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ON APPROXIMATIONS TO FLOQUET SYSTEMS

By A. A. S. Zaghrout and A. A. Ragab

Abstract: A linear system $\dot{x}=A(t)x$, with $A(t+w)=A(t)$ is considered. A step function approximation of a periodic matrix is constructed. The stability criteria is discussed.

1. Introduction

In the mathematical formulation a dynamical system subject to periodic parametric excitation leads to a system of ordinary differential equations with periodic coefficient. There are many literatures on this subject, see for instance [1, 2, 3, 4, 5]. One basic theory is, of course, that of Floquet's which says that the fundamental matrix solution $\Phi(t)$ of the system

$$\dot{x}=A(t)x,\ \ A(t+w)=A(t),\ \ w\in R \tag{1.1}$$

can be expressed as

$$\Phi(t)=P(t)e^{tR},\ \ P(t+w)=P(t),\ \ P(0)=U \tag{1.2}$$

where R is nonsingular constant matrix, and U is the unit matrix, the actual determination of Φ or R, whether analytically or numerically, is not an easy matter. Most analytical methods of stability study involve severe approximations and limitations.

2. Approximations for $\Phi(t)$

Consider a dynamical system for which the equation of motion has been put in the following form

$$\dot{x}(t)=A(t)\,x(t),\ \ x(0)=c, \tag{2.1}$$

where x is n-dimensional vector, $A(t)$ is an n-by-n matrix periodic in t with least positive period w, and c is a constant vector. Let $\Phi(t)$ be the fundamental matrix of (2.1) such that $\Phi(0)=U$. Let

$$B=\Phi(w)=e^{wR}. \tag{2.2}$$

The nonsingular matrix B is called *monodromy matrix*. One has essentially the stability character of the solutions of (2.1) in hand of one can determine B or R. It is well known that there will be stability if and only if all eigenvalues of B have absolute values less than one. The basic idea to approximate $\Phi(t)$ is to divide the period w into n equal and small intervals and to approximate in each interval the system (2.1) by piecewise constant system. The approximating system is constructed in the following manner. Divide each period w into n intervals by t_i, $i=0,1,2,\cdots,n$ with $0=t_0<t_1<t_2<\cdots<t_n=w$. Denote ith interval $[t_{i-1}, t_i]$ by τ_i and its size by $\Delta_i=t_i-t_{i-1}$. In the ith interval replace the coefficient period matrix $A(t)$ by a constant nonsingular matrix C_i which is to have either the value

$$C_i=A(\xi_i),\ \xi_i\in\tau_i, \tag{2.3}$$

or

$$C_i=\frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} A(s)ds. \tag{2.4}$$

Consider the approximating system

$$\dot{y}(t;n)=C(t;n)y(t;n) \tag{2.5}$$

with

$$C(t;n)=\sum_{r=-\infty}^{\infty}\sum_{i=1}^{n} C_i\,[f(t-rw-t_{i-1})-f(t-rw-t_i)], \tag{2.6}$$

where f is the heaviside unit function. The fundamental matrix solution $\Psi(t;n)$ for the system (2.5) with $\Psi(0,\ n)=U$ can be written explicitly as:

$$\Psi(t;n)=\exp(t-t_{n-1})C_n\prod_{i=1}^{n-1}\exp\Delta_i C_i,\ t\in\tau_n. \tag{2.7}$$

It is easy to show that $\Psi(t;n)$ has the extension property (C. F. [4])

$$\Psi(t+iw;n)=\Psi(t;n)\,[\Psi(w;n)]^i \tag{2.8}$$

Equation (2.7) enables us to obtain the monodromy matrix $B_1(n)=\Psi(w;n)$ for the approximating system, i.e.

$$B_1(n)=\prod_{i=1}^{n}\exp\,\Delta_i C_i. \tag{2.9}$$

Since $\Phi(t)$ and $\Psi(t)$ are the fundamental matrices of the systems (2.1) and (2.5) it follows that

$$\dot{\Phi}(t)=A(t)\Phi(t),\ \Phi(0)=U, \tag{2.10}$$

$$\dot{\Psi}(t,\ n)=C(t,\ n)\Psi(t;n),\ \Psi(0,\ n)=U, \tag{2.11}$$

respectively.

Comparing (2.10) and (2.11) and using the definition of $C(t;n)$, (2.6), one could expect that as the number of the intervals n increases, the solution $\Psi(t;n)$ will be a better approximation to $\Phi(t)$ and consequently B_1 is a better approximation to B.

Let

(i) $\Delta = \max_{1 \le i \le n} \Delta_i$,

(ii) Φ and Φ^{-1} be bounded i.e.,

$$m_1 = \sup_{0 \le t \le w} \|\Phi(t)\|, \ m_2 = \sup_{0 \le t \le w} \|\Phi^{-1}(t)\|. \tag{2.12}$$

Now, we are able to prove the following lemma for the limiting case.

LEMMA 1. *$\Psi(t;n) \to \Phi(t)$ as $n \to \infty$, $\Delta \to 0$.*

PROOF. Consider the difference

$$D(t;n) = C(t, n) - A(t).$$

The system (2.11) takes the form:

$$\dot{\Psi}(t;n) = A(t)\Psi(t;n) + D(t;n)\Psi(t;n). \tag{2.13}$$

Solving this equation as nonhomogeneous system and by carrying the norm estimate analysis and using (2.12), we get:

$$\|\Psi(t;n) - \Phi(t)\| \le \int_0^t m_1 m_2 \|D(s;n)\| \, [m_1 + \|\Psi(s;n) - \Phi(s)\|] \, ds.$$

Add m_1 to both sides and applying Gronwall-Reid-Bellman inequality [5], we obtain

$$m_1 + \|\Psi(t;n) - \Phi(t)\| \le m_1 \exp\left\{\int_0^t m_1 m_2 \|D(s, n)\| ds\right\}. \tag{2.14}$$

As $n \to \infty$ (i.e. $\Delta \to 0$), then $\Psi(t, n) \to \Phi(t)$. Thus, we have proved the lemma for $0 \le t \le w$.

