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A STUDY ON THE GENERALIZED NONLINEAR COMPLEMENTARITY PROBLEM

CHANG WOOK KIM

1. Introduction

The nonlinear complementarity problem (CP) is well-known. It can be stated as follows.

(CP): Given a mapping f: $R^n_+ \longrightarrow R$, find an n-vector x_0 such that

$$x_0 \in R^n_+,\ f(x_0) \in R^n_+,\ \text{and } \langle x_0, f(x_0)\rangle = 0.$$

Several authors including Eaves([2]), Karamardian([3], [4]), and N. Megiddo and M. Kojima([5]) have studied existence and uniqueness theorems for(CP).

In this paper, we consider the following generalized nonlinear complementarity problem (GCP);

(GCP): Let C be a closed convex cone in R^n and C^* be the positive polar cone of C. Given a mapping f: $C \longrightarrow R^n$, find an n-vector x_0 such that

$$x_0 \in C,\ f(x_0) \in C^* \text{ and } \langle x_0, f(x_0)\rangle = 0,$$

with establishing existence and uniquenss theorems for (GCP).

2. Preliminaries

Let C be a closed convex cone in R, C^* be the positive polar cone of C and $f: C \longrightarrow R^n$ be a mapping.

DEFINITION 2.1. f is said to be *strictly monotone* if $\langle x-y,\ f(x)-f(y)\rangle \geq 0$ for all $x, y \in C$ and strict inequality holds whenever $x \neq y$.

DEFINITION 2.2. f is called *strongly monotone* if there is a constant $c > 0$ such that $\langle x-y,\ f(x)-f(y)\rangle \geq c\|x-y\|^2$, for all $x, y \in C$.

DEFINITION 2.3. f is said to be *Lipschitzian* if there is a constant $k > 0$ such that $\|f(x)-f(y)\| \leq k\|x-y\|$ for all $x, y \in C$.

DEFINITION 2.4. f is said to be *hemicontinuous* if for all $x, y \in C$, the map $t \longrightarrow f([ty+(1-t)x]$ of $[0, 1]$ to R^n is contimuous.

DEFINITION 2.5. f is said to be *bounded* if there is a constant $k > 0$ such that $\|f(x)\| \leq k\|x\|$ for all $x \in C$.

LEMMA 2.1 ([1]). Let $f: C \longrightarrow R^n$ be hemicontinuous, strictly monotone and bounded and let $\{V_r\}$ be a family of nonempty closed convex sets in C. Then, for each r, there is a unique $x_r \in V_r$ such that $\langle x\ ,\ f(x_r)\rangle \leq \langle z,\ f(x_r)\rangle$ for all $z \in V_r$.

3. Main Results

Now we established existence and uniqueness theorems for (GCP) under certain assumptions.

THEOREM 3.1. *Let $f: C \longrightarrow R^n$ be hemicontinuous, strictly monotone and bounded. Then 0 is the unique solution of (GCP).*

PROOF. For each $r \geq 0$, we write $B_r = \{x \in C : \|x\| \leq r\}$. B_r is a nonempty closed set in C.

By Lemma 2.1, for each $r \geq 0$ there is a unique $x_r \in B_r$ such that $\langle x_r, f(x_r)\rangle \leq \langle z, f(x_r)\rangle$ for all $z \in B_r$. Since $0 \in B_r$, $\langle x_r, f(x_r)\rangle \leq 0$. We can define a function θ from $[0, \infty)$ to $(-\infty, 0]$ by the rule $\theta(r) = \langle x_r, f(x_r)\rangle$. Now suppose that $r \neq 0$ and $r < s$. Then there are unique $x_r \in B_r$ and $x_s \in B_s$ such that

$$\langle x_r, f(x_r)\rangle \leq \langle z, f(x_r)\rangle \text{ for all } z \in B_r,$$

and

$$\langle x_s, f(x_s)\rangle \leq \langle z, f(x_s)\rangle \text{ for all } z \in B_s.$$

Since $(r/s)x_s \in B_r$, $\langle x_r, f(x_r)\rangle \leq (r/s)\langle x_s, f(x_r)\rangle$. Since $(s/r)x_r \in B_s$, $\langle x_s, f(x_s)\rangle \leq (s/r)\langle x_r, f(x_s)\rangle$. Hence we have

$$\begin{aligned}\langle x_r - x_s, f(x_r)\rangle &= \langle x_r, f(x_r)\rangle + \langle x_s, f(x_s)\rangle \\ &\quad - \langle x_s, f(x_r)\rangle - \langle x_r, f(x_s)\rangle\end{aligned}$$

$$\begin{aligned}&\leq\langle x_r,\ f(x_r)\rangle+\langle x_s,\ f(x_s)\rangle\\&\quad-(s/r)\langle x_r,\ f(x_r)\rangle-(r/s)\langle x_s,\ f(x_s)\rangle\\&=[1-(s/r)]\theta(r)+[1-(r/s)]\theta(s)\\&=(s-r)\{[\theta(s)/s]-[\theta(r)/r]\}\end{aligned}$$

Since $s>r$ and f is monotone, $\theta(s)/s\geq\theta(r)/r$. Therefore $\theta(r)/r$ is montonically increasing on $(0,\infty)$. Since f is bounded, $|\theta(r)|=\langle x_r,\ f(x_r)\rangle\leq||x_r||\cdot||f(x_r)||\leq k||x_r||^2$. Hence $|\theta(r)|\leq k_r^2$. Since $\theta(r)<0$, $-\theta(r)\leq k_r^2$, Consequently, $-kr<\theta(r)/r\leq 0$ for all $r\in(0,\infty)$. Since $\lim_{r\to 0+}[\theta(r)/r]=0$ and $\theta(r)/r$ is monotonically increasing, it follows that $\theta(r)=0$ and hence $\theta(r)=0$ for all $r\in(0,\infty)$. So we have $\langle z,\ f(x_r)\geq\rangle 0$ for all $z\in B_r$. Since C is a cone, $\langle z,\ f(x_r)\rangle\geq 0$ for all $z\in C$. Therefore, for each $r\in(0,\infty)$, x_r is a solution of (GCP). Now f is strictly monotone, (GCP) can have at most one solution, say x_0. $x_0=x_r\in B_r$ for each r and $||x_0||=||x_r||\leq r$ for each r. So $x_0=0$.

