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ON SUBCLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

The subclasses $S^*(\alpha,\beta,\mu)$ and $C^*(\alpha,\beta,\mu)$ $(0\leqq\alpha<1,$ $0<\beta\leqq 1$ and $0\leqq\mu\leqq 1)$ of T the class of analytic and univalent functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$$

have been considered. Sharp results concerning coefficients, distortion of functions belonging to $S^*(\alpha,\beta,\mu)$ and $C^*(\alpha,\beta,\mu)$ are determined along with a representation formula for the functions in $S^*(\alpha,\beta,\mu)$. Furthermore, it is shown that the classes $S^*(\alpha,\beta,\mu)$ and $C^*(\alpha,\beta,\mu)$ are closed under arithmetic mean and convex linear combinations. Also in this paper, we find extreme points and support points for these classes.

1. Introduction

Let S denote the class of functions of the form

$$(1.1)\qquad f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

that are analytic and univalent in the unit disc $U=\{z:|z|<1\}$. We denote by $S(\alpha)$ and $C(\alpha)$ the subclasses of S consisting of functions which are, respectively, starlike of order α and convex of order α, $0\leqq\alpha<1$.

Juneja and Mogra [4] introduced the class of starlike functions of order α $(0\leqq\alpha<1)$ and type β $(0<\beta\leqq 1)$, defined as follows:

DEFINITION 1. A function $f\in S$ is in $S(\alpha,\beta)$, $0\leqq\alpha<1$, $0<\beta\leqq 1$, the class of starlike functions of order α and type β, if and only if

$$\left|\frac{\dfrac{zf'(z)}{f(z)}-1}{\dfrac{zf'(z)}{f(z)}+1-2\alpha}\right|<\beta \qquad (|z|<1).$$

Further they [4] defined the class of convex functions of order α and type β which is denoted by $C(\alpha,\beta)$, $0\leqq\alpha<1$, $0<\beta\leqq 1$, as follows:

$$f(z)\in C(\alpha,\beta) \text{ if and only if } zf'(z)\in S(\alpha,\beta).$$

Let T denote the subclass of S consisting of functions whose non zero coefficients, from the second on, are negative; that is, an analytic and univalent function f is in T if and only if it can be expressed as

$$(1.2) \qquad f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n.$$

We denote by $S^*(\alpha)$, $C^*(\alpha)$, $S^*(\alpha,\beta)$, $C^*(\alpha,\beta)$ the classes obtained by taking intersections, respectively, of the classes $S(\alpha)$, $C(\alpha)$, $S(\alpha,\beta)$, and $C(\alpha,\beta)$ with T. In [5], Schild considered a subclass of T consisting of polynomials having $|z|=1$ as radius of univalence. For this class, he obtained a necessary and sufficient condition in terms of the coefficients, and with the aid of this he derived better results for certain quantities connected with conformal mapping of univalent functions. Silverman [6] determined coefficient inequalities, distortion, and covering theorems for the classes $S^*(\alpha)$ and $C^*(\alpha)$. In [3] Gupta and Jain determined sharp results concerning coefficients and distortion theorems for the classes $S^*(\alpha, \beta)$ and $C^*(\alpha, \beta)$. They also proved that these classes are closed under "arithmetic mean" and "convex linear combinations".

The aim of the present paper is first to introduce two subclasses of $S(\alpha, \beta)$ and $C(\alpha, \beta)$, which we denote them by $S(\alpha, \beta, \mu)$ and $C(\alpha, \beta, \mu)$, $0 \leqq \mu \leqq 1$, respectively. We then consider the classes

$$S^*(\alpha, \beta, \mu) = S(\alpha, \beta, \mu) \cap T$$

and

$$C^*(\alpha, \beta, \mu) = C(\alpha, \beta, \mu) \cap T.$$

DEFINITION 2. A function $f \in S$ is in the class $S(\alpha, \beta, \mu)$, $0 \leqq \alpha < 1$, $0 < \beta \leqq 1$ and $0 \leqq \mu \leqq 1$, if and only if

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - 1}{\mu \dfrac{zf'(z)}{f(z)} + 1 - (1+\mu)\alpha} \right| < \beta \qquad (|z| < 1).$$

Further, $f \in S$ is in the class $C(\alpha, \beta, \mu)$ if and only if $zf'(z) \in S(\alpha, \beta, \mu)$.

