ON SUBCLASSES OF UNIVALENT FUNCTIONS
WITH NEGATIVE COEFFICIENTS

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Abstract

In this work we obtain inequalities for coefficients related to functions belonging to two subclasses of univalent functions with negative coefficients, bounds for the modulus of these functions and their derivatives, and the extreme points of these classes. We also show an application of functions belonging to these classes to the fractional calculus.

1. Introduction

Let $A$ denote the class of functions of the form

(1.1) $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. Then a function $f(z)$ belonging to $A$ is said to be starlike of order $\alpha$ if and only if

(1.2) $Re \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha$
for some \( 0 \leq \alpha < 1 \), and for all \( z \in E \). We denote by \( S^*(\alpha) \) the subclass of \( A \) consisting of all starlike functions of order \( \alpha \) in the unit disk \( E \). Further, a function \( f(z) \) belonging to \( A \) is said to be *convex of order* \( \alpha \) if and only if

\[
(1.3) \quad \Re\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha
\]

for some \( 0 \leq \alpha < 1 \), and for all \( z \in E \). Also we denote by \( K(\alpha) \) the subclass of \( A \) consisting of functions which are convex of order \( \alpha \) in the unit disk \( E \).

Let \( T \) be the subclass of \( A \) consisting of the form

\[
(1.4) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).
\]

Further, let

\[
(1.5) \quad T^*(\alpha) = S^*(\alpha) \cap T \quad (0 \leq \alpha < 1)
\]

and

\[
(1.6) \quad C(\alpha) = K(\alpha) \cap T \quad (0 \leq \alpha < 1).
\]

The classes \( T^*(\alpha) \) and \( C(\alpha) \) were introduced by Silverman [3].

Let \( T(\lambda, \alpha) \) be the subclass of \( T \) consisting of functions which satisfy the condition

\[
(1.7) \quad \Re\left\{ \frac{zf''(z)}{\lambda zf''(z) + (1-\lambda)f(z)} \right\} > \alpha
\]

for some \( 0 \leq \alpha < 1 \), \( \lambda \) \( (0 \leq \lambda < 1) \), and for all \( z \in E \).

Let \( C(\lambda, \alpha) \) denote the subclass of \( T \) consisting of all functions satisfying the following condition
for some $\alpha$ ($0 \leq \alpha < 1$), $\lambda$ ($0 \leq \lambda < 1$), and for all $z \in \mathbb{D}$.

Note that $T(0, \alpha) = T^*(\alpha)$ and $C(0, \alpha) = C(\alpha)$, and that $f(z) \in C(\lambda, \alpha)$ if and only if $z f'(z) \in T(\lambda, \alpha)$.

Silverman [3] has obtained coefficient inequalities, extreme points for the classes $T(0, \alpha)$ and $C(0, \alpha)$, the radii of convexity of functions in the class $T(0, \alpha)$, and the order of starlikeness of functions in $C(0, \alpha)$.

In the present paper, the results by Silverman [3] are generalized to the classes $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$.

2. Modulus of function in $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$

We begin with the statement and the proof of the following result.

**Theorem 1.** A function $f(z)$ defined by (1.4) is in the class $T(\lambda, \alpha)$ if and only if

\[
\sum_{n=2}^{\infty} (n - \lambda \alpha n - \alpha + \lambda \alpha) a_n \leq 1 - \alpha.
\]

The result (2.1) is sharp.

**Proof.** Suppose that $f(z) \in T(\lambda, \alpha)$. Then we have from (1.7) that

\[
\begin{align*}
Re \left\{ \frac{zf'(z)}{\lambda zf''(z) + (1 - \lambda) f(z)} \right\} \\
= Re \left\{ \frac{1 - \sum_{n=2}^{\infty} na_n z^{n-1}}{1 - \sum_{n=2}^{\infty} (\lambda n + 1 - \lambda) a_n z^{n-1}} \right\} \\
> \alpha.
\end{align*}
\]
If we choose $z$ real and let $z \to 1^-$, we get

\begin{equation}
1 - \sum_{n=2}^{\infty} n a_n \geq \alpha
\end{equation}

\begin{equation}
1 - \sum_{n=2}^{\infty} (\lambda n + 1 - \lambda) a_n
\end{equation}

which is equivalent to (2.1).

