PUSAN KYŎNGNAM MATHEAMTICAL JOURNAL
Vol. 4, 75~85, 1988.

ORTHOGONALITIES AND CHARACTERIZATIONS OF 2-INNER PRODUCT SPACES

Y. J. CHO AND S. S. KIM

1. Introduction

Let X be a real linear space of dimension greater than 1. Let $(\cdot, \cdot|\cdot)$ be a real-valued function on $X\times X\times X$ which satisfies the following conditions:

(I_1) $(x, x|z)\geq 0$,
$(x, x|z)=0$ if and only if x and z are linearly dependent,

(I_2) $(x, x|z)=(z, z|x)$,

(I_3) $(x, y|z)=(y, x|z)$,

(I_4) $(\alpha x, y|z)=\alpha(x, y|z)$,

(I_5) $(x+x', y|z)=(x, y|z)+(x', y|z)$

for every x, x', y, z in X and for real number α. Then $(\cdot, \cdot|\cdot)$ is called a *2-inner product* and $(X, (\cdot, \cdot|\cdot))$ a 2-inner product space ([4]). The concepts of 2-inner product and 2-inner product space are 2-dimensional analogy of the concepts of inner product and inner product space. R. E. Ehret [4] proved that on any 2-inner product space $(X, (\cdot, \cdot|\cdot))$, $||x, z||^2=(x, x|z)$ defines a 2-norm for which

$$(x, y|z)=\frac{1}{4}(||x+y, z||^2-||x-y, z||^2)$$

and

$$||x+y,z||^2+||x-y,z||^2=2(||x,z||^2+||y,z||^2)$$

for every $x, y, z(\neq 0)$ in X and $z \notin V(x, y)$.

One of the following (A)~(C) listed below is a condition which is necessary and sufficient condition for a linear 2-normed space X to be a 2-inner product space. These characterization (A)~(C) of 2-inner product spaces were proved by C. Diminnie, S. Gähler and A. White ([1]): let $z \in X$ be an arbitrary nonzero element.

(A) If $x, y \in X$, $||x,z||=||y,z||=1$ and $z \notin V(x,y)$, then $||x+y,z||^2 + ||x-y,z||^2 = 4$,

(B) If $x, y \in X$ and a nonzero real number k, $||x,z||=||y,z||$, then $||kx+k^{-1}y,z|| \geq ||x+y,z||$,

(C) If $x, y \in X$ and $||x,z||=||y,z||$, then $||kx+y,z||=||x+ky,z||$ for all real number k.

The main purpose of this paper is to give some new characterizations of 2-inner product spaces and to provide simpler proofs of existing similar characterization.

2. Orthogonalities

Throughout this note, X will denote a linear 2-normed space, x, y, z in X with $z \neq 0$ and $z \notin V(x, y)$.

DEFINITION 2.1. For linear 2-normed space an element x of X is *isosceles orthogonal* to an element y (written $x \perp_i y$) if $||x+y,z||=||x-y,z||$.

DEFINITION 2.2. For a linear 2-normed space an element x of X is *pythagorean orthogonal* to an element y (written $x \perp_p y$) if $||x-y,z||^2=||x,z||^2 + ||y,z||^2$.

DEFINITION 2.3. For a linear 2-normed space an element x of X is *J-orthogonal* to an element y (written $x \perp_J y$) if $\|x+ky, z\| \geq \|x, z\|$ for every real number k.

From [5] we know the following theorem:

THEOREM 2.1. If $x \neq 0$, y in a linear 2-normed space, then there exist numbers a, b, c and d such that $x \perp_I (ax+y)$, $x \perp_p (bx+y)$, $x \perp_J (cx+y)$ and $(dx+y) \perp_J x$. Further, if $\|y\| \leq \|x\|$, then $|a| \leq \|y\|/\|x\|$.

By using the techniques in C. Diminnie, S. Gahler and A. White ([2]), we have the following results: let $z \in X$ and X_z denoted the quotient space $X/V(z)$. For $(x)_z$, $(y)_z$ in X_z define $(x)_z + (y)_z = (x+y)_z$, $(ax)_z = a(x)_z$ and $\|(x)_z\|_z = \|x, z\|$. Then $(X_z, \|\cdot\|_z)$ is a normed linear space. X_z will denote this linear space.

Thus it follows that;

THEOREM 2.2. If $x \neq 0$, y in X, then there exist numbers a, b, c and d such that $x \perp_I (ax+y)$, $x \perp_p (bx+y)$, $x \perp_J (cx+d)$ and $(dx+y) \perp_J x$. Further if $\|y, z\| \leq \|x, z\|$, then $|a| \leq \|y, z\|/\|x, z\|$.

An orthogonality $\perp$ is called left(right) unique if for $x \neq 0$, y in X, there exist only one a such that $(ax+y) \perp x$ $(x \perp (ax+y))$.

