E_j,\ E_j),$$

which together with (2.5) gives

$$\begin{aligned} D_{E_i}(a_jE_{n+1})&=('\nabla_{E_j}h)(E_i,\ E_j)+2h(\nabla_{E_i}E_j,\ E_j)\\ &=D_{E_j}h(E_i,\ E_j)-h(\nabla_{E_j}E_i,\ E_j)-h(E_i,\ \nabla_{E_j}E_j)+2h(\nabla_{E_i}E_j,\ E_j) \end{aligned}$$

for any $j, j=1,2,\cdots,n$. Therefore, for each fixed i, $i=1,2,\cdots,n$ and each $j\neq i$, equation (4.14) and the last equation yield

$$(4.15)\qquad D_{E_i}(a_jE_{n+1})=2h(\nabla_{E_i}E_j,\ E_j)-h(\nabla_{E_j}E_i,\ E_j)-h(E_i,\ \nabla_{E_i}E_j).$$

and consequently

$$(4.16)\qquad \langle D_{E_i}h(E_j,\ E_j),\ E_{n+k}\rangle=0,\ k=2,3,\cdots,m-n.$$

Since the Gauss map is assumed to be regular, $\det(A_{n+1})\neq 0$. So, (4.15) also implies that

$$\langle D_{E_i}E_{n+1},\ E_{n+k}\rangle=0,\ k=2,\cdots,m-n,$$

from which and (4.16) it follows that

$$D_Xh(Y,\ Z)\in \mathrm{Im}h$$

for any vector fields X,Y,Z tangent to M. Thus the first normal space $\mathrm{Im}h$ is parallel in the normal bundle and moreover; it is 1-dimensional by means of Lemma 4.2. Hence, by a theorem due to Erbacher [7], N lies in an $(n+1)$-dimensional subspace E^{n+1} of E^m.

On the other hand, by (4.14) conformality of the Gauss map, we have

$$a_1^2=\cdots=a_n^2\neq 0$$

on N. So we may assume that

$$(4.17)\qquad a_1=\cdots=a_r=a,\ a_{r+1}=\cdots a_n=-a$$

for some positive function a on N. The mean curvature vector H is then gives by

$$(4.18)\qquad H=(2r-n)aE_{n+1}$$

By the way, (4.12) implies that the vector $d\rho^\#$ associated with $d\rho$ is an eigenvector of A_{n+1} with eigenvalue $\pm\frac{1}{2-n}\|H\|$. Thus by using (4.17) and (4.18) we find $r=1$ or $n-1$. Therefore, the second fundamental form h takes the following form

$$h=\{\alpha g-2\alpha u\otimes\mu\}E_{n+1} \text{ on } N \tag{4.19}$$

for some nonzero function α and some 1-form μ. Thus N is conformally flat if $n>3$.

From now on we prove the conformal flatness of N is the case of $n=3$.

Differentiating covariantly both side of (2.4) and making use of

$$\nabla_k h_{ji}^{x}=W_{kj}^{r}h_{ri}^{x},$$

we can easily find

$$\nabla_l K_{kjih}=W_{lk}^{r}K_{rjih}+W_{lj}^{r}K_{krih},$$

from which by contraction

$$\nabla_l K_{kh}=W_{lk}^{r}K_{rh}+W_{ls}^{r}K_{kr}s_h,$$

substituting (4.7) in the last equation we have

$$\nabla_l K_{kh}=2\rho_l K_{kh}+\rho_k K_{lh}-g_{lk}\rho^r K_{rh}+\rho^r K_{khrl}. \tag{4.20}$$

A direct calculation by using (4.12) gives

$$\rho^r K_{khrl}=\sum\rho^r(h_{kl}^{x}h_{hr}^{x}-h_{kr}^{x}h_{hl}^{x})=\frac{1}{n-2}\{\rho_k(K_{hl}+G_{hl})-\rho_h(K_{kl}+G_{kl})\}, \tag{4.21}$$

where we have used

$$K_{ji}=\sum h_{ji}^{x}h^{x}-G_{ji} \tag{4.22}$$

which is a direct consequence of (2.4) and (4.1). Moreover, transvecting (4.22) with ρ^j and taking account of $G_{ji}=e^{2\rho}g_{ji}$, we have

$$\rho^r K_{ri}=-\left[\frac{1}{n-2}\|H\|^2+e^{2\rho}\right]\rho_i. \tag{4.23}$$

Combining (4.20) with (4.21) and (4.23), we obtain

$$\begin{aligned}\nabla_k K_{ji}=2\rho_k K_{ji}+\rho_j K_{ki}+\left\{\frac{1}{n-2}\|H\|^2+e^{2\rho}\right\}g_{kj}\rho_i\\ -\frac{1}{n-2}\{\rho_i(K_{jk}+G_{jk})-\rho_j(K_{ik}+G_{ik})\}.\end{aligned} \tag{4.24}$$