Conversely, suppose that (2.1) holds true. Then, adding

\[-(1-\alpha) \sum_{n=2}^{\infty} (\lambda n + 1 - \lambda) a_n\]

to both sides of (2.1), we obtain

\begin{equation}
(1 - \lambda) \sum_{n=2}^{\infty} (n-1) a_n \leq (1 - \alpha) (1 - \sum_{n=2}^{\infty} (\lambda n + 1 - \lambda) a_n).
\end{equation}

On the other hand, we see that

\begin{equation}
\left| \frac{zf'(z)}{\lambda zf'(z) + (1 - \lambda)f(z)} - 1 \right| = \left| \frac{(1 - \lambda) \sum_{n=2}^{\infty} (n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} (\lambda n + 1 - \lambda) a_n} \right|
\end{equation}

It follows from (2.4) that the last expression in (2.5) is bounded above by $(1 - \alpha)$. This implies $f(z) \in T(\lambda, \alpha)$.

Finally, taking the function

\begin{equation}
f(z) = z - \frac{1 - \alpha}{n - \lambda \alpha n - \alpha + \lambda \alpha} z^n \quad (n \geq 2),
\end{equation}

We can show that the result (2.1) is sharp.
Corollary 1. If \( f(z) \in T(\lambda, \alpha) \), then

\[
(2.7) \quad a_n \leq \frac{1-\alpha}{n-\lambda \alpha n - \alpha + \lambda \alpha} \quad (n \geq 2).
\]

The equality in (2.7) holds for the function \( f(z) \) defined by (2.6).

Theorem 2. A function \( f(z) \) defined by (1.4) is in the class \( C(\lambda, \alpha) \) if and only if

\[
(2.8) \quad \sum_{n=2}^{\infty} n(n-\lambda \alpha n - \alpha + \lambda \alpha) a_n \leq 1-\alpha.
\]

The result (2.8) is sharp.

Proof. Note that \( f(z) \in C(\lambda, \alpha) \) if and only if \( z f'(z) \in T(\lambda, \alpha) \). Hence, replacing \( a_n \) by \( na_n \) in Theorem 1, we have the inequality (2.8). Furthermore, the result (2.8) is sharp for the function

\[
(2.9) \quad f(z) = z - \frac{1-\alpha}{n(n-\lambda \alpha n - \alpha + \lambda \alpha)} z^n \quad (n \geq 2).
\]

Corollary 2. If \( f(z) \in C(\lambda, \alpha) \), then

\[
(2.10) \quad a_n \leq \frac{1-\alpha}{n(n-\lambda \alpha n - \alpha + \lambda \alpha)} \quad (n \geq 2).
\]

The equality in (2.10) holds for the function \( f(z) \) defined by (2.9).

Applying Theorem 1, we prove

Theorem 3. If \( f(z) \in T(\lambda, \alpha) \), then

\[
(2.11) \quad r - \frac{1-\alpha}{2-\lambda \alpha - \alpha} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{2-\lambda \alpha - \alpha} r^2.
\]
and

\[(2.12) \quad 1 - \frac{2(1-\alpha)}{2 - \lambda \alpha - \alpha} r \leq |f'(z)| \leq 1 + \frac{2(1-\alpha)}{2 - \lambda \alpha - \alpha} r\]

for $|z| = r < 1$. The equalities in (2.11) and (2.12) hold for the function

\[(2.13) \quad f(z) = z - \frac{1-\alpha}{2 - \lambda \alpha - \alpha} z^2.\]

\textbf{Proof.} In view of Theorem 1, we have

\[(2.14) \quad (2 - \lambda \alpha - \alpha) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} (n - \lambda \alpha n + \lambda \alpha - \alpha) a_n \leq 1 - \alpha\]

or

\[(2.15) \quad \sum_{n=2}^{\infty} a_n \leq \frac{1 - \alpha}{2 - \lambda \alpha - \alpha}.\]

Using (2.15), we obtain

\[(2.16) \quad |f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \leq r + \frac{1 - \alpha}{2 - \lambda \alpha - \alpha} r^2\]

and

\[(2.17) \quad |f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq r - \frac{1 - \alpha}{2 - \lambda \alpha - \alpha} r^2.\]