In this paper, sharp results concerning coefficients and distortion theorems for the classes $S^*(\alpha, \beta, \mu)$ and $C^*(\alpha, \beta, \mu)$ are determined. We also show that these classes are closed under "arithmetic mean" and "convex linear combinations". Also we find extreme points and support points for these classes.

2. Coefficient theorems

We begin with the statement and the proof of the following result.

THEOREM 1. A function $f(z)$ defined by (1.2) is in the class $S^*(\alpha, \beta, \mu)$ if and only if

$$\sum_{n=2}^{\infty}\{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}\}|a_n|\leqq(1+\mu)\beta(1-\alpha).$$

The result is sharp.

PROOF. Let $|z|=1$. Then

$$\begin{aligned}
&|zf'(z)-f(z)|-\beta|\mu zf'(z)+f(z)\{1-(1+\mu)\alpha\}| \\
&\quad=|\sum_{n=2}^{\infty}(1-n)|a_n|z^n|-\beta|(1+\mu)(1-\alpha)z \\
&\qquad-\sum_{n=2}^{\infty}\{\mu n+1-(1+\mu)\alpha\}|a_n|z^n| \\
&\quad\leqq\sum_{n=2}^{\infty}\{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}\}|a_n|-(1+\mu)\beta(1-\alpha) \\
&\quad\leqq 0.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f\in S^*(\alpha, \beta, \mu)$.

For the converse, assume that

$$\left|\frac{\frac{zf'(z)}{f(z)}-1}{\mu\frac{zf'(z)}{f(z)}+1-(1+\mu)\alpha}\right|$$
$$=\left|\frac{\sum_{n=2}^{\infty}(n-1)|a_n|z^n}{(1+\mu)(1-\alpha)z-\sum_{n=2}^{\infty}\{\mu n+1-(1+\mu)\alpha\}|a_n|z^n}\right|.$$

Since $|Re(z)|\leqq|z|$ for all z, we have

$$(2.1)\quad Re\left(\frac{\sum_{n=2}^{\infty}(n-1)|a_n|z^n}{(1+\mu)(1-\alpha)z-\sum_{n=2}^{\infty}\{\mu n+1-(1+\mu)\alpha\}|a_n|z^n}\right)<\beta.$$

Choose values of z on the real axis so that $zf'(z)/f(z)$ is real. Upon clearing the denominator in (2.1) and letting $z\rightarrow 1^-$ through real values, we obtain

$$\sum_{n=2}^{\infty}(n-1)|a_n|\leqq\beta\{(1+\mu)(1-\alpha)z$$
$$-\sum_{n=2}^{\infty}\{\mu n+1-(1+\mu)\alpha\}|a_n|\}.$$

This gives the required condition.

Finally, the function

$$f(z)=z-\frac{(1+\mu)\beta(1-\alpha)}{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}}z^n\quad(n\geqq 2)$$

is an extremal function for the theorem.

THEOREM 2. A function $f(z)$ defined by (1.2) is in the class $C^*(\alpha,\ \beta,\mu)$ if and only if

$$\sum_{n=2}^{\infty}n\{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}\}|a_n|\leqq(1+\mu)\beta(1-\alpha).$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z-\frac{(1+\mu)\beta(1-\alpha)}{n\{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}\}}z^n \quad (n\geqq 2).$$

PROOF. Note that $f(z)\in C^*(\alpha,\beta,\mu)$ if and only if $zf'(z)\in S^*(\alpha,\beta,\mu)$. Therefore, the proof follows from Theorem 1.

3. A representation formula

We now proceed to prove a theorem which gives a representation for functions in the class $S^*(\alpha,\beta,\mu)$.

THEOREM 3. A function $f(z)$ defined by (1.2) is in the class $S^*(\alpha,\beta,\mu)$ if and only if

$$f(z)=z\ \exp\left\{(1+\mu)(1-\alpha)\int_0^z\frac{\phi(t)}{1-\mu t\phi(t)}dt\right\}, \tag{3.1}$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)|\leqq\beta$, for $|z|<1$.

