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ON HILBERT SEMIGROUP RINGS*

CHOLON KIM

A ring R is called a Hilbert ring if every prime ideal of R is an intersection of primitive ideals. When R is commutative, Gilmer [3] shows that equivalent conditions for $R[\{x_i\}_{i\in I}]$ to be a Hilbert ring. But the weakness of his results is including commutativity.

In this paper, we shall discuss Hilbert semigroup ring with noncommutative coefficients rings. Actually, when S is a cancellative monoid and the coefficient ring R is a PI ring, the condition that the semigroup ring $R[S]$ to be Hilbert will be observed. All monoid considered are assumed to be commutative.

We begin the following.

LEMMA 1. Let R bea PI ring and P be an ideal of R. Then P is a primitive ideal if and only if P is a maximal ideal.

PROOF. Suppose P is a maximal ideal of R. Then the factor ring R/P is a simple ring. So it is primitive and

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therefore P is a primitive ideal.

Conversely, if P is a primitive ideal of R, then R/P is a primitive ring. Since R is PI, R/P is PI. Thus the factor ring R/P is a primitive PI ring and so it is simple by Kaplansky's Theorem [4].

We denote the center and the classical quotient ring of R by $Z(R)$ and $Q(R)$, respectively.

LEMMA 2. Let R be a prime PI ring and $Q(R)$ the classical quotient ring of R. Then $Q(R)=R[a^{-1}]$ for some $0\neq a$ in $Z(R)$ if and only if $A\cap R=0$ for some maximal A of $R[x]$.

PROOF. Suppose $Q(R)=R[a^{-1}]$ with $0\neq a\in Z(R)$. Then *the map* σ *from* $R[x]$ *to* $R[a^{-1}]$ induced from the map sending x to a^{-1} is a ring epimorphism. Now since $Q(R)=R[a^{-1}]$ is simple Artinian, $A=\ker\sigma$ is a maximal ideal of $R[x]$. In this case $R[x]/A\cong R[a^{-1}]=Q(R)$ and $A\cap R=0$ since the map sending r to $r+A$ is one to one.

Conversely, assume that $A\cap R=0$ for some maximal ideal A of $R[x]$. Let $u=x+A$ in the ring $R[x]/A$. Then since $A\cap R=0$, $R\subseteq R[u]$ and $R[u]=R[x]/A$ is simple Artinian. So $Q(R)[u]=R[u]=Q(R[u])$. If $u=0$, then $R=R[x]$ is simple Artinian and so we are done. Hence we may assume that $u\neq 0$. Since u is a central element of the simple Artinian ring $R[u]$, u is in the center of $R[u]$. But note that the center of a simple Artinian ring is a field. So u is invertible in $R[u]$. Actually $u^{-1}\in Z(R[u])=Z(Q(R)[u])$. say

$$u^{-1}=\alpha_0+\alpha_1 u+\cdots+\alpha_n u^n$$

with $\alpha_0, \alpha_1, \cdots, \alpha_n \in Q(R)$ and $\alpha_n \neq 0$. But since $Q(R)$ is simple, $Q(R)\alpha_n Q(R) = Q(R)$ and so there exist $q_1, q_1, \cdots, q_s$ and $q_1', q_2', \cdots, q_s'$ in $Q(R)$ such that

$$\sum_{i=1}^{s} q_i \alpha_n q_i' = 1.$$

Thus $\sum_{i=1}^{s} q_i(\alpha_n u^{n+1} + \cdots + \alpha_1 u^2 + \alpha_0 u - 1)q_i' = 0$. Therefore $\beta_0 + \beta_1 u + \cdots + \beta_n u^n + u^{n+1} = 0$ with some $\beta_i \in Q(R)$. Now let k be the least positive integer such that

$$u^k + \beta_{k-1}u^{k-1} + \cdots + \beta_1 u + \beta_0 = 0$$

with $\beta_i \in Q(R)$, $i = 0, 1, \cdots, k-1$. In this case our *claim* is that $\beta_0, \beta_1, \cdots, \beta_{k-1}$ are in the center $Z(Q(R))$ of $Q(R)$. Now for $r \in Q(R)$, we have

$$\begin{aligned} 0 &= r(u^k + \beta_{k-1}u^{k-1} + \cdots + \beta_1 u + \beta_0) \\ &\quad - (u^k + \beta_{k-1}u^{k-1} + \cdots + \beta_1 u + \beta_0)r \\ &= (r\beta_{k-1} - \beta_{k-1}r)u^{k-1} + \cdots + (r\beta_1 - \beta_1 r)u + (r\beta_0 - \beta_0 r). \end{aligned}$$