Next, it follows from (2.1) and (2.15) that

\[(2.18) \quad (1 - \lambda \alpha) \sum_{n=2}^{\infty} n a_n \leq 1 - \alpha + (1 - \lambda) \alpha \sum_{n=2}^{\infty} a_n\]

or

\[(2.19) \quad \sum_{n=2}^{\infty} n a_n \leq \frac{2(1 - \alpha)}{2 - \lambda \alpha - \alpha}.\]

This derives that

\[(2.20) \quad |f'(z)| \leq 1 + |z| \sum_{n=2}^{\infty} n a_n \leq 1 + \frac{2(1 - \alpha)}{2 - \lambda \alpha - \alpha} r.\]
and

\[(2.21) \quad |f'(z)| \geq 1 - |z| \sum_{n=2}^{\infty} na_n \geq 1 - \frac{2(1 - \alpha)}{2 - \lambda \alpha - \alpha} r.\]

Further, with the help of Theorem 2, we have

**Theorem 4.** If \( f(z) \in C(\lambda, \alpha) \), then

\[(2.22) \quad r - \frac{1 - \alpha}{2(2 - \lambda \alpha - \alpha)} r^2 \leq |f(z)| \leq r + \frac{1 - \alpha}{2(2 - \lambda \alpha - \alpha)} r^2\]

and

\[(2.23) \quad 1 - \frac{1 - \alpha}{2 - \lambda \alpha - \alpha} r \leq |f'(z)| \leq 1 + \frac{1 - \alpha}{2 - \lambda \alpha - \alpha} r^2\]

for \(|z| = r < 1\). The equalities in (2.22) and (2.23) hold for the function

\[(2.24) \quad (z) = z - \frac{1 - \alpha}{2(2 - \lambda \alpha - \alpha)} z^2.\]

**Proof.** Since \( f(z) \in C(\lambda, \alpha) \) if and only if \( zf'(z) \in T(\lambda, \alpha) \), (2.23) follows from (2.11). By using (2.8), we have

\[(2.25) \quad \sum_{n=2}^{\infty} a_n \leq \frac{1 - \alpha}{2(2 - \lambda \alpha - \alpha)}.\]

Therefore, we can see that

\[(2.26) \quad |f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \leq r + \frac{1 - \alpha}{2(2 - \lambda \alpha - \alpha)} r^2\]

and

\[(2.27) \quad |f'(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq r - \frac{1 - \alpha}{2(2 - \lambda \alpha - \alpha)} r^2.\]
3. Properties of $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$

By using Theorem 1 and Theorem 2, we prove

**Theorem 5.** If $f(z) \in C(\lambda, \alpha)$, then

$$f(z) \in T\left(\lambda, \frac{2-2\lambda\alpha}{3-\lambda\alpha-\lambda-\alpha}\right).$$

The result is sharp with the extremal function $f(z)$ given by (2.24).

**Proof.** By virtue of Theorem 1 and Theorem 2, we have to show that if

$$\sum_{n=2}^{\infty} \frac{n(n-\lambda n+\lambda \alpha -\alpha)}{1-\alpha} a_n \leq 1$$

then

$$\sum_{n=2}^{\infty} \frac{n-(\lambda n-\lambda +1)}{1-(2-2\lambda \alpha)/(3-\lambda \alpha-\lambda-\alpha)} \cdot \frac{(2-2\lambda \alpha)/(3-\lambda \alpha-\lambda-\alpha)}{a_n} \leq 1.$$  

In order to show the above, we must have the following inequality

$$\frac{n-(\lambda n-\lambda +1)}{1-(2-2\lambda \alpha)/(3-\lambda \alpha-\lambda-\alpha)} \leq \frac{n(n-\lambda n+\lambda \alpha -\alpha)}{1-\alpha}$$

for $n \geq 2$. But the above inequality is equivalent to

$$(3.4) \quad (1-\lambda)(1-\lambda \alpha)(n^2-3n+2) \geq 0 \quad (n \geq 2)$$

which is always true. Hence we complete the proof of Theorem 5.

Next, we prove
Theorem 6. If \( f(z) \in T(\lambda, \alpha) \), then \( f(z) \in C(\lambda, 0) \) in the disk \( |z| < r_1 < 1 \), where

\[
(3.5) \quad r_1 = \inf_{n \geq 2} \left( \frac{n - \lambda \alpha n + \lambda \alpha - \alpha}{n^2 (1 - \alpha)} \right)^{1/(n-1)}.
\]

The result is sharp with the extremal function \( f(z) \) given by (2.6).

