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ON SUBCLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

In this work we obtain inequalities for coefficients related to functions belonging to two subclasses of univalent functions with negative coefficients, bounds for the modulus of these functions and their derivatives, and the extreme points of these classes. We also show an application of functions belonging to these classes to the fractional calculus.

1. Introduction

Let A denote the class of functions of the form

$$(1.1)\quad f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

which are analytic in the unit disk $E=\{z:|z|<1\}$. Then a function $f(z)$ belonging to A is said to be *starlike of order* α if and only if

$$(1.2)\quad Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha$$

for some α $(0 \leqq \alpha < 1)$, and for all $z \in E$. We denote by $S^*(\alpha)$ the subclass of A consisting of all starlike functions of order α in the unit disk E. Further, a function $f(z)$ belonging to A is said to be *convex of order* α if and only if

$$(1.3) \quad Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha$$

for some α $(0 \leqq \alpha < 1)$, and for all $z \in E$. Also we denote by $K(\alpha)$ the subclass of A consisting of functions which are convex of order α in the unit disk E.

Let T be the subclass of A consisting of the form

$$(1.4) \qquad f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0).$$

Further, let

$$(1.5) \qquad T^*(\alpha)=S^*(\alpha)\cap T \qquad (0 \leqq \alpha < 1)$$

and

$$(1.6) \qquad C(\alpha)=K(\alpha)\cap T \qquad (0 \leqq \alpha < 1).$$

The classes $T^*(\alpha)$ and $C(\alpha)$ were introduced by Silverman [3].

Let $T(\lambda, \alpha)$ be the subclass of T consisting of functions which satisfy the condition

$$(1.7) \qquad Re\left\{\frac{zf'(z)}{\lambda z f'(z)+(1-\lambda) f(z)}\right\}>\alpha$$

for some α $(0 \leqq \alpha < 1)$, λ $(0 \leqq \lambda < 1)$, and for all $z \in E$.

Let $C(\lambda, \alpha)$ denote the subclass of T consisting of all functions satisfying the following condition

$$(1.8) \qquad Re\left\{\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf''(z)}\right\}>\alpha$$

for some α $(0\leqq\alpha<1)$, λ $(0\leqq\lambda<1)$, and for all $z\in E$.

Note that $T(0,\alpha)=T^*(\alpha)$ and $C(0,\alpha)=C(\alpha)$, and that $f(z)\in C(\lambda,\alpha)$ if and only if $zf'(z)\in T(\lambda,\alpha)$.

Silverman [3] has obtained coefficient inequalities, extreme points for the classes $T(0,\alpha)$ and $C(0,\alpha)$, the radii of convexity of functions in the class $T(0,\alpha)$, and the order of of starlikeness of functions in $C(0,\alpha)$.

In the present paper, the results by Silverman [3] are generalized to the classes $T(\lambda,\alpha)$ and $C(\lambda,\alpha)$.

2. Modulus of function in $T(\lambda,\alpha)$ and $C(\lambda,\alpha)$

We begin with the statement and the proof of the following result.

THEOREM 1. A function $f(z)$ defined by (1.4) is in the class $T(\lambda,\alpha)$ if and only if

$$(2.1) \qquad \sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)a_n\leqq 1-\alpha.$$

The result (2.1) is sharp.

PROOF. Suppose that $f(z)\in T(\lambda,\alpha)$. Then we have from (1.7) that

$$(2.2) \qquad Re\left\{\frac{zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\right\}$$

$$=Re\left\{\frac{1-\sum_{n=2}^{\infty}na_nz^{n-1}}{1-\sum_{n=2}^{\infty}(\lambda n+1-\lambda)a_nz^{n-1}}\right\}$$

$$>\alpha.$$

If we choose z real and let $z\to 1^-$, we get

$$\text{(2.3)} \qquad \frac{1-\sum_{n=2}^{\infty} na_n}{1-\sum_{n=2}^{\infty}(\lambda n+1-\lambda)a_n} \geqq \alpha$$

which is equivalent to (2.1).