If $r\beta_{k-1} - \beta_{k-1}r \neq 0$, then since $Q(R)$ is simple, we have

$$Q(R)(r\beta_{k-1} - \beta_{k-1}r)Q(R) = Q(R).$$

So there exist $l_1, l_2, \cdots, l_n$ and $l_1', l_2', \cdots, l_n'$ in $Q(R)$ such that

$$\sum_{i=1}^{n} l_i(r\beta_{k-1} - \beta_{k-1}r)l_i' = 1.$$

So we have

$$0 = \sum_{i=1}^{n} l_i(r\beta_{k-1}\beta - \beta_{k-1}r)l_i' u^{k-1} + \cdots + \sum_{i=1}^{n} l_i(r\beta_0 - \beta_0 r)l_i'.$$

Therefore

$$0 = u^{k-1} + \cdots + \varepsilon_1 u + \varepsilon_0$$

with $\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{k-1}$ in $Q(R)$. But thit is impossible by the choice of k. Hence $r\beta_{k-1}-\beta_{k-1}r=0$. Similarly $r\beta_{k-2}-\beta_{k-2}r=0, \cdots$, and $r\beta_0-\beta_0 r=0$. Therefore $r\beta_i=\beta_i r$ for every $r\in Q(R)$ and $i=0, 1, \cdots, k-1$. This means that

$$u^k+\beta_{k-1}u^{k-1}+\cdots+\beta_1 u+\beta_0=0$$

with $\beta_i\in Z(Q(R)), i=0, 1, \cdots, k-1$.

Now since $F[u]$ is central in $R[u]$, $F[u]$ is a domain. But since u is algebraic over F, $F[u]$ is an algebraic domain over the field F. So $F[u]$ is a field. Consider the canonical map σ from $Q(R)\otimes_F F[u]$ onto $Q(R)[u]$:

$$Q(R) \otimes_F F[u] \xrightarrow{\sigma} Q(R)[u]$$
$$\Sigma q_i \otimes a_i \longrightarrow \Sigma q_i a_i.$$

Then since $Q(R)$ is a central simple F-algebra and $F[u]$ is also simple F-algebra, $Q(R)\otimes_F F[u]$ is a simple F-algebra by Therefore the nonzero map σ has the zero kernel. Thus σ is an isomorphism. So we have $Q(R)\otimes_F F[u]\cong Q(R)[u]$. Since $F=Z(Q(R))$ is the field of fractions of $Z(R)$, we have

$$a_k u^k+a_{k-1}u^{k-1}+\cdots+a_1 u+a_0=0,$$

with $a_i\in Z(R)$ and $a_k\neq 0$ from the realtion

$$u^k+\beta_{k-1}u^{k-1}+\cdots+\beta_1 u+\beta_0=0.$$

Observe the ring $R[a_k^{-1}]$. Since

$$u^k=(a_k^{-1})a_{k-1}u^{k-1}+\cdots+(a_k^{-1})a_0$$

with $a_k^{-1}a_i\in R[a_k^{-1}]$, we have that $R[a_k^{-1}][u]$ is a finitely generated $R[a_k^{-1}]$-module by $\{1, u, \cdots, u^{k-1}\}$. Of course in the case $Q(R)[u]=R[a_k^{-1}][u]$. Since $\dim{}_F F[u]=k$,

$Q(R)\otimes_F F[u]\cong Q(R)[u]$ is a free $Q(R)$-module with basis $\{1, u, \cdots, u^{k-1}\}$ by the standard tenssor product property.