REMARK. For isosceles and pythagorean orthogonalities, left and right uniqueness are equivalent.

For J-orthogonality, the following was proved

THEOREM 2.3 ([6]). J-orthogonality, $\perp_J$, is left(right) unique if and only if X is strictly convex (smooth).

DEFINITION 2.4([3]). X is strictly convex if $\|x, z\| = \|y, z\| = \left\|\frac{x+y}{2}, z\right\| = 1$ imply $y = x$.

THEOREM 2.4. An isosceles orthogonality, $\perp_I$, in X is unique if and only if X is strictly convex.

PROOF. Suppose that X is not strictly convex and isosceles orthogonality is not unique. Then, by Theorem 2.2, there exist $x \neq 0$, y in X and a real number $a > 0$ such that $x \perp_I y$ and $x \perp_I (ax+y)$. The function $f(t) = \|y+tx, z\|$, $-\infty < t < \infty$, is a strictly convex function with $f(1) = f(-1)$ and $f(a+1) = f(a-1)$ because $\perp$ is isosceles orthogonality.

In the case $0 < a \leq 2$, we have

$$\begin{aligned} f(a-1) &= f\Big(\frac{2-a}{2}(-1) + \frac{a}{2}\Big) \\ &< \frac{2-a}{2} f(-1) + \frac{a}{2} f(1) = f(1) \\ &= f\Big(\frac{a}{2}(a-1) + (1-\frac{a}{2})(a+1)\Big) < f(a+1). \end{aligned}$$

This contradicts $f(a-1) = f(a+1)$.

In other case $a > 2$, f will have two distinct local minima, one each in $[-1, 1]$ and $[a-1, a+1]$. This contradicts that f is strictly convex function.

Conversely, suppose that X is not strictly convex. Then there exist x, y in X such that $\|x, z\| = \|y, z\| = \left\|\frac{x+y}{2}, z\right\| = 1$ implies $y \neq x$. We get $\|x+y, z\| = \|x+y+(x-y), z\| = \|x+y-(x-y), z\|$. Put $x' = x+y$ and $y' = x-y$. Then we have $\|x', z\| = \|x'+y', z\| = \|x'-y', z\|$, $y' \neq 0$. Hence,

$$\left\|x'+\frac{y'}{2}-\frac{y'}{2},\ z\right\|=\left\|x'+\frac{y'}{2}+\frac{y'}{2},\ z\right\|=\left\|x'-\frac{y'}{2}-\frac{y'}{2},\ z\right\|.$$

Thus

$$\frac{y'}{2}\perp_i\left(x'+\frac{y'}{2}\right)\ \text{and}\ \frac{y'}{2}\perp_i\left(x'-\frac{y'}{2}\right).$$

This contradicts the uniqueness of $\perp_i$.

3. Characterizations of 2-inner product spaces

THEOREM 3.1. For a linear 2-normed space X, the following statements are equivalent:

(1) X is a 2-inner product space,

(2) x, y in X, $x \perp_p y$ imply $x \perp_i y$,

(3) x, y in X, $x \perp_i y$ imply $x \perp_p y$.

At first, we shall prove lemma.

LEMMA 3.2. If pythagorean orthogonality implies isosceles orthogonality in a linear 2-normed space X, then X is strictly convex.

PROOF. Suppose that X is not strictly convex. Then there exist x, y in X such that $\|x, z\|=\|y, z\|=\left\|\frac{x+y}{2},\ z\right\|=1$ implies $y \neq x$ and $x \not\perp_p y$ (called x is not pythagorean orthogonality to y). By Theorem 2.2., there exists a nonzero real number a such that $x \perp_p ax+y$, that is,

$$\begin{aligned}\|x-(ax+y), z\|^2 &= \|x, z\|^2+\|ax+y, z\|^2 \\ &= 1+\|ax+y, z\|^2 \cdots\cdots\cdots\cdots\cdots (*)\end{aligned}$$

and by the fact that $\perp_p$ implies $\perp_i$, $x \perp_i (ax+y)$. Further, $|a| \leq 1$ by Theorem 2.2.

From (*), we get

$$\begin{aligned}1 &\le 1+\|ax+y,z\|^2\\ &=(2+a)^2\left\|\frac{ax+x+y}{2+a},\ z\right\|^2\\ &=\|ax+x+y,z\|^2\\ &\le\{\|ax+x,z\|+\|y,z\|\}^2\\ &=(a+1)^2\|x,z\|^2+2|a+1|\,\|x,z\|\,\|y,z\|+\|y,z\|^2\\ &=(a+2)^2.\end{aligned}$$

Thus, we obtain $a=-1$. Apply $a=-1$ to (*). Then $1=\|y,z\|^2=\|x-y,z\|^2+1$ and therefore $\|x-y,z\|=0$. Hence $x-y$ and z are linearly dependent. That is, $z=\alpha(x-y)$ for some $\alpha\in R$, or $x-y=0$.