Conversely, suppose that (2.1) holds true. Then, adding

$$-(1-\alpha)\sum_{n=2}^{\infty}(\lambda n+1-\lambda)a_n$$

to both sides of (2.1), we obtain

$$\text{(2.4)} \qquad (1-\lambda)\sum_{n=2}^{\infty}(n-1)a_n \leqq (1-\alpha)\left(1-\sum_{n=2}^{\infty}(\lambda n+1-\lambda)a_n\right).$$

On the other hand, we see that

$$\text{(2.5)} \qquad \left|\frac{zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right|$$

$$=\left|\frac{(1-\lambda)\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}(\lambda n+1-\lambda)a_n z^{n-1}}\right|$$

$$\leqq \frac{(1-\lambda)\sum_{n=2}^{\infty}(n-1)a_n}{1-\sum_{n=2}^{\infty}(\lambda n+1-\lambda)a_n}.$$

It follows from (2.4) that the last expression in (2.5) is bounded above by $(1-\alpha)$. This implies $f(z)\in T(\lambda,\alpha)$.

Finally, taking the function

$$\text{(2.6)} \qquad f(z)=z-\frac{1-\alpha}{n-\lambda\alpha n-\alpha+\lambda\alpha}z^n \qquad (n\geqq 2),$$

We can show that the result (2.1) is sharp.

COROLLARY 1. If $f(z) \in T(\lambda, \alpha)$, then

$$a_n \leqq \frac{1-\alpha}{n-\lambda\alpha n-\alpha+\lambda\alpha} \qquad (n \geqq 2). \tag{2.7}$$

The equality in (2.7) holds for the function $f(z)$ defined by (2.6).

THEOREM 2. A function $f(z)$ defined by (1.4) is in the class $C(\lambda, \alpha)$ if and only if

$$\sum_{n=2}^{\infty} n(n-\lambda\alpha n-\alpha+\lambda\alpha)a_n \leqq 1-\alpha. \tag{2.8}$$

The result (2.8) is sharp.

PROOF. Note that $f(z) \in C(\lambda, \alpha)$ if and only if $zf'(z) \in T(\lambda, \alpha)$. Hence, replacing a_n by na_n in Theorem 1, we have the inequality (2.8). Furthermore, the result (2.8) is sharp for the function

$$f(z) = z - \frac{1-\alpha}{n(n-\lambda\alpha n-\alpha+\lambda\alpha)} z^n \qquad (n \geqq 2). \tag{2.9}$$

COROLLARY 2. If $f(z) \in C(\lambda, \alpha)$, then

$$a_n \leqq \frac{1-\alpha}{n(n-\lambda\alpha n-\alpha+\lambda\alpha)} \qquad (n \geqq 2). \tag{2.10}$$

The equality in (2.10) holds for the function $f(z)$ defined by (2.9).

Applying Theorem 1, we prove

THEOREM 3. If $f(z) \in T(\lambda, \alpha)$, then

$$r - \frac{1-\alpha}{2-\lambda\alpha-\alpha} r^2 \leqq |f(z)| \leqq r + \frac{1-\alpha}{2-\lambda\alpha-\alpha} r^2 \tag{2.11}$$

and

$$(2.12)\qquad 1-\frac{2(1-\alpha)}{2-\lambda\alpha-\alpha}r \leqq |f'(z)| \leqq 1+\frac{2(1-\alpha)}{2-\lambda\alpha-\alpha}r$$

for $|z|=r<1$. The equalities in (2.11) and (2.12) hold for the function

$$(2.13)\qquad f(z)=z-\frac{1-\alpha}{2-\lambda\alpha-\alpha}z^2.$$

PROOF. In view of Theorem 1, we have

$$(2.14)\qquad (2-\lambda\alpha-\alpha)\sum_{n=2}^{\infty}a_n \leqq \sum_{n=2}^{\infty}(n-\lambda\alpha n+\lambda\alpha-\alpha)a_n \leqq 1-\alpha$$

or

$$(2.15)\qquad \sum_{n=2}^{\infty}a_n \leqq \frac{1-\alpha}{2-\lambda\alpha-\alpha}.$$

Using (2.15), we obtain

$$(2.16)\qquad |f(z)| \leqq |z|+|z|^2\sum_{n=2}^{\infty}a_n \leqq r+\frac{1-\alpha}{2-\lambda\alpha-\alpha}r^2$$

and

$$(2.17)\qquad |f(z)| \geqq |z|-|z|^2\sum_{n=2}^{\infty}a_n \geqq r-\frac{1-\alpha}{2-\lambda\alpha-\alpha}r^2.$$

