DISTRIBUTIVE, STANDARD AND NEUTRAL ELEMENTS IN THE JOINSEMILATTICE OF CONVEX SUBLATTICES OF A LATTICE

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1. Introduction and basic concepts

The purpose of this paper is to generalize the properties of distributive, standard and neutral ideals of a lattice. A generalization of standard ideals is given by Fried and Schmidt in [1], where the concept of a standard convex sublattice is defined and related properties are described. The main difficulty in the generalization work is the lack of a suitable algebra for describing the convex sublattices of a lattice. We shall first introduce an algebra for convex sublattices of a lattice and thereafter consider distributive, standard and neutral convex sublattices by means of the properties of the algebra.

A \( \mathcal{L}_{ab} \)-lattice \( H = (H, \vee, \wedge) \) is a joinsemilattice, where \( a \vee b = \text{lub} \{a, b\} \), the least upper bound of \( a \) and \( b \), for every two elements \( a, b \in H \), and \( a \wedge b = \text{glb} \{a, b\} \), the greatest lower bound of \( a \) and \( b \), when the set \( \text{lb} \{a, b\} \) of lower bounds of \( a \) and \( b \) is nonempty. If \( \text{lb} \{a, b\} = \emptyset \), we put \( a \wedge b = a \vee b \). Thus the operation \( \vee \) behaves like the corresponding operation in a joinsemilattice, i.e. it is associative and \( c \leq a \) and \( d \leq b \) imply \( c \vee d \leq a \vee b \). Unfortunately, \( c \leq a \) and \( d \leq b \) need not imply \( c \wedge d \leq a \wedge b \), and \( \wedge \) need not be associative. On the other hand, \( a \wedge a = a \) and \( a \wedge b = b \wedge a \) for all \( a, b \in H \). As easily seen, every lattice is also a \( \mathcal{L}_{ab} \)-lattice, but not every joinsemilattice \( S \) need not be a \( \mathcal{L}_{ab} \)-lattice, because the property \( \text{lb} \{a, b\} \neq \emptyset \) in \( S \) need not imply the existence of an element \( \text{glb} \{a, b\} \) in \( S \). A \( \mathcal{L}_{ab} \)-lattice \( H \) is called distributive (modular) if the conditions \( D_1 \) and \( D_2 \) (\( M_1 \) and \( M_2 \)) below hold:

\[
D_1 \quad a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c) \text{ for all } a, b, c \in H;
D_2 \quad a \vee (b \wedge c) \geq (a \vee b) \wedge (a \vee c) \text{ for all } a, b, c \in H;
M_1 \quad a \wedge (b \vee (c \wedge a)) \leq (a \wedge b) \vee (a \wedge c) \text{ for all } a, b, c \in H;
M_2 \quad a \vee (b \wedge (c \vee a)) \geq (a \vee b) \wedge (a \vee c) \text{ for all } a, b, c \in H.
\]

Clearly every distributive \( \mathcal{L}_{ab} \)-lattice is also modular. Note that the equality sign need not always hold in \( D_1, D_2, M_1 \) and \( M_2 \). Appropriate examples one can find by considering e.g. finite trees.
Let $C_{\text{sub}}(L)$ be the set of nonempty convex sublattices of a lattice $L$ and $A, B \in C_{\text{sub}}(L)$. As well known, the least convex sublattice of $L$ containing $A$ and $B$ is $A \lor B = \{x | x \in L, \ a_1 \land b_1 \leq x \leq a_2 \lor b_2 \text{ for some } a_1, a_2 \in A \text{ and } b_1, b_2 \in B\}$. Moreover, if $A \cap B \neq \emptyset$, then there is a greatest convex sublattice of $L$ contained in $A$ and $B$, namely $A \land B = A \cap B$. Now, by putting $A \land B = A \lor B$ for $A$ and $B$ with $A \cap B = \emptyset$, we see that the convex sublattices of a lattice constitute a $\emptyset_{\text{lub}}$-lattice $C_{\text{sub}}(L)$, where $A \lor B$ is given above, $A \land B = A \cap B$ when $A \cap B \neq \emptyset$ and otherwise $A \land B = A \lor B$. Note that when $A \cap B \neq \emptyset$ and $w \in A \cap B$, then $a \land b \in A \land B$ for $a, b \geq w$ and $a \lor b \in A \lor B$ for $a, b \leq w$, where $a$ is from $A$ and $b$ from $B$. As well known, every ideal of $L$ is also a convex sublattice of $L$. If $I$ and $J$ are two ideals of $L$, then $I \land J = I \cap J$ in the lattice $I(L)$ of all ideals of $L$ and $I \lor J = \{x | x \in L, \ i \leq x \lor j \text{ for some } i \in I \text{ and } j \in J\}$ in $I(L)$. Thus the meet and join of two ideals in $I(L)$ and $C_{\text{sub}}(L)$ coincide and we shall use a single sign $\lor (\land)$ for the join (the meet) in $I(L)$ as well as in $C_{\text{sub}}(L)$.