Proof. With the aid of Theorem 1, we have

\[
(3.6) \quad \sum_{n=2}^{\infty} \frac{n - \lambda \alpha n + \lambda \alpha - \alpha}{1 - \alpha} a_n \leq 1
\]

for \( f(z) \in T(\lambda, \alpha) \). It is clear that \( f(z) \in C(\lambda, 0) \) if

\[
(3.7) \quad \frac{|(1-\lambda)zf'''(z)|}{|f'(z) + \lambda zf''(z)|} < 1.
\]

If

\[
(1-\lambda) \sum_{n=2}^{\infty} n(n-1) a_n |z|^{n-1} \leq 1
\]

\[
1 - \sum_{n=2}^{\infty} n(1 + \lambda n - \lambda) a_n |z|^{n-1}
\]

or

\[
\sum_{n=2}^{\infty} n^2 a_n |z|^{n-1} \leq 1,
\]

then (3.7) holds true. Therefore, we see that (3.7) is satisfied if

\[
(3.10) \quad n^2 |z|^{n-1} \leq \frac{n - \lambda \alpha n + \lambda \alpha - \alpha}{1 - \alpha}
\]

or

\[
(3.11) \quad |z| \leq \left( \frac{n - \lambda \alpha n + \lambda \alpha - \alpha}{1 - \alpha} \right)^{1/(n-1)}.
\]
THEOREM 7. Let

\begin{equation}
(3.12) \quad f_1(z) = z
\end{equation}

and

\begin{equation}
(3.13) \quad f_n(z) = z - \frac{1-\alpha}{n - \lambda \alpha n + \lambda \alpha - \alpha} z^n \quad (n \geq 2).
\end{equation}

Then \( f(z) \in T(\lambda, \alpha) \) if and only if

\begin{equation}
(3.14) \quad f(z) = \sum_{n=1}^{\infty} k_n f_n(z),
\end{equation}

where \( k_n \geq 0 \) and \( \sum_{n=1}^{\infty} k_n = 1 \).

PROOF. Suppose that (3.14) is satisfied. Then we have

\begin{equation}
(3.15) \quad f(z) = z - \sum_{n=2}^{\infty} k_n \frac{1-\alpha}{n - \lambda \alpha n + \lambda \alpha - \alpha} z^n = z - \sum_{n=2}^{\infty} a_n z^n,
\end{equation}

where

\begin{equation}
(3.16) \quad a_n = k_n \frac{1-\alpha}{n - \lambda \alpha n + \lambda \alpha - \alpha} \quad (n \geq 2).
\end{equation}

It follows from (3.16) that

\begin{equation}
(3.17) \quad \sum_{n=2}^{\infty} \frac{n - \lambda \alpha n + \lambda \alpha - \alpha}{1-\alpha} a_n = \sum_{n=2}^{\infty} k_n 1 - k_1 \leq 1.
\end{equation}

This implies that \( f(z) \in T(\lambda, \alpha) \) with the help of Theorem 1.

Conversely, suppose that \( f(z) \in T(\lambda, \alpha) \). Then, by Theorem 1, we may put

\begin{equation}
(3.18) \quad k_n = \frac{n - \lambda \alpha n + \lambda \alpha - \alpha}{1-\alpha} a_n \quad (n \geq 2)
\end{equation}

and
(3.19) \( k_1 = 1 - \sum_{n=2}^{\infty} k_n. \)

Consequently, we can see that

\[(3.20) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n = z - \sum_{n=2}^{\infty} k_n \frac{1-\alpha}{n-\lambda \alpha n + \lambda \alpha - \alpha} z^n = \sum_{n=1}^{\infty} k_n f_n(z)\]

which completes the proof of Theorem 7.

With the help of Theorem 7, we have

**Corollary 3.** The extreme points of \( T(\lambda, \alpha) \) are the functions \( f_n(z) \) \( (n \geq 1) \) defined in Theorem 7.