The extension of the lemma to all values of $t > 0$ is then made by using the extension property (2.8) and a similar one satisfied by $\Phi(t)$. This completes the proof of the lemma.

REMARK 1. It is clear that in the limiting case ($n \to \infty$, $\Delta \to 0$), the monodromy matrix B of the system (2.1) is the limit of the monodromy matrix of the approximating system (2.5). Also, it is well known that the stability properties of any periodic system are determined by its monodromy matrix.

Lemma 1 merely says that we can study the stability character of the periodic system (2.1) by studying the approximating system (2.5) as $n \to \infty$. In order for the method of approximation to have any practical value we need to show that if n is sufficiently large then the systems (2.1) and (2.5) will have certain stability character in common. Using the definition of $D(t;n)$, the system (2.10) takes the form:

$$\dot{\Phi}(t)=C(t,\ n)\Phi(t)-D(t,\ n)\Phi(t). \tag{2.15}$$

Using Floquet theorem, we have

$$\Psi(t;n)=Q(t;n)e^{tR_1}.$$

The solution of the nonhomogeneous equation (2.15) has the form

$$\Phi(t)=Q(t;n)e^{tR_1}-\int_0^t Q(t,\ n)e^{(t-s)R_1}Q^{-1}(s;n)D(s;n)\Phi(s)ds. \tag{2.16}$$

Since the system (2.5) is asymptotically stable, it follows that all its solutions go to zero as $t\to\infty$, i.e.

(i) $\|e^{tR_1}\|\leq a_1 e^{-C_1 t}$, for some $C_1>0$ (2.17)

(ii) $a_2=\sup\limits_{0\leq t\leq w}\|Q(t,\ n)\|$, $a_3=\sup\limits_{0\leq t\leq w}\|Q^{-1}(t,\ n)\|$. (2.18)

Now, we are able to prove the following:

THEOREM 1. *If the approximating system* (2.5) *is asymptotically stable, and*

$$\frac{1}{t}\int_0^t\|C(s,\ n)-A(s)\|ds<C_1/a_1a_2a_3, \tag{2.19}$$

then the system (2.1) *is asymptotically stable.*

PROOF. Using (2.15), (2.17), (2.18) and carrying out the norm estimate analysis on both sides of (2.16), then, applying Gronwall-Reid-Bellman inequality we obtain

$$\|\Phi(t)\|e^{C_1 t}\leq a_1a_2\ \exp\left\{\int_0^t a_1a_2a_3\|D(s)\|ds\right\}. \tag{2.20}$$

If $\frac{1}{t}\int_0^t\|D(s)\|ds<\frac{C_1}{a_1a_2a_3}$, then $\|\Phi(t)\|\to 0$ as $t\to\infty$. Thus, all solutions of the system (2.1) tends to zero as $t\to\infty$, and hence the system (2.1) is asymptotically stable. This completes the proof of the theorem.

Denote $\varepsilon=\sup\limits_{0\leq t\leq w}\|D(t,\ n)\|$. Then, we have a weaker result (which can be proved similarly).

THEOREM 2. *If the approximating system* (2.5) *is asymptotically stable and*

$$\varepsilon<\frac{C_1}{a_1a_2a_3},$$

then the system (2.1) *is asymptotically stable.*

Now, we process to discuss the converses of theorems 1 and 2. For this purpose, according to Floquet theorem for system (2.1), let $\Phi(t)=P(t)e^{tR}$.

Since the system (2.1) is asymptotically stable, it follows

$$\|\Phi(t)\| \to 0 \text{ as } t \to \infty$$

i. e.,

(i) $|e^{tR}| \leq b_1 e^{-a_2 t}$, $b_1 > 0$, $a_2 > 0$.

(ii) $b_2 = \sup_{0 \leq t \leq w} \|P(t)\|$, $b_3 = \sup_{0 \leq t \leq w} \|P^{-1}(t)\|$.

Consider the fundamental matrix solution $\Psi(t, n)$ of (2.11). Then, by carrying out a norm estimate analysis on $\Psi(t;n)$ similar to that given above for $\Phi(t)$, and applying Gronwall-Reid-Bellman inequality, we can easily prove the following:

THEOREM 3. *If the system* (2.1) *is asymptotically stable and*

$$\frac{1}{t}\int_0^t \|C(s, n) - A(s)\| ds \leq a_2/b_1 b_2 b_3,$$

then the approximating system (2.6) *is asymptotically stable.*

THEOREM 4. *If the system* (2.1) *is asymptotically stable and*

$$\varepsilon < a_2/b_1 b_2 b_3,$$

then the approximating system (2.5) *is asymptotically stable.*

REMARK 2. It is clear that we have established the validity of this approximate method of replacing a general periodic $A(t)$ by a piecewise constant function. For an arbitrary $A(t)$ might be difficult to ascertain the stability of (2.1), but replacing an approximating system is easy to deal with.

REMARK 3. The approximation method used above requires the evaluation of the matrices exp $\Delta_i C_i$ in $B_1(n)$ as shown in (2.9) which is for higher order system is long and tedious.

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Faculty of Science
Al-Azhar University
Nasr City, Cairo
Egypt