An element $A \in C_{\text{sub}}(L)$ is

1. **distributive** if and only if $A \lor (X \land Y) > (A \lor X) \land (A \lor Y)$ for all $X, Y \in C_{\text{sub}}(L)$;

2. **standard** if and only if it is distributive and (3) and (4) hold, where

   (3) when $A \cap X \neq \emptyset$, then $A \land X = A \land Y$ and $A \lor X = A \lor Y$ imply $X = Y$;

   (4) when $A \cap X = \emptyset$, then $(A) \land (X) = (A) \land (Y)$ and $(A) \lor (X) = (A) \lor (Y)$ imply $(X) = (Y)$, $X, Y \in C_{\text{sub}}(L)$ and $(X) = \{z | z \in L \text{ and } z \leq x \text{ for some } x \in X\}$;

3. **neutral** if and only if it is standard and dually distributive.

### 2. Distributive convex sublattices

In this section we shall describe distributive convex sublattices of a lattice $L$. At first we write a lemma, the proof of which is obvious and hence omitted.

**Lemma 1.** Let $A, X, Y \in C_{\text{sub}}(L)$. If $X \cap Y = \emptyset$, then $A \lor (X \land Y) > (A \lor X) \land (A \lor Y)$.

The following theorem shows a connection between convex sublattices, ideals and dual ideals of $L$.

**Theorem 1.** Let $A \in C_{\text{sub}}(L)$. Then $A$ is distributive in $C_{\text{sub}}(L)$ if and only if $(A)$ is distributive in $I(L)$ and $(A)$ is distributive in the lattice $D(L)$ of dual ideals of $L$. 
PROOF. Assume first that \( A \) is distributive in \( \text{Csub}(L) \). We shall show the distributivity of \( (A) \) in \( I(L) \) only; the proof for \( [A] \) is analogous and hence omitted.

Let \( (A) = \{ x \mid x \leq a, a \in A \} \). Because \( A \) is a sublattice of \( L \), \( x_1 \land x_2 \leq a_1 \land a_2 \leq A \) for any two elements \( x_1, x_2 \in (A) \), and thus \( (A) \) is an ideal of \( L \). Let \( X, Y \in I(L) \subseteq \text{Csub}(L) \). Because \( A \) is distributive in \( \text{Csub}(L) \), \( A \lor (X \land Y) \geq (A \lor X) \land (A \lor Y) \) in \( \text{Csub}(L) \). If \( z \in ((A) \lor X) \setminus ((A) \lor Y) \), then \( z \in (A) \lor X \), \( (A) \lor Y \), and so \( z \leq a_1 \lor x_1, a_2 \lor y_2 \), where \( a_1, a_2 \in A \), \( x_1 \in X \), \( y_2 \in Y \) and \( (a_1 \lor x_1) \land (a_2 \lor y_2) \leq (A \lor X) \land (A \lor Y) \). Because \( A \lor (X \land Y) \geq (A \lor X) \land (A \lor Y) \), \( (a_1 \lor x_1) \land (a_2 \lor y_2) \leq A \lor (X \land Y) \). But then \( a_3 \lor (x_3 \land y_3) \geq (a_1 \lor x_1) \land (a_2 \lor y_2) \geq z \), and thus \( z \in (A) \lor (X \land Y) \).