Similarly, we have

**Corollary 4.** The extreme points of \( C(\lambda, \alpha) \) are the functions \( f_1(z) = z \) and

\[(3.21) \quad f_n(z) = z - \frac{1-\alpha}{n(n-\lambda \alpha n + \lambda \alpha - \alpha)} z^n \quad (n \geq 2).\]

4. Application to the fractional calculus

The following definitions of the fractional calculus (fractional integrals and fractional derivatives) due to Owa ([1], [2]).

**Definition 1.** The fractional integral of order \( \delta \) is defined by

\[(4.1) \quad D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\xi)}{(z-\xi)_{1-\delta}} d\xi,\]
where \( \delta > 0 \), \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin, and the multiplicity of \((z-\zeta)^{\delta-1}\) is removed by requiring \( \log(z-\zeta) \) to be real when \((z-\zeta) > 0\).

**Definition 2.** The fractional derivative of order \( \delta \) is defined by

\[
D_{\delta}^{\gamma} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_{0}^{z} \frac{f(\xi)}{(z-\xi)^\delta} \, d\xi,
\]

where \( 0 \leq \delta < 1 \), \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin, and the multiplicity of \((z-\zeta)^{\delta} \) is removed by requiring \( \log(z-\zeta) \) to be real when \((z-\zeta) > 0\).

**Definition 3.** Under the hypotheses of Definition 2, the fractional derivative of order \((n+\delta)\) is defined by

\[
D_{\delta}^{n+\delta} f(z) = \frac{d^n}{dz^n} D_{\delta}^{\gamma} f(z),
\]

where \( 0 \leq \delta < 1 \) and \( n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \).

Applying the above definitions, we prove

**Theorem 8.** If \( f(z) \in T(\lambda, \alpha) \), then

\[
|D_{\delta}^{-\gamma} f(z)| \leq \frac{r^{1+\delta}}{\Gamma(2+\delta)} \left\{ 1 - \frac{2(1-\alpha)}{(2+\delta)(2-\lambda\alpha-\alpha)} \right\} r
\]

and

\[
|D_{\delta}^{-\gamma} f(z)| \leq \frac{r^{1+\delta}}{\Gamma(2+\delta)} \left\{ 1 + \frac{2(1-\alpha)}{(2+\delta)(2-\lambda\alpha-\alpha)} \right\} r
\]

for \( \delta > 0 \) and \(|z|=r<1\). The equalities in (4.4) and (4.5) are attained for the function \( f(z) \) given by (2.13).

**Proof.** It is easy to see that
(4.6) \( \Gamma(2+\delta)z^{-\delta}D_{z}^{-\delta}f(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} a_n z^n \).

Defining \( \phi(n) \) by

(4.7) \( \phi(n) = \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} \) \( (n \geq 2) \),

we know that \( \phi(n) \) is a decreasing function of \( n \), that is, that

(4.8) \( 0 < \phi(n) \leq \phi(2) = \frac{2}{2+\delta} \).

Thus, by using (2.15) and (4.8), we have

(4.9) \( |\Gamma(2+\delta)z^{-\delta}D_{z}^{-\delta}f(z)| \geq |z| - \phi(2)|z|^{2} \sum_{n=2}^{\infty} a_n \)

\[ \geq r - \frac{2(1-\alpha)}{(2+\delta)(2-\lambda\alpha-\alpha)} r^2 \]

which gives (4.4), and

(4.10) \( |\Gamma(2+\delta)z^{-\delta}D_{z}^{-\delta}f(z)| \leq |z| + \phi(2)|z|^{2} \sum_{n=2}^{\infty} a_n \)

\[ \leq r + \frac{2(1-\alpha)}{(2+\delta)(2-\lambda\alpha-\alpha)} r^2 \]

which shows (4.5). Further, the equalities in (4.4) and (4.5) are attained for the function \( f(z) \) defined by

(4.11) \( D_{z}^{-\delta}f(z) = \frac{z^{1+\delta}}{\Gamma(2+\delta)} \left\{ 1 - \frac{2(1-\alpha)}{(2+\delta)(2-\lambda\alpha-\alpha)} z \right\} \),

or, defined by (2.13).