On the other hand, differentiating covariantly both side of (4.22) with $G_{ji}=e^{2\rho}g_{ji}$, we can see that

$$\nabla_k K_{ji}=2\rho_k K_{ji}+\rho_j(K_{ki}+G_{ki})+\frac{1}{n-2}\|H\|^2 g_{kj}\rho_i \tag{4.25}$$

where we have used

$$\nabla_k h^{x}=\rho_k h^{x} \tag{4.26}$$

which is direct consequence of (4.9). Comparing (4.24) with (4.25), we have

(4.27) $$\frac{1}{n-2}\{\rho_i(K_{jk}+G_{jk})-\rho_j(K_{ik}+G_{ik})\}=\rho_i G_{kj}-\rho_j G_{ki}.$$

Transvecting (4.27) with g^{ki} and using (4.23), we can esily verify

$$\left\{e^{2\rho}-\frac{1}{(n-2)^2}\|H\|^2\right\}\rho_j=0,$$

which implies

(4.28) $$e^{2\rho}=\frac{1}{(n-2)^2}\|H\|^2 \text{ on } N.$$

Transvecting again (4.27) with ρ^i, we obtain

$$\frac{1}{n-2}\|d\rho\|^2 K_{kj}+\left\{e^{2\rho}+\frac{1}{(n-2)^2}\|H\|^2\right\}\rho_k\rho_j-\frac{n-3}{n-2}\|d\rho\|^2 G_{kj}=0,$$

or equivalently, by using (4.28)

(4.29) $$\|d\rho\|^2 K_{kj}+2(n-2)e^{2\rho}\rho_k\rho_j-(n-3)\|d\rho\|^2 G_{kj}=0.$$

It follows from (4.25) that

$$\nabla_k K_{ji}-\nabla_j K_{ki}=\rho_k K_{ji}-\rho_j K_{ki}+\rho_j G_{ki}-\rho_k G_{ji},$$

from which, taking account of (4.27),

(4.30) $$\nabla_k K_{ji}-\nabla_j K_{ki}=(n-4)(\rho_k G_{ji}-\rho_j G_{ki}) \text{ on } N.$$

On the other hand, equations (4.12) and (4.13) yield

$$\|d\rho\|^2\left\{\|h_{ji}^{x}\|^2-\frac{n}{(n-2)^2}\|H\|^2\right\}=0.$$

Thus

$$K=g^{ji}K_{ji}=\|H\|^2-\|h_{ji}^{x}\|^2=\frac{(n-2)^2-n}{(n-2)^2}\|H\|^2$$

on N and consequently

$$\nabla_k K=\frac{2\{(n-2)^2-n\}}{(n-2)^2}\|H\|^2\rho_k$$

with the help of (4.26). Therefore, it follows from (4.28) that

(4.31) $$\frac{1}{2(n-1)}-(g_{ji}\nabla_k k-g_{ki}\nabla_j k)=\frac{(n-2)^2-n}{n-1}(\rho_k G_{ji}-\rho_j G_{ki}),$$

from which and (4.30), if $n=3$,

$$\nabla_k K_{ji}-\nabla_j K_{ki}-\frac{1}{2(n-1)}(g_{ji}\nabla_k K-g_{ki}\nabla_j K)=0.$$

Hence N is conformally flat when $n=3$.

If $\text{int}(N_2)=\phi$, then, by the regularity of Gauss map and the discussi above, the statement (2) holds.

Now, we assume that $\text{int}(N_2)\neq\phi$. Then, by the Lemma 4.2, the Gauss m is homothetic on int(N_2), in particular, the Gauss map is affine on int(N_2). Therefore $\nabla^G=\nabla$ and consequently $'\nabla h\equiv 0$ on int(N_2) by (4.9). Hence int(N_2) is locally symmetric. If dim(Imh)=1 on int(N_2), dim(Imh)≡1 on M. Thus, by the regularity of Gauss map and codazzi's, M is a hypersurface of an affine $(n+1)$-subspace E^{n+1} of E^m. In this case, a similar argument as before shows that the statement (2) holds.

If dim(Imh)>1 on some point of int(N_2), some component V of int(N_2) lies fully in an affine $(n+p)$-subspace E^{n+p} of E^m with $p>1$. Since $'\nabla h\equiv 0$ on int(N_2). V is a locally symmetric space immersed in E^{n+p} in a standard fashion (see Ferus [8]). From this we may conclude that $N_2=M$ and the theorem follows.

REMARK. The proof of Theorem 4.3 for $n>3$ is given by Chen and Yamaguchi [5].