PROOF. Let $f(z)\in S^*(\alpha,\beta,\mu)$. Then

$$\left|\frac{\dfrac{zf'(z)}{f(z)}-1}{\mu\dfrac{zf'(z)}{f(z)}+1-(1+\mu)\alpha}\right|<\beta \qquad (|z|<1).$$

Since the absolute value vanishes for $z=0$, we have

$$\frac{\dfrac{zf'(z)}{f(z)}-1}{\mu\dfrac{zf'(z)}{f(z)}+1-(1+\mu)\alpha}=V(z), \tag{3.2}$$

where $V(z)$ is analytic and $|V(z)|\leqq\beta$ for $|z|<1$. The "only if" part is easily obtained by integrating (3.2) with $V(z)=z\phi(z)$, and the other part by differentiating (3.1).

4. Distortion theorems

THEOREM 4. If $f(z)\in S^*(\alpha,\beta,\mu)$, then

$$r-\frac{(1+\mu)\beta(1-\alpha)}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}}r^2 \leqq |f(z)| \leqq r+\frac{(1+\mu)\beta(1-\alpha)}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}}r^2$$

and

$$1-\frac{2(1+\mu)\beta(1-\alpha)}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}}r \leqq |f'(z)| \leqq 1+\frac{2(1+\mu)\beta(1-\alpha)}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}}r$$

for $|z|=r<1$.

PROOF. In view of Theorem 1, we have

$$\begin{aligned}&\{1+\beta\{2\mu+1-(1+\mu)\alpha\}\}\sum_{n=2}^{\infty}|a_n|\\ &=\{1+\beta\{(1+\mu)(1-\alpha)+\mu\}\}\sum_{n=2}^{\infty}|a_n|\\ &\leqq\sum_{n=2}^{\infty}\{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}\}|a_n|\\ &\leqq(1+\mu)\beta(1-\alpha),\end{aligned}$$

which gives

$$\sum_{n=2}^{\infty}|a_n|\leqq\frac{(1+\mu)\beta(1-\alpha)}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}}.$$

Therefore we have

$$\begin{aligned}|f(z)|&\leqq r+r^2\sum_{n=2}^{\infty}|a_n|\\ &\leqq r+\frac{(1+\mu)\beta(1-\alpha)}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}}r^2\end{aligned}$$

and

$$\begin{aligned}|f(z)| &\geqq r - r^2 \sum_{n=2}^{\infty}|a_n| \\ &\geqq r - \frac{(1+\mu)\beta(1-\alpha)}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}} r^2\end{aligned}$$

which prove the first part of the theorem. Further we have

$$1 - r\sum_{n=2}^{\infty} n|a_n| \leqq |f'(z)| \leqq 1 + r\sum_{n=2}^{\infty} n|a_n|.$$

Using Theorem 1, we note that

$$\sum_{n=2}^{\infty} n|a_n| \leqq \frac{2(1+\mu)\beta(1-\alpha)}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}}.$$

Therefore, the second part of the theorem follows from the above inequality.

Finally, since the equalities in the theorem are attained for the function

$$(4.1) \quad f(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}} z^2 \quad (z=\pm r),$$

the bounds in the theorem are sharp.

Using the same technique as in the proof of Theorem 4, we obtain

THEOREM 5. If $f(z) \in C^*(\alpha, \beta, \mu)$, then

$$\begin{aligned}r - \frac{(1+\mu)\beta(1-\alpha)}{2\{1+\beta\{(1+\mu)(1-\alpha)+\mu\}\}} r^2 &\leqq |f(z)| \\ &\leqq r + \frac{(1+\mu)\beta(1-\alpha)}{2\{1+\beta\{(1+\mu)(1-\alpha)+\mu\}\}} r^2\end{aligned}$$

and

$$\begin{aligned}1 - \frac{(1+\mu)\beta(1-\alpha)}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}} r &\leqq |f'(z)| \\ &\leqq 1 + \frac{(1+\mu)\beta(1-\alpha)}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}} r\end{aligned}$$

for $|z|=r<1$. The bounds are attained for the function

$$(4.2) \qquad f(z) = z - \frac{(1+\mu)\beta(1-\alpha)}{2\{1+\beta\{(1+\mu)(1-\alpha)+\mu\}\}} z^2 \quad (z=\pm r).$$