This shows that \( (A) \lor (X \land Y) \geq ((A) \lor X) \land ((A) \lor Y) \), and the distributivity of \( (A) \) in \( I(L) \) follows.

Conversely, let \( A \subseteq \text{Csub}(L) \), \( (A) \) be distributive in \( I(L) \) and \( [A] \) distributive in \( D(L) \). We shall show the distributivity of \( A \) in \( \text{Csub}(L) \).

Let \( z \in (A \lor X) \setminus (A \lor Y) \), where \( X \) and \( Y \) are two arbitrary elements of \( \text{Csub}(L) \). Because \( \phi \neq A \subseteq (A \lor X) \setminus (A \lor Y) \), then \( z \in A \lor X \), \( A \lor Y \), whence \( a_1 \lor x_1, a_2 \lor y_2 \leq z \leq a_3 \lor x_3, a_4 \lor y_4 \) with \( a_1, a_2, a_3, a_4 \in A \), \( x_1, x_3 \in X \) and \( y_2, y_4 \in Y \). According to Lemma 1 we may assume that \( X \setminus Y \neq \emptyset \). When \( w \in X \setminus Y \), we may clearly choose the elements \( x_1 \) and \( y_2 \) above so that \( x_1, y_2 \leq w \leq x_3, y_4 \). Obviously \( (a_3 \lor x_3) \land (a_4 \lor y_4) \) belongs to the ideal \( ((A) \lor X) \setminus ((A) \lor Y) \), to which thus \( z \) also belongs. According to the distributivity of \( (A) \) in \( I(L) \), \( ((A) \lor X) \setminus ((A) \lor Y) = ((A) \lor X) \setminus (A \lor Y) \), whence \( (a_3 \lor x_3) \land (a_4 \lor y_4) \leq z \) in \( (X \setminus Y) \) with \( a' \in (A) \), \( x' \in (X) \) and \( y' \in (Y) \). We can clearly choose new elements \( a, x \) and \( y \) from \( A, X \) and \( Y \), respectively, such that \( a' \lor (x' \land y') \leq z \) in \( (X \setminus Y) \). According to the distributivity of \( [A] \) in \( D(L) \), the dual argumentation, we obtain that \( a'' \land (x'' \lor y'') \leq z \) in \( (X \setminus Y) \). Consequently, \( z \leq a \lor (x \land y) \). The results above and Lemma 1 imply now that \( A \lor (X \setminus Y) \geq (A \lor X) \land (A \lor Y) \) for all \( X, Y \in \text{Csub}(L) \), which proves the distributivity of \( A \) in \( \text{Csub}(L) \).

As a corollary we can write

**Corollary 1.** Let \( a \leq b \) in \( L \). The interval \([a, b] \in \text{Csub}(L) \) is distributive in \( \text{Csub}(L) \) if and only if \( b \) is distributive and a dually distributive in \( L \). Moreover, \([b] \) is distributive in \( \text{Csub}(L) \) if and only if \( b \) is distributive as well.
as dually distributive in $L$.

3. Standard elements of $\text{Csub}(L)$.

The standardness of an element in a lattice has many equivalent definitions. Although one can modify these definitions for $\text{Csub}(L)$, the equivalence need not hold any more. As an example we consider the condition

(6) $X \land (A \lor Y) \leq (X \land A) \lor (X \land Y)$ for all $X, Y \in \text{Csub}(L)$.

At first we prove

**LEMMA 2.** Let $A, X, Y \in \text{Csub}(L)$. The inequality $X \land (A \lor Y) \leq (X \land A) \lor (X \land Y)$ holds if $X=L$ or $A \leq Y$ or $X \cap A = \phi$ or $X \cap Y = \phi$ or $X \cap (A \lor Y) = \phi$.