5. Submanifolds with totally umbilical Gauss image

Let M be a submanifold of E^m whose Gauss image is totally umbilical. Then, by means of Lemma 3.1, we have

$$G_{ji}C^x_l=\beta^r_l(\nabla_j h_{ir}^{\ x}-W_{ji}^{\ s}h_{sr}^{\ x}) \tag{5.1}$$

for some normal bundle valued 1-forms C^x_l. Contracting (5.1) with β^h_l and using $\sum_l \beta^r_l\beta^s_l=g^{rs}$, we obtain

$$G_{ji}D^{mx}=\nabla_j h_i^{mx}-W^s_{ji}h_s^{mx}, \tag{5.2}$$

or equivalently

$$G_{ji}D_m^{\ x}=\nabla_j h_{im}^{\ x}-W^s_{ji}h^x_{sm}, \tag{5.2$'$}$$

where $h_i^{mx}=g^{mj}h^x_{ij}$ and we put $\sum_l C^x_l\beta^m_l=D^{mx}$, $D_m^{\ x}=g_{ms}D^{sx}$.

In addition, we assume that the Gauss map is conformal with $G=e^{2\rho}g$. Then as already shown in the previous section, we have

$$W_{ji}h=\rho_j\delta^h_i+\rho_i\delta^h_j-g_{ji}\rho^h.$$

Therefore, (5.2)$'$ reduces to

$$\nabla_k h_{ji}^{\ x}=e^{2\rho}g_{jk}D^x_i+\rho_k h_{ji}^{\ x}+\rho_j h_{ki}^{\ x}-g_{kj}\rho^r h_{ri}^{\ x}, \tag{5.3}$$

which and the equation (2.5) if codazzi give

$$(e^{2\rho}D^x_i-\rho^r h_{ri}^{\ x})g_{kj}-(e^{2\rho}D^x_j-\rho^r h_{rj}^{\ x})g_{ki}+\rho_j h^x_{ki}-\rho_i h^x_{kj}=0.$$

Transvecting this with g^{kj}, we find

$$(n-1)\ (e^{2\rho}D^x_i-\rho^r h_{ri}^{\ x})+\rho^r h_{ri}^{\ x}-\rho_i h^x=0. \tag{5.4}$$

Hence, if the Gauss map is homothetic, it follows that

$$D_i^x=0$$

on M, which and (5.3) yield that the submanifold M is a parallel submanifold in E^m, *i.e.*, that the second fundamental form h of M in E^m is parallel. Thus we have

LEMMA 5.1. *If the Gauss image $\Gamma(M)$ is totally umbilical and if Γ is homothetic, then M is a parallel submanifold in E^m.*

Combining Lemma 4.1 with Lemma 5.1, we have

THEOREM 5.2. *If the Gauss image $\Gamma(M)$ is totally umbilical, and if Γ is conformal and harmonic, then M is a parallel submanifold in E^m, provided* $\dim M>2$.

On the other hand, substituting (5.4) in (5.3), we have

$$\nabla_k h_{ji}^{\ x}=\rho_k h_{ji}^{\ x}+\rho_j h_{ki}^{\ x}-\frac{1}{n-1}g_{kj}(\rho^r h_{ri}^{\ x}-\rho_i h^x). \tag{5.3$'$}$$

Substituting (5.3)′ in (4.2) and using (4.1), we obtain

$$(n-2)e^{2\rho}(\rho_j g_{ki}+\rho_i g_{jk})+g_{ki}\rho^t h_{jt}^{\ x}h_x+g_{kj}\rho^t h_{it}^{\ x}h_x=0,$$

from which, transvecting with g^{ji},

$$(n-2)e^{2\rho}\rho_k+\rho^t h_{kt}^{\ x}h_x=0. \tag{5.5}$$

Using this equatin, we shall prove

TFHEOREM 5.3. *Let M be a submanifold of E^m whose Gauss image $\Gamma(M)$ is totally umbilical and let Γ be conformal. If the immersion is isotropic, then Γ is homothetic and M is a parallel submanifold in E^m, provided* $\dim M=n>3$.

PROOF. Let the immersion be isotropic. Then as already mentioned in section 2, the second fundamental tensors of M in E^m satisfy (2.9) for a continuous function λ, that is,

$$h_{kj}^{\ x}h_{ihx}+h_{ik}^{\ x}h_{jhx}+h_{ji}^{\ x}h_{khx}=\lambda^2(g_{kj}g_{ih}+g_{ik}g_{jh}+g_{ji}g_{kh}).$$

Transvecting this equation with g^{ih}, we have

$$h_{kj}^{\ x}h_x=\{(n+2)\lambda^2-2e^{2\rho}\}g_{kj}, \tag{5.6}$$

from which, substituting in (5.5),

$$\{(n+2)\lambda^2+(n-4)e^{2\rho}\}\rho_k=0.$$

If $n>3$, since the Gauss map Γ is assumed to be regular, $\rho_k=0$ which means that Γ is homothetic. Therefore our assertion is followed by Lemma 5.1.

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