Letting $r \to 1$ in Theorem 4 and Theorem 5, we have

THEOREM 6. Let $f(z) \in S^*(\alpha, \beta, \mu)$. Then the unit disc $|z|<1$ is mapped onto a domain that contains the disc

$$|w| < \frac{1+\mu\beta}{1+\beta\{(1+\mu)(1-\alpha)+\mu\}}.$$

The result is sharp with the extremal function given by (4.1).

THEOREM 7. Let $f(z) \in C^*(\alpha, \beta, \mu)$. Then the unit disc $|z|<1$ is mapped onto a domain that contains the disc

$$|w| < \frac{2+\beta\{1+\mu)(1-\alpha)+2\mu\}}{2\{1+\beta\{(1+\mu)(1-\alpha)+\mu\}\}}.$$

The result is sharp with the extremal function given by (4.2).

5. Closure theorems

In this section, we shall prove that the classes $S^*(\alpha, \beta, \mu)$ and $C^*((\alpha, \beta, \mu)$ are closed under "arithmetic mean" and "convex linear combinations".

THEOREM 8. If the functions

$$f(z) = z - \sum_{r=2}^{\infty} |a_n| z^n$$

and

$$g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$$

are in the class $S^*(\alpha, \beta, \mu)$, then the function

$$h(z) = z - \frac{1}{2}\sum_{n=2}^{\infty}|a_n+b_n|z^n$$

is also in the class $S^*(\alpha, \beta, \mu)$.

PROOF. The proof follows directly by appealing to Theorem 1. In fact, $f(z)$ and $g(z)$ being in the class $S^*(\alpha, \beta, \mu)$, we have

$$\sum_{n=2}^{\infty}\{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}\}|a_n| \leqq (1+\mu)\beta(1-\alpha) \tag{5.1}$$

and

$$\sum_{n=2}^{\infty}\{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}\}|b_n| \leqq (1+\mu)\beta(1-\alpha). \tag{5.2}$$

It is sufficient, for $h(z)$ to be a member of the class $S^*(\alpha, \beta, \mu)$, to show

$$\frac{1}{2}\sum_{n=2}^{\infty}\{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}\}|a_n+b_n| \leqq (1+\mu)\beta(1-\alpha),$$

which will follow immediately by the use of (5.1) and (5.2).

THEOREM 9. Let $f_1(z)=z$ and

$$f_n(z)=z-\frac{(1+\mu)\beta(1-\alpha)}{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}}z^n \qquad (n\geqq 2).$$

Then $f(z)\in S^*(\alpha, \beta, \mu)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z),$$

where $\lambda_n \geqq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

PROOF. Suppose

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$$

$$= z - \sum_{n=2}^{\infty} \frac{(1+\mu)\beta(1-\alpha)}{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}} \lambda_n z^n.$$

Then

$$\sum_{n=2}^{\infty} \left(\frac{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}}{(1+\mu)\beta(1-\alpha)} \lambda_n \right) \cdot \left(\frac{(1+\mu)\beta(1-\alpha)}{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}} \right) \leqq 1.$$

Thus, by Theorem 1, we have $f(z) \in S^*(\alpha, \beta, \mu)$.

Conversely, suppose $f(z) \in S^*(\alpha, \beta, \mu)$. Again, by Theorem 1, we have

$$|a_n| \leqq \frac{(1+\mu)\beta(1-\alpha)}{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}} \quad (n \geqq 2).$$

Setting

$$\lambda_n = \frac{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}}{(1+\mu)\beta(1-\alpha)} |a_n| \quad (n \geqq 2)$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n,$$

we have

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

which completes the proof of Theorem 9.