**PROOF.** If $X=L$, then $X \land (A \lor Y) = A \lor Y = (X \land A) \lor (X \land Y)$. If $A \leq Y$, then $X \land (A \lor Y) = X \land Y \leq (X \land A) \lor (X \land Y)$. If $X \cap A = X \cap Y = \phi$, then $(X \land A) \lor (X \land Y) = (X \lor A) \lor (X \lor Y) = X \lor A \lor Y \geq X \land (A \lor Y)$. If $X \cap A = \phi$ and $X \cap Y \neq \phi$, then $(X \land A) \lor (X \land Y) = X \lor A \lor Y > X \land (A \lor Y)$. If $X \cap (A \lor Y) = \phi$, then $X \cap A = X \cap Y = \phi$, and this case is proved above.

Now we can prove

**THEOREM 2.** Let $A \in \text{Csub}(L)$ be standard. Then (6) holds.

**PROOF.** Let $B = X \land (A \lor Y)$ and $C = (X \land A) \lor (X \land Y)$, and according to Lemma 2 we may assume that $X \cap A \neq \phi = X \cap Y$, $X \cap (A \lor Y) \neq \phi$, $X \neq L$ and $A \neq Y$. We show first that $A \land B = A \land C$. After showing $A \lor B = A \lor C$ we can conclude by (3) that $B = C$. This and Lemma 2 prove then the validity of (6).

Now $A \land X = A \cap X$, $X \cap A \subset C$ and $X \cap A \cap A \cap C = A \land C$. Further, $A \land B = A \land B = A \cap (X \cap (A \lor Y)) = (A \cap X) \cap (A \lor Y) = A \cap X = A \land X$. Moreover, $X \cap A = X \cap Y \subset X \cap (A \lor Y) = B$, whence $C \subset B$ and $A \cap C \subset A \cap B$. By combining these results we obtain $A \land X \subset A \land C \subset A \land B \subset A \land X$, and thus $A \land C = A \land B$, where $A \land B \neq \phi = A \land C$.

Secondly we prove $A \lor B = A \lor C$. Clearly $A \lor (X \cap Y) \subset A \lor X$, $A \lor Y$, whence $A \lor (X \cap Y) \subset (A \lor X) \lor (A \lor Y)$. Thus, under the assumptions made above, the distributivity of $A$ in $\text{Csub}(L)$ implies the equality $A \lor (X \land Y) = (A \lor X) \land (A \lor Y)$. Now $A \lor B = A \lor (X \land (A \lor Y)) = (A \lor X) \land (A \lor Y) = A \lor (X \land Y) = A \lor (X \land A) \lor (X \land Y) = A \lor C$. This completes the proof.

The converse does not hold; this will be shown by an example after the next theorem.
THEOREM 3. Let \( A \in \text{Csub}(L) \). If \( A \) is standard in \( \text{Csub}(L) \), then \( [A] \) is standard in \( I(L) \).

PROOF. We shall show that \( [A] \) is distributive in \( I(L) \), and \( [A] \land X = [A] \land Y \) and \( [A] \lor X = [A] \lor Y \) imply \( X = Y \) for all \( X, Y \in I(L) \), from which the standardness of \( [A] \) in \( I(L) \) follows [2, Theorem III.3.5].

Because \( A \) is distributive in \( \text{Csub}(L) \), the distributivity of \( [A] \) in \( I(L) \) follows from Theorem 1. Let now \( [A] \land X = [A] \land Y \) and \( [A] \lor X = [A] \lor Y \) for some \( X, Y \in I(L) \). If \( A \cap X \neq \phi \), then also \( A \cap Y \neq \phi \), because \( [A] \land X = [A] \land Y \) and \( [A] \cap X \) contains at least one element from \( A \). By similar argument we see that \( A \cap X = A \cap Y \), and thus in this case \( A \cap X = A \cap Y \) in \( \text{Csub}(L) \). When \( X \in I(L) \), then \( A \lor X = [A] \lor X \) in \( \text{Csub}(L) \), whence the equation \( A \lor X = A \lor Y \) follows from \( [A] \lor X = [A] \lor Y \). Because \( A \) is standard in \( \text{Csub}(L) \), (3) implies now \( X = Y \).