Now finally let $q\in Q(R)$. Then $q\in Q(R)[u]=R[a_k^{-1}][u]$. Then we have

$$q=\alpha_0+\alpha_1 u+\cdots+\alpha_{k-1}u^{k-1}$$

with $\alpha_i\in R[a_k^{-1}]\subseteq Q(R)$. But since $Q(R)[u]$ is free over $Q(R)$ with basis $\{1, u, \cdots, u^{k-1}\}$, we have

$$0=(\alpha_0-q)+\alpha_1 u+\cdots+\alpha_{k-1}u^{k-1}.$$

Hence $\alpha_0-q=0$, $\alpha_1=\cdots=\alpha_{k-1}=0$. Thus $q=\alpha_0$ is in $R[a_k^{-1}]$. So we have $Q(R)=R[a_k^{-1}]$ completing our tedious proof.

For a *PI* ring R, an ideal Q of R is called *G-ideal* if $Q=M\cap R$ for some maximal ideal M of $R[x]$. A ring R is called *G-ring* if 0 is *G-ideal*. By Lemma 2, we easily have the following.

LEMMA 3. A prime *PI* ring R is *G-ring* if and only if $Q(R)=R[u]$ for some u in $Q(Z(R))$.

The following may already be well known but for completeness we collect some characterizations of Hilberts rings.

PROPOSITION 4. Let R be *PI* ring. Then the following conditions are equivalent.

(1) R is a Hilbert ring.

(2) Every maximal ideal A of $R[x]$, $A\cap R$ is maximal ideal of R.

(3) Every simple $R[x]$-module is finitely generated R-module.

(4) Every *G-ideal* of R is maximal.

PROOF. (1) implies (2): Suppose R is a Hilbert ring. For a maximal ideal A of $R[x]$, $A\cap R$ is a prime ideal of R. Now by passing R to $R/A\cap R$, we may assume that $A\cap R=0$ and R is a prime PI ring. But since R is Hilbert, so its homomorphic image $R/A\cap R$ is Hilbert. So again we may assume that R is a prime PI Hilbert ring. Therefore R is semiprimitive and hence the intersection of maximal ideals $\bigcap_\alpha M_\alpha$ is zero. Since $A\cap R=0$, $Q(R)=R[a^{-1}]$ for some $0\neq a$ in $Z(R)$ by Lemma 2.

Assume $M_\alpha\neq 0$ for every α. Then since R is prime PI, $M_\alpha\cap Z(R)\neq 0$ for every α by Rowen [5]. Take $0\neq b_\alpha$ in $M_\alpha\cap Z(R)$. Then $b_\alpha^{-1}\in Q(R)=R[a^{-1}]$. So there is a positive integer $n(\alpha)$ such that $b_\alpha^{-1}=c_\alpha a^{-n(\alpha)}$ with $c_\alpha\in R$. Therefore $a^{n(\alpha)}=b_\alpha c_\alpha$ is in M_α. Since M_α is a maximal ideal and a is central, we have $a\in M_\alpha$ for every α. For, since $a^{n(\alpha)}\in M_\alpha$, $Ra^{n(\alpha)}\subseteq M$. Thus

$$Ra^{n(\alpha)}R=\underbrace{(RaR)\cdots(RaR)}_{n(\alpha)\text{-times}}\subseteq M_\alpha.$$

But since M_α is maximal, M_α is prime and so $RaR=aR\subseteq M_\alpha$. Thus $a\in M_\alpha$ for every α. Therefore $a\in\bigcap_\alpha M_\alpha=0$ is a contradiction. So $M_\alpha=0$ for some α. That is, 0 is a maximal ideal of R and so R is a simple ring. In other words, $A\cap R$ is a maximal ideal.

(2) implies (3): Let N be a simple $R[x]$-module. Then there is a maximal right ideal I of $R[x]$ such that $N=R[x]/I$. In this case, we can choose a two-sided ideal I_0 of $R[x]$ which is maximal with respect to the fact that I_0 is sitting inside I. Indeed I_0 is the right annihilator of

N in $R[x]$. So I_0 is a primitive ideal of $R[x]$ with N as a faithful irreducible $R[x]/I_0$-module. But since $R[x]$ is a PI ring, I_0 is a maximal ideal by Lemma 1. and obviously it is nonzero. Furtheremore $R[x]/I_0$ is simple Artinian and $R[x]/I$ is isomorphic to a minimal right ideal which is of course a direct summand of the ring $R[x]/I_0$. Now by our assumption (2), $R\cap I_0$ is a maximal ideal and so we have $R[x]/I_0=\bar{R}[x]/\bar{I}_0$, where $R=R/R\cap I_0$ and $I_0=I_0/(R\cap I_0)[x]$. But since R is simple, $R[x]/I_0=\bar{R}[x]/\bar{I}_0$ is finitely generated as R module. So $R[x]/I_0$ is finitely generated as R-module which shows (3) holds.