(i) $\alpha=0$, then $z=0$. This contradicts $\|x,z\|=1=\|y,z\|$.
(ii) $\alpha\neq 0$ and $x-y\neq 0$, then $z=\alpha(x-y)\neq 0$.

This contradicts $z\notin V(x,y)$. Consequently, $x-y=0$ which contradicts $x\neq y$.

PROOF of THEOREM 3.1. (1) implies (2) is trivial, (2) implies (3): Suppose that (2) does not imply (3). Then there exist x,y in X such that $x\perp_i y$ but $x\not\perp_p y$. By Theorem 2.2., choose a nonzero real number a such that $x\perp_p(ax+y)$. But by (2), $x\perp_i(ax+y)$. Hence, by Lemma 3.2 and (2), X is strictly convex. Also, by Theorem 2.4, an isosceles orthogonality, $\perp_i$, is unique. This contradicts $a\neq 0$.

(3) implies (1): Let $\|x,z\|=\|y,z\|=1$. Then, since $\|x+y+x-y,z\|=\|x+y-x+y,z\|$, $x+y\perp_i x-y$ and so $x+y\perp_p x-y$. Thus, we get $\|x+y,z\|^2+\|x-y,z\|^2=\|x+y+x-y,z\|^2=4$. By (A), X is a 2-inner product space.

THEOREM 3.3. For a linear 2-normed space, the following statements are equivalent:

(1) X is a 2-inner product space,

(2) x, y in X, $x \perp_p y$ implies $x \perp_J y$,

(3) x, y in X, $x \perp_J y$ implies $x \perp_p y$.

At first, we shall prove lemma.

LEMMA 3.4. If pythagorean orthogonality, $\perp_p$, implies J-orthogonality, $\perp_J$, in a linear 2-normed space, then X is strictly convex.

PROOF. Suppose that X is not strictly convex. Then there exist x, y in X such that $\|x,z\|=\|y,z\|=\left\|\frac{x+y}{2}, z\right\|=1$ implies $y \neq x$ and $x \not\perp_p \frac{x+y}{2}$. By Theorem 2.2, there exists a nonzero real number a such that $\frac{x+y}{2} \perp_p \left(a \frac{x+y}{2}+x\right)$, that is,

$$\left\|\frac{x+y}{2}-\left(a \frac{x+y}{2}+x\right), z\right\|^2=\left\|\frac{x+y}{2}, z\right\|^2 +\left\|a \frac{x+y}{2}+x, z\right\| =1+\left\|a \frac{x+y}{2}+x, z\right\|^2 \cdots (**)$$

and since $\perp_p$ implies $\perp_i$,

$$\left\|\frac{x+y}{2}+k\left(a \frac{x+y}{2}+x\right), z\right\| \geq \left\|\frac{x+y}{2}, z\right\|=1 \cdots\cdots (***)$$

for every real k.

With $k=-\frac{1}{a}$, (***) yields $|a| \leq 1$. Again putting

$$k=-\frac{1}{a+2}, \qquad \left\|\frac{x+y}{2}-\frac{1}{a+2}\left(a\frac{x+y}{2}+x\right),\ z\right\| = \left\|\frac{1}{a+2}y,\ z\right\| = \left|\frac{1}{a+2}\right| \geq 1.$$

Thus $a=-1$. We apply $a=-1$ to (**)

$$\begin{aligned} 1 &= \left\|\frac{x+y}{2}+\frac{x-y}{2},\ z\right\|^2 \\ &= \left\|\frac{x+y}{2},\ z\right\|^2+\left\|\frac{x-y}{2},\ z\right\|^2 \\ &= 1+\left\|\frac{x-y}{2},\ z\right\|^2. \end{aligned}$$

Therefore $||x-y,z||=0$. The rest of the argument is same as in proof of lemma 3.2.

PROOF of THEOREM 3.3. (1) implies (2) is trivial. (2) implies (3): Suppose that (2) does not imply (3). Then there exist x, y in X such that $x \perp_J y$ but $x \not\perp_p y$. By Theorem 2.2, choose a nonzero real number a such that $(ay+x) \perp_p y$. By (2), $(ay+x) \perp_J y$. Also, by Lemma 3.4, X is strictly convex and by Theorem 2.3, J-orthogonality is a left unique. This contradicts $a \neq 0$.