When \( A \cap X = \phi \), then \( X = Y \) follows by (4) from \( [A] \land X = [A] \land Y \) and \( [A] \lor X = [A] \lor Y \). This completes the proof.

Now we show that (6) does not imply the standardness of an element \( A \in \text{Csub}(L) \). Let \( L \) be the well known least modular and nondistributive lattice with elements \( 0 < a, b, c < 1 \). We put \( A = [a] \) and show that \( X \land (A \lor Y) \leq (X \land A) \lor (X \land Y) \) for all \( X, Y \in \text{Csub}(L) \). According to Lemma 2 we may assume that \( X \neq L \), \( A \leq Y \), \( X \cap A \neq \phi \neq X \cap Y \) and \( X \cap (A \lor Y) \neq \phi \). \( X \cap A \neq \phi \) implies that \( a \in X \), and \( A \leq Y \) that \( a \in Y \). Since \( X = [a] \) contradicts \( X \cap Y \neq \phi \), we have \( X = [a] \) or \([a] \). If \( X = [a] \), then \( X \cap Y = [0] \), whence \( (X \land A) \lor (X \land Y) = [a] \lor [0] = X \land (A \lor Y) \). If \( X = [a] \), then \( X \land Y = [1] \), whence \( (X \land A) \lor (X \land Y) = [a] \lor [1] = X \lor (A \lor Y) \). Hence \( A \) satisfies (6).

We call an element \( A \in \text{Csub}(L) \) double standard if \( A \) is distributive in \( \text{Csub}(L) \) and satisfies (3), (4) and (7), where

(7) when \( A \cap X = \phi \), then \( [A] \land [X] = [A] \land [Y] \) and \( [A] \lor [X] = [A] \lor [Y] \) imply \( [X] = [Y] \) for all \( X, Y \in \text{Csub}(L) \).

Now we can prove

THEOREM 4. Let \( A \in \text{Csub}(L) \). \( A \) is double standard in \( \text{Csub}(L) \) if and only if \( [A] \) is standard in \( I(L) \) and \( [A] \) standard in \( D(L) \).

PROOF. Let \( A \) be double standard. The standardness of \( [A] \) in \( I(L) \) is already proved in Theorem 3. The standardness of \( [A] \) in \( D(L) \) can be proved
dually, and in the dual proof (4) is substituted by (7), as easily seen. Thus we concentrate on the converse proof, only.

Let \([A]\) be standard in \(I(L)\). \([A]\) standard in \(D(L)\) and \(X, Y \subseteq C_{sub}(L)\). Because \([A]\) is then distributive in \(I(L)\) and \([A]\) distributive in \(D(L)\), \(A\) is distributive in \(C_{sub}(L)\) by Theorem 1. Thus it remains to show the validity of (4), (7) and (3). If \(A \cap X = \emptyset\), then \((A \cap X) = (A \cap Y)\) and \((A \cup X) = (A \cup Y)\) imply \((X) = (Y)\) because \([A]\) is standard in \(I(L)\) [2, Theorem III.3.5]. The validity of (7) is proved dually. Assume now that \(A \cap X \neq \emptyset\), \(A \cap X = A \cup Y\) and \(A \cup X = A \cup Y\). As one can easily see, these equations imply \((A \cap X) = (A \cap Y)\), \((A \cup X) = (A \cup Y)\), \((A \cap X) = (A \cap Y)\) and \((A \cup X) = (A \cup Y)\). Because \([A]\) is standard in \(I(L)\), the first two equations imply \((X) = (Y)\), and because \([A]\) is standard in \(D(L)\), the remaining two equations imply \((X) = (Y)\). But then \(X = (X) \cap (X) = (Y) \cap (Y) = Y\), which proves (3). Accordingly, \(A\) is double standard in \(C_{sub}(L)\), and the theorem follows.