The analogous of Theorem 8 and Theorem 9 for the class $C^*(\alpha, \beta, \mu)$ are

THEOREM 10. Let the functions $f(z)$, $g(z)$ and $h(z)$ be defined as in Theorem 8. If $f(z)$ and $g(z)$ belong to the class $C^*(\alpha, \beta, \mu)$, then $h(z)$ is also in the class $C^*(\alpha, \beta, \mu)$.

THEOREM 11. Let $f_1(z)$ and

$$f_n(z)=z-\frac{(1+\mu)\beta(1-\alpha)}{n\{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}\}}z^n \quad (n \geqq 2).$$

Then $f(z)\in C^*(\alpha, \beta, \mu)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \geqq 0$ and $\sum_{n=1}^{\infty} \lambda_n=1$.

REMARKS. (i) Putting $\mu=1$ and $\beta=1$ in the above results, we get the results obtained by Silverman [6].

(ii) Putting $\mu=1$ in the above results, we get the results obtained by Gupta and Jain [3].

6. Support points

A fuction $f(z)$ in $S^*(\alpha, \beta, \mu)$ is said to be a *support point* of $S^*(\alpha, \beta, \mu)$ if there exists a continuous linear functional J on T such that $Re\{J(f)\} \geqq Re\{J(g)\}$ for all $g(z)\in S^*(\alpha, \beta, \mu)$, and $Re\{J\}$ is non constant on $S^*(\alpha, \beta, \mu)$. We denote by *Supp* $S^*(\alpha, \beta, \mu)$ the set of support points of $S^*(\alpha, \beta, \mu)$, and by *Ext* $S^*(\alpha, \beta, \mu)$ the set of extreme points of $S^*(\alpha, \beta, \mu)$.

Let F be a subfamily of univalents in U whose set of extreme points is countable, suppose f_0 is a support point of F, and let J be a corresponding continuous linear functional. Defining G_J by

$$G_J=\{f\in F:\ Re\{J(f)\}=Re\{J(f_0)\}.$$

Deeb [2] showed the following result.

LEMMA 1. Let G_J be defined by the above. Then G_J is convex, Ext $G_J\subset$ Ext F, and

$$G_J=\{f\in F: f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z),\ \lambda_n\geqq 0,\ \sum_{n=1}^{\infty}\lambda_n=1,\ f_n\in\text{Ext } G_J\}.$$

Let A be the class of functions of the form

$$f(z)=\sum_{n=0}^{\infty}a_n z^n$$

which are analytic in U. Then, Brickman, MacGregor and Wilken [1] proved the following lemma.

LEMMA 2. Let $\{b_n\}$ be a sequence of complex numbers such that

$$\lim_{n\to\infty}\sup|b_n|^{1/n}<1,$$

and set

$$J(f)=\sum_{n=0}^{\infty}a_n b_n$$

for $f(z)\in A$. Then J is a continuous linear functional on A. Conversely, any continuons linear functional on A is given such a sequence $\{b_n\}$.

In order to give our theorems for support points, we need the following results.

LEMMA 3. The extreme points of $S^*(\alpha,\beta,\mu)$ are $f_1(z)=z$ and

$$f_n(z)=z-\frac{(1+\mu)\beta(1-\alpha)}{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}}z^n \qquad (n\geqq 2).$$

LEMMA 4. The extreme points of $C^*(\alpha, \beta, \mu)$ are $f_1(z)=z$ and

$$f_n(z)=z-\frac{(1+\mu)\beta(1-\alpha)}{n\{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}\}}z^n \quad (n\geqq 2).$$

Lemma 3 and Lemma 4 follow from Theorem 9 and Theorem 11, respectively.

Now, with the help of the above lemmas, we prove

THEOREM 12. The set Supp $S^*(\alpha, \beta, \mu)$ of support points of $S^*(\alpha, \beta, \mu)$ given by

$$\begin{aligned}\text{Supp } S^*(\alpha,\beta,\mu)=\{f\in S^*(\alpha,\beta,\mu):\\ f(z)=z-\sum_{n=2}^{\infty}\frac{(1+\mu)\beta(1-\alpha)\lambda_n}{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}}z^n,\\ \lambda_n\geqq 0,\ \sum_{n=2}^{\infty}\lambda_n\leqq 1,\ \lambda_j=0 \text{ for some } j\}.\end{aligned}$$