(3) implies (1): Let $||x,z||=||y,z||=1$. If $x \perp_J y$ and $(x+y) \perp_J (x-y)$, then $4 = ||x+y+x-y,z||^2 = ||x+y,z||^2+||x-y,z||^2$. Thus by (A), X is 2-inner product space. If $x \not\perp_J y$, then choose $w \in X$ such that $x \perp_J w$ and $(x+w) \perp_J (x-w)$. Hence

$$\begin{aligned}||w,z||^2&=\left\|\frac{(x+w)-(x-w)}{2},\ z\right\|^2\\&=\left\|\frac{x+w}{2},\ z\right\|^2+\left\|\frac{x-w}{2},\ z\right\|^2\\&=\left\|\frac{x}{2},\ z\right\|^2+\left\|\frac{w}{2},\ z\right\|^2+\left\|\frac{x}{2},\ z\right\|^2+\left\|\frac{w}{2},\ z\right\|^2.\end{aligned}$$

This means $||x,z||=||w,z||=1$. Let a and b be such that $y=ax+bw$. Then

$$\begin{aligned}||y,z||^2&=||ax+bw,z||^2\\&=||ax,z||^2+||bw,z||^2\\&=a^2+b^2,\\||x+y,z||^2&=||(1+a)x+bw,z||^2\\&=(1+a)^2+b^2,\end{aligned}$$

and

$$\begin{aligned}||x-y,z||^2&=||(1-a)x-bw,z||^2\\&=(1-a)^2+b^2.\end{aligned}$$

Therefore, $\begin{aligned}[t]||x+y,z||^2+||x-y,z||^2&=2(a^2+b^2)+2\\&=2||y,z||^2+2\\&=4.\end{aligned}$

Hence, by(A), X is a 2-inner product space.

LEMMA 3.5. If isosceles orthogonality is homogenous in linear 2-normed space X, then X is a 2-inner product space.

PROOF. If $||x,z||=||y,z||$, x,y in X, then $||x+y+x-y,z||=||x+y-(x-y),z||$ and so $(x+y)\perp_i(x-y)$. If isosceles orthogonality is homogenous in X, then

$$\begin{aligned}&||(a+1)(x+y)+(a-1)(x-y),z||\\=&||(a+1)(x+y)-(a-1)(x-y),z||\end{aligned}$$

or

$$||ax+y,z||=||x+ay,z|| \quad \text{for all real } a.$$

Hence, by(C), X is a 2-inner product space.

THEOREM 3.6. For a linear 2-normed space X, the following statements are equivalent:

(1) X is a 2-inner product space.

(2) x, y in X, $x\perp_J y$ implies $x\perp_i y$.

PROOF. (1) implies (2) is trivial. (2) implies (1): Let $x\neq 0$, y in X. By [6, Theorem], there exists a real number a such that $x\perp_J(ax+y)$. Since J-orthogonality is homogenous, $x\perp_J k(ax+y)$ for every real number k. Also, by (2), $x\perp_i k(ax+y)$ for every real number k. Thus by Lemma 3.5, we obtain (2) implies (1).

THEOREM 3.7. For a linear 2-normed space X, the following statements are equivalent:

(1) X is a 2-inner product space.

(2) x, y in X, $x\perp_i y$ implies $x\perp_J y$.

PROOF. (1) implies (2) is trivial. (2) implies (1): Suppose that

$$||x,z||=||y,z||$$

for every x, y in X. Then

$$||x+y+x-y,z||=||x+y-(x-y),z||,$$

that is, $(x+y)\perp_i(x-y)$. Therefore $(x+y)\perp_J(x-y)$. Thus we have $||x+y+k(x-y),z||\geq||x+y,z||$ for all real number k. In particular for all $a>1$ we have

$$\left|\left|x+y+\frac{a^2-1}{a^2+1}(x-y),\ z\right|\right|\geq||x+y,z||.$$

Therefore

$$\begin{aligned}||ax+a^{-1}y,\ z||&\geq\frac{a^2+1}{2a}||x+y,z||\\&\geq||x+y,z|| \quad \text{for all } a>1.\end{aligned}$$

Hence by (B), X is a 2-inner product space.

References

1. C. Diminnie, S. Gahler and A. White, 2-inner product spaces, Demonstractio Math. 2 (1973), 525-536.
2. C. Diminnie, S. Gähler and A. White, Strictly convex linear 2-normed Spaces. Math. Nachr. 59(1974), 316-324.
3. C. Diminnie, S. Gahler and A. White, Remarks on strictly convex and strictly 2-convex normed spaces, Math. Nachr. 88 (1979), 363-372.
4. R.E. Ehret, Linear 2-normed spaces, Doctoral Dissertation, St. Louis Univ., 1969.
5. R.C. James, Orthogonality and linear functionals, Trans. Amer. Math. Soc. 62(1947), 265-292.
6. A. Khan and A. Siddiqui, B-orthognality in 2-normed spaces, Bull. Calcutta Math. Soc. 74(1982), 216-222.

Gyeongsang National University
Jinju 660-300
Korea

and

Dongeui University
Pusan 614-010
Korea

Received March 5, 1988