Next, it follows from (2.1) and (2.15) that

$$(2.18)\qquad (1-\lambda\alpha)\sum_{n=2}^{\infty}na_n \leqq 1-\alpha+(1-\lambda)\alpha\sum_{n=2}^{\infty}a_n$$

or

$$(2.19)\qquad \sum_{n=2}^{\infty}na_n \leqq \frac{2(1-\alpha)}{2-\lambda\alpha-\alpha}.$$

This derives that

$$(2.20)\qquad |f'(z)| \leqq 1+|z|\sum_{n=2}^{\infty}na_n \leqq 1+\frac{2(1-\alpha)}{2-\lambda\alpha-\alpha}r$$

and

$$(2.21)\qquad |f'(z)| \geqq 1-|z|\sum_{n=2}^{\infty} na_n \geqq 1-\frac{2(1-\alpha)}{2-\lambda\alpha-\alpha}r.$$

Further, with the help of Theorem 2, we have

THEOREM 4. If $f(z)\in C(\lambda,\alpha)$, then

$$(2.22)\qquad r-\frac{1-\alpha}{2(2-\lambda\alpha-\alpha)}r^2 \leqq |f(z)| \leqq r+\frac{1-\alpha}{2(2-\lambda\alpha-\alpha)}r^2$$

and

$$(2.23)\qquad 1-\frac{1-\alpha}{2-\lambda\alpha-\alpha}r \leqq |f'(z)| \leqq 1+\frac{1-\alpha}{2-\lambda\alpha-\alpha}r^2$$

for $|z|=r<1$. The equalities in (2.22) and (2.23) hold for the function

$$(2.24)\qquad (z)=z-\frac{1-\alpha}{2(2-\lambda\alpha-\alpha)}z^2.$$

PROOF. Since $f(z)\in C(\lambda,\alpha)$ if and only if $zf'(z)\in T(\lambda,\alpha)$, (2.23) follows from (2.11). By using (2.8), we have

$$(2.25)\qquad \sum_{n=2}^{\infty} a_n \leqq \frac{1-\alpha}{2(2-\lambda\alpha-\alpha)}.$$

Therefore, we can see that

$$(2.26)\qquad |f(z)| \leqq |z|+|z|^2\sum_{n=2}^{\infty} a_n \leqq r+\frac{1-\alpha}{2(2-\lambda\alpha-\alpha)}r^2$$

and

$$(2.27)\qquad |f(z)| \geqq |z|-|z|^2\sum_{n=2}^{\infty} a_n \geqq r-\frac{1-\alpha}{2(2-\lambda\alpha-\alpha)}r^2.$$

3. Properties of $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$

By using Theorem 1 and Theorem 2, we prove

THEOREM 5. If $f(z) \in C(\lambda, \alpha)$, then

$$f(z) \in T\left(\lambda, \frac{2-2\lambda\alpha}{3-\lambda\alpha-\lambda-\alpha}\right).$$

The result is sharp with the extremal function $f(z)$ given by (2.24).

PROOF. By virtue of Theorem 1 and Theorem 2, we have to show that if

$$\sum_{n=2}^{\infty} \frac{n(n-\lambda\alpha n+\lambda\alpha-\alpha)}{1-\alpha} a_n \leqq 1 \tag{3.1}$$

then

$$\sum_{n=2}^{\infty} \frac{n-(\lambda n-\lambda+1)\{(2-2\lambda\alpha)/(3-\lambda\alpha-\lambda-\alpha)\}}{1-(2-2\lambda\alpha)/(3-\lambda\alpha-\lambda-\alpha)} a_n \leqq 1. \tag{3.2}$$

In order to show the above, we must have the following inequality

$$\frac{n-(\lambda n-\lambda+1)\{(2-2\lambda\alpha)/(3-\lambda\alpha-\lambda-\alpha)\}}{1-(2-2\lambda\alpha)/(3-\lambda\alpha-\lambda-\alpha)} \leqq \frac{n(n-\lambda\alpha n+\lambda\alpha-\alpha)}{1-\alpha} \tag{3.3}$$

for $n \geqq 2$. But the above inequality is equivalent to

$$(1-\lambda)(1-\lambda\alpha)(n^2-3n+2) \geqq 0 \qquad (n \geqq 2) \tag{3.4}$$

which is always true. Hence we complete the proof of Theorem 5.