Using the same manner with (2.25), we have

**Theorem 9.** If \( f(z) \in C(\lambda, \alpha) \), then

(4.12) \( |D_{z}^{-\delta}f(z)| \geq \frac{r^{1+\delta}}{\Gamma(2+\delta)} \left\{ 1 - \frac{1-\alpha}{(2+\delta)(2-\lambda\alpha-\alpha)} r \right\} \).
and

\[ |D_z^{-\delta}f(z)| \leq \frac{r^{1+\delta}}{\Gamma(2+\delta)} \left\{ 1 + \frac{1 - \alpha}{(2+\delta)(2-\lambda \alpha - \alpha)} r \right\} \]  

(4.13)

for \( \delta > 0 \) and \(|z|=r<1\). The equalities in (4.12) and (4.13) are attained for the function \( f(z) \) given by (2.24).

Next, we prove

**Theorem 10.** If \( f(z) \in T(\lambda, \alpha) \), then

\[ |D_z^\delta f(z)| \geq \frac{r^{1-\delta}}{\Gamma(2-\delta)} \left\{ 1 - \frac{2(1-\alpha)}{(2-\delta)(2-\lambda \alpha - \alpha)} r \right\} \]  

(4.14)

and

\[ |D_z^\delta f(z)| \leq \frac{r^{1-\delta}}{\Gamma(2-\delta)} \left\{ 1 + \frac{2(1-\alpha)}{(2-\delta)(2-\lambda \alpha - \alpha)} r \right\} \]

(4.15)

for \( 0 \leq \delta < 1 \) and \(|z|=r<1\). The equalities in (4.14) and (4.15) are attained for the function \( f(z) \) given by (2.13).

**Proof.** It is clear that

\[ \Gamma(2-\delta) z^\delta D_z^\delta f(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} a_n z^n \]

(4.16)

\[ = z - \sum_{n=2}^{\infty} \phi(n) n a_n z^n, \]

where

\[ \phi(n) = \frac{\Gamma(n)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}. \]

(4.17)

Since \( \phi(n) \) is decreasing in \( n \), we have

\[ 0 < \phi(n) \leq \phi(2) = \frac{1}{2-\delta}. \]

(4.18)

Therefore, with the aid of (2.19) and (4.18), we show that
(4.19) \[ |I'(2-\delta)z^\delta D_z^\delta f(z)| \geq |z| - \phi(2) |z| 2 \sum_{n=2}^{\infty} n a_n \]
\[ \geq r - \frac{2(1-\alpha)}{(2-\delta)(2-\lambda\alpha-\alpha)} r^2 \]
which is equivalent to (4.14), and

(4.20) \[ |I'(2-\delta)z^\delta D_z^\delta f(z)| \leq |z| + \phi(2) |z| 2 \sum_{n=2}^{\infty} n a_n \]
\[ \leq r + \frac{2(1-\alpha)}{(2-\delta)(2-\lambda\alpha-\alpha)} r^2 \]
which gives (4.15). Furthermore, since the equalities in (4.14) and (4.15) are attained for the function \( f(z) \) defined by

(4.21) \[ D_z^\delta f(z) = \frac{z^{1-\delta}}{I(2-\delta)} \left\{ 1 - \frac{2(1-\alpha)}{(2-\delta)(2-\lambda\alpha-\alpha)} z \right\}, \]
we complete the proof of Theorem 10.

Remark 1. Letting \( \delta=0 \) in Theorem 10, we have (2.11) of Theorem 3, and letting \( \delta=1 \) in Theorem 10, we have (2.12) of Theorem 3.

Finally, we have

Theorem 11. If \( f(z) \in C(\lambda, \alpha) \), then

(4.22) \[ |D_z^\delta f(z)| \geq \frac{r^{1-\delta}}{I(2-\delta)} \left\{ 1 - \frac{1-\alpha}{(2-\delta)(2-\lambda\alpha-\alpha)} r \right\} \]
and

(4.23) \[ |D_z^\delta f(z)| \leq \frac{r^{1-\delta}}{I(2-\delta)} \left\{ 1 + \frac{1-\alpha}{(2-\delta)(2-\lambda\alpha-\alpha)} r \right\} \]
for \( 0 \leq \delta < 1 \) and \( |z|=r < 1 \). The equalities in (4.22) and (4.23) are attained for the function \( f(z) \) given by (2.24).

Remark 2. Taking \( \delta=0 \), Theorem 11 becomes (2.22) of
Theorem 4. Further, letting \( \delta \rightarrow 1 \) in Theorem 11, we have (2.23) of Theorem 4.

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