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MIXED PROBLEM OF SEMILINEAR HYPERBOLIC SYSTEMS

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Abstract: In this paper we consider the semilinear hyperbolic symmetric system of the first-order. The existence and uniqueness of the solution are proved, under certain conditions, some properties of the solution are investigated.

1. Introduction

The linear symmetric hyperbolic system of the first-order have been studied in [2], [3], [4] and others. Here we will consider the mixed problem of the semilinear system

$$\frac{\partial u(x,t)}{\partial t}-\sum_{j=1}^{n} a_j(x,t)\frac{\partial u(x,t)}{\partial x_j}=f(t,u(x,t)) \qquad (1.1)$$
$$x\in\Omega,\ 0\leqslant t\leqslant T$$

with the mixed conditions

$$u(x,0)=\phi(x),\ x\in\Omega \qquad (1.2)$$
$$u(x,t)|_{\Gamma}=0,\ 0\leqslant t\leqslant T \qquad (1.3)$$

where the unknown $u(x,t)=(u_1,\cdots,u_n)$ is a real vector-valued function, the coefficients, $a_j(x,t)$ are real $N\times N$ symmetric matrix-valued functions, and of class C^2 on $\overline{\Omega}\times[0,T]$. Ω is a bounded open subset of R^n with boundary Γ.

The non linear function $f(t,u(x,t))\in C([0,T],\ H^m)$ and satisfies the Lipschitz condition

$$\|f(t,u_1(x,t))-f(t,u_2(x,t))\|_{H^m}\leqslant k\|u_1(x,t)-u_2(x,t)\|_{H^m} \qquad (1.4)$$

where H^m is the Banach space of all matrix-valued functions with components belongs to the Sobolev space. $W_2^m(\Omega)$, and k is a positive constant.

2. Existence and Uniqueness theorem

Now the solution of the mixed problem (1.1), (1.2) and (1.3) can be expressed in the form

$$u(x,t)=G(t,0)+\phi(x)+\int_0^t G(t,s)*f(s,u(x,s))ds. \qquad (2.1)$$

where $G(t,s)$ is the fundamental solution of (1.1) (with zero right hand side) and (1.3), which satisfies [2], [3] for $v\in H^m$

$$\sum_{k=0}^{m} \left\| \frac{\partial^k}{\partial t^k}(G(t,s)\times v(x,t))\right\|_{H^{m-k}} \leq C\|v\|_{H^m} \tag{2.2}$$

with $C>0$.

$$G(t,t)*v(x,t)=v(x,t) \tag{2.3}$$

$$\frac{\partial}{\partial t}(G(t,s)\times v(x,t))=\sum_{j=1}^{n} a_j(x,t)\frac{\partial}{\partial x_j}(G(t,s)\times v(x,s)) \tag{2.4}$$

$$G(t,s) * v(x,t)|_{\Gamma}=0 \tag{2.4}$$

THEOREM 2.1. *If $\phi(x)\in H^m$, then for sufficiently small T there exists one and only one solution of the mixed problem* (1.1), (1.2) *and* (1.3), $u(x,t)\in H^m$ *and* $\frac{\partial u(x,t)}{\partial t}\in H^{m-1}$.

PROOF. It is clear that the integral equation (2.1) and the mixed problem (1.1), (1.2) and (1.3) are equivalent.

Now let us write

$$Fu=G(t,0)*\phi(x)+\int_0^t G(t,s)*f(s,\ u(x,s))ds. \tag{2.5}$$

Then

$$\|Fu_1-Fu_2\|_{H^m}\leq C\int_0^t\|f(s,u_1)-f(s,u_2)\|_{H^m}ds$$
$$\leq Ck\int_0^t\|u_1(x,s)-u_2(x,s)\|_{H^m}ds.$$

If we let

$$X=C([0,T]\times H^m)$$

where

$$\|u\|_X=\max_t\|u\|_{H^m},$$

then it is clear that the operator F defined by (2.5) maps X to X, and satisfies [5]

$$\|Fu_1-Fu_2\|_X\leq Tck\|u_1(x,t)-u_2(x,t)\|_X.$$

If T is sufficiently small such that $Tck<1$, then F is a contraction map, so by the contraction fixed point theorem we conclude that there exists a unique solution $u(x,t)$ of (2.1) and consequently of the mixed problem (1.1), (1.2) and (1.3), satisfies

$$u(X, t) \in X$$

i.e. $$u(x, t) \in H^{m},\ x \in \Omega\ ;\ 0 < t < T$$

From (2.1), (2.2) and (1.4) we have

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial t} G(t,0) * \phi(x) + f(t, u(x,t)) +$$
$$+ \int_0^t \frac{\partial}{\partial t} G(t,s) * f(s, u(x,s)) ds$$

and $$\left\| \frac{\partial u(x,t)}{\partial t} \right\|_{H^{m-1}} < C \|\phi\|_{H^m} + \|f(t, u(x,t))\|_{H^{m-1}} + \int_0^t \|f(s, u(x,s))\|_{H^m} ds$$

which shows that $\frac{\partial u(x,t)}{\partial t} \in H^{m-1}$, $0 < t < T$ and completes the prove.

Now from the Sobolev's Embedding theorems we have [1].

COROLLARY 2.1. *If* $2m-2>n$, *and the conditions of theorem* 2.1 *satisfied, then the solution* $u(x,t)$ *of the mixed problem* (1.1), (1.2) *and* (1.3) *equivalent to a function of* $C(\Omega)$, *and the derivative* $\frac{\partial u(x,t)}{\partial t}$ *exists in the usual sense.*

THEOREM 2.2. *If the solution* $u(x,t)$ *of the mixed problem* (1.1), (1.2) *and* (1.3) *exists for all values of* t, *then it is satisfied*

$$\|u(x,t)\|_{H^m} < Me^{at}$$

for all values of t.

PROOF. From (2.1), (2.2) and (1.4) we have

$$\|u(x,t) - u(y,t)\|_{H^m} < C \|\phi(x) - \phi(x)\|_{H^m} +$$
$$+ C \int_0^t \|f(s,\ u(x,s)) - f(s,\ u(y,s))\|_{H^m} ds$$
$$< C \|\phi(x) - \phi(y)\|_{H^m} + Ck \int_0^t \|u(x,s) - u(y,s)\|_{H^m} ds$$

from which we can deduce that

$$\|u(x,t) - u(y,t)\|_{H^m} < C\ k\ e^{ckt} \|\phi(x) - \phi(y)\|_{H^m}.$$

Let us take $y \in \Gamma$, consequently we get

$$\|u(x,t)\|_{H^m} < Me^{at},\ a = ck$$

and M is a constant depend upon $\|\phi(x)\|_{H^m}$ and a.

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