Next, we prove

THEOREM 6. If $f(z)\in T(\lambda,\alpha)$, then $f(z)\in C(\lambda,0)$ in the disk $|z|<r_1<1$, where

$$r_1=\inf_{n\geqq 2}\left(\frac{n-\lambda\alpha n+\lambda\alpha-\alpha}{n^2(1-\alpha)}\right)^{1/(n-1)}. \tag{3.5}$$

The result is sharp with the extremal function $f(z)$ given by (2.6).

PROOF. With the aid of Theorem 1, we have

$$\sum_{n=2}^{\infty}\frac{n-\lambda\alpha n+\lambda\alpha-\alpha}{1-\alpha}a_n\leqq 1 \tag{3.6}$$

for $f(z)\in T(\lambda,\alpha)$. It is clear that $f(z)\in C(\lambda,0)$ if

$$\left|\frac{(1-\lambda)zf''(z)}{f'(z)+\lambda zf''(z)}\right|<1. \tag{3.7}$$

If

$$\frac{(1-\lambda)\sum_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}n(1+\lambda n-\lambda)a_n|z|^{n-1}}\leqq 1 \tag{3.8}$$

or

$$\sum_{n=2}^{\infty}n^2a_n|z|^{n-1}\leqq 1, \tag{3.9}$$

then (3.7) holds true. Therefore, we see that (3.7) is satisfied if

$$n^2|z|^{n-1}\leqq\frac{n-\lambda\alpha n+\lambda\alpha-\alpha}{1-\alpha} \tag{3.10}$$

or

$$|z|\leqq\left(\frac{n-\lambda\alpha n+\lambda\alpha-\alpha}{1-\alpha}\right)^{1/(n-1)}. \tag{3.11}$$

THEOREM 7. Let

$$f_1(z)=z \tag{3.12}$$

and

$$f_n(z)=z-\frac{1-\alpha}{n-\lambda\alpha n+\lambda\alpha-\alpha}z^n \quad (n\geqq 2). \tag{3.13}$$

Then $f(z)\in T(\lambda,\alpha)$ if and only if

$$f(z)=\sum_{n=1}^{\infty}k_n f_n(z), \tag{3.14}$$

where $k_n\geqq 0$ and $\sum_{n=1}^{\infty}k_n=1$.

PROOF. Suppose that (3.14) is satisfied. Then we have

$$\begin{aligned} f(z)&=z-\sum_{n=2}^{\infty}k_n\frac{1-\alpha}{n-\lambda\alpha n+\lambda\alpha-\alpha}z^n \\ &=z-\sum_{n=2}^{\infty}a_n z^n, \end{aligned} \tag{3.15}$$

where

$$a_n=k_n\frac{1-\alpha}{n-\lambda\alpha n+\lambda\alpha-\alpha} \quad (n\geqq 2). \tag{3.16}$$

It follows from (3.16) that

$$\sum_{n=2}^{\infty}\frac{n-\lambda\alpha n+\lambda\alpha-\alpha}{1-\alpha}a_n=\sum_{n=2}^{\infty}k_n=1-k_1\leqq 1. \tag{3.17}$$

This implies that $f(z)\in T(\lambda,\alpha)$ with the help of Theorem 1.

Conversely, suppose that $f(z)\in T(\lambda,\alpha)$. Then, by Theorem 1, we may put

$$k_n=\frac{n-\lambda\alpha n+\lambda\alpha-\alpha}{1-\alpha}a_n \quad (n\geqq 2) \tag{3.18}$$

and

(3.19) $$k_1=1-\sum_{n=2}^{\infty}k_n.$$

Consequently, we can see that

(3.20) $$\begin{aligned} f(z) &= z-\sum_{n=2}^{\infty}a_n z^n \\ &= z-\sum_{n=2}^{\infty}k_n\frac{1-\alpha}{n-\lambda\alpha n+\lambda\alpha-\alpha}z^n \\ &= \sum_{n=1}^{\infty}k_n f_n(z) \end{aligned}$$

which completes the proof of Theorem 7.