COROLLARY 3.1. Let $f:R^n_+\longrightarrow R$ be hemicontinuous, strictly monotone and bounded. Then 0 is the unique solution of (CP).

PROOF. R^n_+ is a closed convex cone in R^n. By Theorem 3.1, the above result holds.

THEOREM 3.2. Let $f:C\to R^n$ be strongly monotone and Lipschitzian with $k^2<2c<k^2+1$. Then there is the unique solution of (GCP).

PROOF. Since C is a nonempty closed convex set in R^n, for every $x\in C$ there is a unique $y\in C$ closest to

$x-f(x)$; that is $||y-x+f(x)|| \leq ||z-x+f(x)||$ for all $z \in C$.

Let the correspondence $x \longrightarrow y$ be denoted by θ. Let z be any element of C and let $0 \leq \lambda \leq 1$.

Since C is convex, $(1-\lambda)y+\lambda z \in C$. We define a map h: $[0, 1] \longrightarrow R_+$ by the rule

$$h(\lambda)=||x-f(x)-(1-\lambda)y-\lambda z||^2.$$

Then h is a twice continuously differentiable function of λ and $h'(\lambda)=2\langle x-f(x)-\lambda z-(1-\lambda)y, \ y-z\rangle$. Since y is the unique element closet to $x-f(x)$, $h'(a) \geq 0$. So we have

(1) $\langle x-f(x)-y, \ y-z\rangle \geq 0$ for all $z \in C$.

Let x_1 and x_2 be two elements of C with $x_1 \neq x_2$. Put $\theta(x_1)=y_1$ and $\theta(x_2)=y_2$. From (1), we get

$$\langle x_1-f(x_1)-\theta(x_1), \ \theta(x_1)-\theta(x_2)\rangle \geq 0$$

and

$$\langle x_2-f(x_2)-\theta(x_2), \ \theta(x_2)-\theta(x_1)\rangle \geq 0.$$

From these two inequalities, we have

$$\langle x_1-f(x_1)-\theta(x_1)-x_2+f(x_2)+\theta(x_2), \\ \theta(x_1)-\theta(x_2)\rangle \geq 0.$$

Hence,

$$\begin{aligned} &\langle x_1-f(x_1)-x_2+f(x_2), \theta(x_1)-\theta(x_2)\rangle \\ &\geq \langle \theta(x_1)-\theta(x_2), \ \theta(x_1)-\theta(x_2)\rangle \\ &=||\theta(x_1)-\theta(x_2)||^2. \end{aligned}$$

Therefore,

$$\begin{aligned}\|\theta(x_1)-\theta(x_2)\|^2 &\leq |\langle x_1-f(x_1)-x_2+f(x_2),\\ &\quad \theta(x_1)-\theta(x_2)\rangle|\\ &\leq \|x_1-f(x_1)-x_2+f(x_2)\|\cdot\\ &\quad \|\theta(x_1)-\theta(x_2)\|\end{aligned}$$

Thus, $\|\theta(x_1)-\theta(x_2)\| \leq \|f(x_1)-f(x_2)-x_1+x_2\|$.

Since f is strongly monotone and Lipschitzian, we have

$$\begin{aligned}\|\theta(x_1)-\theta(x_2)\|^2 &\leq f(x_1)-f(x_2)x_1+x_2\|^2\\ &= \langle f(x_1)-f(x_2)-x_1+x_2,\\ &\quad f(x_1)-f(x_2)-x_1+x_2\rangle\\ &= \|f(x_1)-f(x_2)\|^2+\|x_1-x_2\|^2\\ &\quad -2\langle x_1-x_2,\ f(x_1)-f(x_2)\rangle\\ &\leq k^2\|x_1-x_2\|^2+\|x_1-x_2\|^2\\ &\quad -2c\|x_1-x_2\|^2\\ &= (k^2+1-2c)\|x_1-x_2\|^2\end{aligned}$$

Since $k^2 < 2c < k^2+1$, we have $0 < k^2+1-2c < 1$.

Letting $\alpha = k^2+1-2c$ in the above inequality, we obtain $\|\theta(x_1)-\theta(x_2)\| \leq \alpha\|x_1-x_2\|$ with $0 < \alpha < 1$.

By the Banach contraction principle, θ has the unique fixed point, say x_0. Now putting $x=x_0$ in (1), We get $\langle z-x_0,\ f(x_0)\rangle > 0$ for all $z \in C$. Since $0 \in C$, $\langle x_0,\ f(x_0)\rangle \leq 0$. Since C is a cone, $2x_0 \in C$ and $\langle x_0,\ f(x_0)\rangle \geq 0$. So $\langle x_0,\ f(x_0)\rangle = 0$ and $\langle z,\ f(x_0)\rangle > 0$ for all $z \in C$. Therefore, x_0 is the unique solutionof (GCP).

COROLLARY 3.2. Let $f\colon R^n_+ \longrightarrow R$ be strongly monotone and Lipschitzian with $k^2 < 2c < k^2+1$. Then there is the unique solution of (CP).

PROOF. R^n_+ is a closed convex cone in R. By Theorem 3.1, the above result holds.

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Korea Maritime University
Pusan 606
Korea

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