PROOF. Assume that

$$f_0(z)=z-\sum_{n=2}^{\infty}\frac{(1+\mu)\beta(1-\alpha)\lambda_n}{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}}z^n$$

is in the class $S^*(\alpha, \beta, \mu)$, where $\lambda_n\geqq 0$, $\sum_{n=2}^{\infty}\lambda_n\leqq 1$, and $\lambda_j=0$ for some $j\geqq 2$. Since

$$\lim_{n\to\infty}\ \sup|b_n|^{1/n}<1$$

for $b_n=0$ $(n\geqq 2,\ n\neq j)$ and $b_1=b_j=1$, with Lemma 2, we define the continuous linear functional J given by $\{b_n\}$. Then $J(f_0)=1$ and $J(f)=1-|a_j|\leqq 1$ for $f(z)$ belonging to the class $S^*(\alpha,\beta,\mu)$. This shows that $Re\{J(f_0)\} \geqq Re\{J(f)\}$ for all $f(z) \in S^*(\alpha,\beta,\mu)$. Thus, we see that $f_0(z)$ is a support point of $S^*(\alpha,\beta,\mu)$.

Conversely, assume that $f_0(z)$ is a support point of $S^*(\alpha,\beta,\mu)$ and that its continuous linear functional J is given by $\{b_n\}$. Note that $Re\{J\}$ is also continuous and linear on $S^*(\alpha,\beta,\mu)$. Therefore, by the Krein-Milman theorem, there exists an extreme point $f_n(z)$ of $S^*(\alpha,\beta,\mu)$ such that

$$\begin{aligned} Re\{J(f_0)\} &= \text{Max}\{Re\{J(f)\}:\ f\in S^*(\alpha,\beta,\mu)\} \\ &= Re\{J(f_n)\}. \end{aligned}$$

Let

$$G_J=\{f_n:\ Re\{J(f_0)\}=Re\{J(f_n)\},\ f_n\in \text{Ext } S^*(\alpha,\beta,\mu)\}.$$

Note that Ext $S^*(\alpha,\beta,\mu)$ is countable by Lemma 3. If $G_J=$ Ext $S^*(\alpha,\beta,\mu)$, then $Re\{J\}$ must be constant on $S^*(\alpha,\beta,\mu)$. This contradicts that $f_0(z)$ is a support point of $S^*(\alpha,\beta,\mu)$. Therefore, there exists a j such that $Re\{J(f)\} > Re\{J(f_j)\}$. It follows from this fact that

$$\text{Ext } G_J \subset \{f_n:\ f_n\in\text{Ext } S^*(\alpha,\beta,\mu),\ n\geqq 2,\ n\neq j\}.$$

Consequently, by using Lemma 1, we have

$$f_0(z)=\sum_{n=2}^{\infty}\lambda_n f_n(z),$$

where $\lambda_n \geqq 0$, $\sum_{n=2}^{\infty}\lambda_n=1$, and $f_n\in$ Ext G_J, $n\geqq 2$, $n\neq j$. With the help of Lemma 3, this gives

$$f_0(z)=z-\sum_{\substack{n=2\\ n\neq j}}^{\infty}\frac{(1+\mu)\beta(1-\alpha)\lambda_n}{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}}z^n,$$

which completes the assertion of Theorem 12.

Using the same technique as in the proof of Theorem 12, with Lemma 4, we have

THEOREM 13. The set Supp $C^*(\alpha,\beta,\mu)$ of support points of $C^*(\alpha,\beta,\mu)$ is given by

$$\text{Supp } C^*(\alpha,\beta,\mu)=\{f\in C^*(\alpha,\beta,\mu):$$
$$f(z)=z-\sum_{n=2}^{\infty}\frac{(1+\mu)\beta(1-\alpha)\lambda_n}{n\{(n-1)+\beta\{\mu n+1-(1+\mu)\alpha\}\}}z^n,$$
$$\lambda_n\geqq 0,\ \sum_{n=2}^{\infty}\lambda_n\leqq 1,\ \lambda_j=0 \text{ for some } j\}.$$

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