With the help of Theorem 7, we have

COROLLARY 3. The extreme points of $T(\lambda,\alpha)$ are the functions $f_n(z)$ $(n\geqq 1)$ defined in Theorem 7.

Similarly, we have

COROLLARY 4. The extreme points of $C(\lambda,\alpha)$ are the functions $f_1(z)=z$ and

(3.21) $$f_n(z)=z-\frac{1-\alpha}{n(n-\lambda\alpha n+\lambda\alpha-\alpha)}z^n \qquad (n\geqq 2).$$

4. Application to the fractional calculus

The following definitions of the fractional calculus (fractional integrals and fractional derivatives) due to Owa ([1], [2]).

DEFINITION 1. The *fractional integral of order* δ is defined by

(4.1) $$D_z^{-\delta}f(z)=\frac{1}{\Gamma(\delta)}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{1-\delta}}d\zeta,$$

where $\delta > 0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

DEFINITION 2. The *fractional derivative of order* δ is defined by

$$(4.2)\quad D_z^{\delta} f(z)=\frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\delta}} d\zeta,$$

where $0 \leqq \delta < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containg the origin, and the multiplicity of $(z-\zeta)^{-\delta}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

DEFINITION 3. Under the hypotheses of Definition 2, the *fractional derivative of order* $(n+\delta)$ is defined by

$$(4.3)\qquad D_z^{n+\delta} f(z)=\frac{d^n}{dz^n} D_z^{\delta} f(z),$$

where $0 \leqq \delta < 1$ and $n \in N_0=\{0, 1, 2, \cdots\}$.

Applying the above definitions, we prove

THEOREM 8. If $f(z) \in T(\lambda, \alpha)$, then

$$(4.4)\qquad |D_z^{-\delta} f(z)| \geqq \frac{r^{1+\delta}}{\Gamma(2+\delta)}\left\{1-\frac{2(1-\alpha)}{(2+\delta)(2-\lambda\alpha-\alpha)} r\right\}$$

and

$$(4.5)\qquad |D_z^{-\delta} f(z)| \leqq \frac{r^{1+\delta}}{\Gamma(2+\delta)}\left\{1+\frac{2(1-\alpha)}{(2+\delta)(2-\lambda\alpha-\alpha)} r\right\}$$

for $\delta > 0$ and $|z|=r<1$. The equalities in (4.4) and (4.5) are attained for the function $f(z)$ given by (2.13).

PROOF. It is easy to see that

$$\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}a_n z^n. \tag{4.6}$$

Defining $\phi(n)$ by

$$\phi(n)=\frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} \qquad (n\geqq 2), \tag{4.7}$$

we know that $\phi(n)$ is a decreasing function of n, that is, that

$$0<\phi(n)\leqq\phi(2)=\frac{2}{2+\delta}. \tag{4.8}$$

Thus, by using (2.15) and (4.8), we have

$$\begin{aligned}|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)|| &\geqq |z|-\phi(2)|z|^2\sum_{n=2}^{\infty}a_n \\ &\geqq r-\frac{2(1-\alpha)}{(2+\delta)(2-\lambda\alpha-\alpha)}r^2\end{aligned} \tag{4.9}$$

which gives (4.4), and

$$\begin{aligned}|\Gamma(2+\delta)\,z^{-\delta}D_z^{-\delta}f(z)| &\leqq |z|+\phi(2)|z|^2\sum_{n=2}^{\infty}a_n \\ &\leqq r+\frac{2(1-\alpha)}{(2+\delta)(2-\lambda\alpha-\alpha)}r^2\end{aligned} \tag{4.10}$$

which shows (4.5). Further, the equalities in (4.4) and (4.5) are attained for the function $f(z)$ defined by

$$D_z^{-\delta}f(z)=\frac{z^{1+\delta}}{\Gamma(2+\delta)}\left\{1-\frac{2(1-\alpha)}{(2+\delta)(2-\lambda\alpha-\alpha)}z\right\}, \tag{4.11}$$

or, defined by (2.13).