(3) implies (5): Assume that every $R[x]$ module is a finitely generated R-module. For a given prime ideal of R by passing to its factor ring, we may assume that R is prime PI. In this situation we need to show that the intersection of maximal ideals (equivalently, primitive ideals) is 0 from the definition of Hilbert ring.

Let $\{M_\alpha\}$ be the set of all maximal ideals of R. Suppose $\bigcap_\alpha M_\alpha \neq 0$. Then $\bigcap M_\alpha Z(R) \neq 0$. Take $0 \neq a \in \bigcap M_\alpha \cap Z(R)$. Then $\{a^n\}_{n=0}^{\infty}$ is an m-system with $a^i \neq 0$ for every i, because $Z(R)$ is a commutative domain. Let P be an ideal of R with $P\cap \{a^n\}_{n=0}^{\infty}=\phi$ and P is maximal with such property. Then as we already noted, P is a prime ideal of R. Of course the exstence of such P is assured by Zorn's lemma. Let $\bar{a}$ be the image of a in the factor ring $R=R/P$. Then $\bar{a}$ is in the center of R. For a nonzero $\bar{b}$ in $Z(R)$, $\bar{b}R$ is a nonzero ideal of R. Now by the definition of P, we have $\bar{a}^k \in \bar{b}R$ for some positive integer k. Hence $\bar{a}^k=\bar{b}\bar{c}$ with $\bar{c}$ in R and so $\bar{c}=\bar{b}^{-1}\bar{a}^k \in Q(Z(R))\cap R=Z(R)$. So $\bar{b}^{-1}$

$=\bar{a}^{-n}\in R[\bar{a}^{-1}]$. Therefore $R[\bar{a}^{-1}]=Q(R)$. So R is a G-ring. Thus P is a G-ideal and hence $P=A\cap R$ for some maximal ideal A of $R[x]$. So P is a maximal ideal of R. Since $a\notin P$, we have $a\notin M_\alpha$ for some α. But this is a contradiction since $a\in Z(R)\cap\ \cap M_\alpha$. So $Z(R)\cap\ \cap M_\alpha=0$. Thus we have $\bigcap_\alpha M_\alpha=0$. Hence R is a Hilbert ring.

(2) implies (4) and (4) implies (2): Obvious.

We recall that the pseudoradical of the ring R is the intersection of all nonzero prime ideals of R. Now we are in the situation to characterize Hilbert ring $R[x]$ that the coefficient ring R satisfies a polynomial identity.

THEOREM 5. Assume that $X=\{x_i\}$ is a set of commuting indeterminates of cardinality α over the PI ring R. Denote by Z^α the direct sum of α copiesof the additive group Z of integers. Then the following conditions are equivalent.

(1) $R[X]$ is not a Hilbert ring.
(2) There exists a prime ideal P of R such that R/P admist an α-generated extension ring that is a G-ring but not a simple.
(3) The group ring $R[Z^\alpha]$ is not a Hilbert ring.

PROOF. (1) implies (2): Let $R[X]$ be not a Hilbert ring. Then there is *G-ideal* A of $R[X]$ that is not maximal. So there a maximalideal M of $R[X][y]$ such that $A=M\cap R[X]$. Since R is a PI ring, $R[X][y]$ is a PI ring. Of $V=\{x+A|x\in A\}$, then $|V|\leq\alpha$ and $R[X]/A=(R/R\cap A)[V]$ is *G-ring* because A is *G-ideal* clealy $R\cap A$ is prime ideal of R. Of $P=R\cap A$, then $(R/P)[V]=R[X]/M\cap R[X]$ is an

α-generated extension ring of R/P because x is the set of commuting indeterminates. Since $M \cap R[X]$ is not maximal. it is not simple.