Because \(L\) is standard in \(I(L)\) as well as in \(D(L)\), \([I] = L\) is standard in \(D(L)\) for every \(I \subseteq I(L)\) and \([D] = L\) is standard in \(I(L)\) for every \(D \subseteq D(L)\). Thus by Theorem 4 every standard ideal \(I\) (standard dual ideal \(D\)) of \(L\) is double standard in \(C_{sub}(L)\). The convex sublattice \([b]\) is standard in \(C_{sub}(L)\) if \(b\) is standard and dually distributive in \(L\). Indeed, the standardness and the dual distributivity of \(b\) in \(L\) imply the distributivity of \([b]\) in \(C_{sub}(L)\), and (4) holds by the standardness of \(b\) in \(L\). If \([b] \cap X \neq \emptyset\), then \(b \subseteq X, Y\), and thus \(X = [b] \cap X = [b] \cap Y = Y\), which proves (3). Now we can write a corollary

**COROLLARY 2.** Every standard ideal (dual ideal) of \(L\) is double standard in \(C_{sub}(L)\). An interval \([a, b]\) is double standard in \(C_{sub}(L)\) if and only if \(b\) is standard and a dually standard in \(L\). \([b]\) is double standard in \(C_{sub}(L)\) if and only if \(b\) is neutral (i.e. standard and dually standard) in \(L\), and \([b]\) is standard in \(C_{sub}(L)\), if \(b\) is standard and dually distributive in \(L\).

As well known, in a modular lattice \(L\) an ideal (a dual ideal) is distributive if and only if it is standard [2, Theorem III.2.6]. This and Theorems 1 and 4 imply

**THEOREM 5.** Let \(L\) be a modular lattice and \(A \subseteq C_{sub}(L)\). \(A\) is distributive in \(C_{sub}(L)\) if and only if \(A\) is standard in \(C_{sub}(L)\). Moreover, \(A\) is standard in \(C_{sub}(L)\) if and only if \(A\) is double standard in \(C_{sub}(L)\).
4. Neutral convex sublattices

At first we like to show a connection between the neutrality of \( A \) in \( C_{sub}(L) \) and the neutrality of \( (A) \) in \( I(L) \).

**Theorem 6.** Let \( A \subseteq C_{sub}(L) \). If \( A \) is neutral in \( C_{sub}(L) \), then \( (A) \) is neutral in \( I(L) \).

**Proof.** When \( A \) is neutral, it is also standard, and thus by Theorem 3 we know that \( (A) \) is standard in \( I(L) \). The neutrality of \( (A) \) in \( I(L) \) is proved by showing the dual distributivity of \( (A) \) in \( I(L) \) [2, Theorem III.3.6]. The dual proof of Theorem 1 shows that the dual distributivity of \( A \) in \( C_{sub}(L) \) implies the dual distributivity of \( (A) \) in \( C_{sub}(L) \), and thus the neutrality of \( A \) in \( C_{sub}(L) \) implies the neutrality of \( (A) \) in \( I(L) \).

By modifying the definition of double standardness, the concept of double neutrality in \( C_{sub}(L) \) can be defined and an analogy of Theorem 4 proved. This generalization is obvious, and hence we omit it. Also an analogy of Corollary 2 as well as that of Theorem 5 can be easily presented.

As a last observation of this paper we like to give another immediate generalization. When \( L \) is a distributive lattice then \( (A) \) is distributive in \( I(L) \) as well as \( (A) \) in \( D(L) \) for every \( A \subseteq C_{sub}(L) \). According to Theorem 1 and its dual we see that then \( C_{sub}(L) \) is a distributive \( \chi_{lub} \)-lattice. Conversely, when \( C_{sub}(L) \) is a distributive \( \chi_{lub} \)-lattice, then \( I \lor (J \land K) \supseteq (I \lor J) \land (I \lor K) \) for all three ideals \( I, J, K \) of \( L \) in \( C_{sub}(L) \). Hence \( I(L) \) is a distributive lattice and thus \( L \), too. Accordingly we can write

**Theorem 7.** A lattice \( L \) is distributive if and only if \( C_{sub}(L) \) is a distributive \( \chi_{lub} \)-lattice.
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