Using the same manner with (2.25), we have

THEOREM 9. If $f(z)\in C(\lambda,\alpha)$, then

$$|D_z^{-\delta}f(z)|\geqq\frac{r^{1+\delta}}{\Gamma(2+\delta)}\left\{1-\frac{1-\alpha}{(2+\delta)(2-\lambda\alpha-\alpha)}r\right\} \tag{4.12}$$

and

$$(4.13)\qquad |D_z^{-\delta}f(z)|\leqq\frac{r^{1+\delta}}{\Gamma(2+\delta)}\left\{1+\frac{1-\alpha}{(2+\delta)(2-\lambda\alpha-\alpha)}r\right\}$$

for $\delta>0$ and $|z|=r<1$. The equalities in (4.12) and (4.13) are attained for the function $f(z)$ given by (2.24).

Next, we prove

THEOREM 10. If $f(z)\in T(\lambda,\alpha)$, then

$$(4.14)\qquad |D_z^{\delta}f(z)|\geqq\frac{r^{1-\delta}}{\Gamma(2-\delta)}\left\{1-\frac{2(1-\alpha)}{(2-\delta)(2-\lambda\alpha-\alpha)}r\right\}$$

and

$$(4.15)\qquad |D_z^{\delta}f(z)|\leqq\frac{r^{1-\delta}}{\Gamma(2-\delta)}\left\{1+\frac{2(1-\alpha)}{(2-\delta)(2-\lambda\alpha-\alpha)}r\right\}$$

for $0\leqq\delta<1$ and $|z|=r<1$. The equalities in (4.14) and (4.15) are attained for the function $f(z)$ given by (2.13).

PROOF. It is clear that

$$(4.16)\qquad \Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}a_nz^n$$

$$=z-\sum_{n=2}^{\infty}\phi(n)na_nz^n,$$

where

$$(4.17)\qquad \phi(n)=\frac{\Gamma(n)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}.$$

Since $\phi(n)$ is decreasing in n, we have

$$(4.18)\qquad 0<\phi(n)\leqq\phi(2)=\frac{1}{2-\delta}.$$

Therefore, with the aid of (2.19) and (4.18), we show that

$$(4.19)\qquad |\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)|\geqq|z|-\phi(2)\,|z|^2\sum_{n=2}^{\infty}na_n$$
$$\geqq r-\frac{2(1-\alpha)}{(2-\delta)(2-\lambda\alpha-\alpha)}r^2$$

which is equivalent to (4.14), and

$$(4.20)\qquad |\Gamma(2-\delta)z^{\delta}D_z^{\delta}f(z)|\leqq|z|+\phi(2)|z|^2\sum_{n=2}^{\infty}na_n$$
$$\leqq r+\frac{2(1-\alpha)}{(2-\delta)(2-\lambda\alpha-\alpha)}r^2$$

which gives (4.15). Furthermore, since the equalities in (4.14) and (4.15) are attained for the function $f(z)$ defined by

$$(4.21)\qquad D_z^{\delta}f(z)=\frac{z^{1-\delta}}{\Gamma(2-\delta)}\left\{1-\frac{2(1-\alpha)}{(2-\delta)(2-\lambda\alpha-\alpha)}z\right\},$$

we complete the proof of Theorem 10.

REMARK 1. Letting $\delta=0$ in Theorem 10, we have (2.11) of Theorem 3, and letting $\delta\to1$ in Theorem 10, we have (2.12) of Theorem 3.

Finally, we have

THEOREM 11. If $f(z)\in C(\lambda,\alpha)$, then

$$(4.22)\qquad |D_z^{\delta}f(z)|\geqq\frac{r^{1-\delta}}{\Gamma(2-\delta)}\left\{1-\frac{1-\alpha}{(2-\delta)(2-\lambda\alpha-\alpha)}r\right\}$$

and

$$(4.23)\qquad |D_z^{\delta}f(z)|\leqq\frac{r^{1-\delta}}{\Gamma(2-\delta)}\left\{1+\frac{1-\alpha}{(2-\delta)(2-\lambda\alpha-\alpha)}r\right\}$$

for $0\leqq\delta<1$ and $|z|=r<1$. The equalities in (4.22) and (4.23) are attained for the function $f(z)$ given by (2.24).

REMARK 2. Taking $\delta=0$, Theorem 11 becomes (2.22) of

Theorem 4. Further, letting $\delta \to 1$ in Theorem 11, we have (2.23) of Theorem 4.

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