(2) imples (3): Let $D=R/P$ and let $J=D[\{a_i\}_{\iota\varepsilon I}]$ be α-generated extension of D that is a *G-ring* but not simple. We will show that if T is a subring of $D[\{x_i\},\{x_\iota^{-1}\}]\cong D[Z^\alpha]$ containing $D[X]$, then T is not a Hilbert ring. Since J is a *G-ring*, it has nonzero pseudoradical by gilmer [2]. Of J has a zero maximal ideal, then J is simple. Thus the pseudoradical of J is contained in the intersection of nonzero maximal ideal of J choose a nonzero element b in the pseudoradical. Then, for each i, $1+ba_\iota$ is invertible in J because the pseudoradical of J is in Rad J by Lemma 1. So

$$D[\{1+bai\},\ \{(1+bai)^{-1}\}]\subseteq J.$$

The D-homomorphism of $D[X]=D[\{x_\iota\}]$ onto $D[\{1+ba_\iota\}]$ determined by $i\longrightarrow 1+ba_i$ and $d\longrightarrow d$ forall $d\in D$ induces a D-homomorphism σ of $D[\{x_\iota\},\ \{x_\iota^{-1}\}=D[Z^\alpha]$ onto

$$D[\{1+ba_i],\ \{(1+ba_\iota)^{-1}\}]$$

and under σ we have

$$J\supseteq\sigma(T)\supseteq D[\{1+ba_\iota\}]=D[\{ba_\iota\}].$$

Now

$$\begin{aligned}J[b^{-1}]\supseteq\sigma(T)[b^{-1}]\supseteq D[\{ba_\iota\},b^{-1}]&=D[\{a_\iota\},\ b^{-1}]\\&=D[\{a_i\}][b^{-1}]=J[b^{-1}]=Q(J)\end{aligned}$$

Consequently,

$$J[b^{-1}]=\sigma(T)[b^{-1}]=Q(J).$$

We claim that $\sigma(T)$ is not simple. To do this, if $\sigma(T)$ is simple, then $\sigma(T)$ is a simple PI ring. Since the quotient ring of simple PI is itself,

$$Q(\sigma(T)) = \sigma(T) = J[b^{-1}] \subseteq J$$

Hence

$$J = J[b^{-1}] = Q(J).$$

It means that J is simple and this contradicts the fact that J is not simple. Therefore, $\sigma(T)$, and hence T, is not a Hilbert ring.

Finally, let $T = D[\{x_i\}, \{x_i^{-1}\}] = D[Z^{\alpha}]$. Then

$$\sigma(T) = D[\{1+bai\}, \{(1+bai)^{-1}\}] \subseteq J$$

is not simple. Since

$$Q(\sigma(T)) = \sigma(T)[b^{-1}] \subseteq J[b^{-1}] = Q(J)$$

by Lemma 3, $\sigma(T)$ is a G-ring. Now if $\sigma(T)$ is a Hilbert ring, then so is its homomorphic image $\sigma(T)$ and zero is maximal ideal of $\sigma(T)$. Hence $\sigma(T)$ is simple. Therefore, in particular $T = D[Z^{\alpha}]$, $D[Z^{\alpha}]$ is not a Hilbert ring.

(3) implies (1): By Armendariz, Koo and Park [1].

An overring S of a ring R with the same identity is called a *finite centralizing extension* of R if there is a finite subset $\{u_1, u_2, \cdots u_n\}$ of S such that $S = u_1R + u_2R + \cdots + u_nR$ and $n_i r = r u_i$ to all $i = 1, 2, \cdots, n$ and $r \in R$. Schelter [6] shows that every finite centralizing extension of a PI ring R is an integral extensions. Also he proves that such extensions enjoy Lying over, Going up and Incomparablity. So for a finite centralizing extension S of R, S is Hilbert if and only if R is Hilbert. By the help of Theorem 5 we can investigate monoid rings over PI rings.

THEOREM 6. Let R be a PI ring, S be a cancellative monoid of torsion-free rank α and $G=SS^{-1}$ be its quotient group. The the following anditions are equivalent:

(1) $R[x]$ is a Hilbert ring with $|x|=\alpha$.
(2) $R[G]$ is a Hilbert ring.
(3) $R[S]$ is a Hilbert ring.

PROOF. For α we consider two cases:

Case 1. α is infinite.

(1) implies (2): Suppore that $R[x]$ is a Hilbert ring. Let H be the sub group of G generatedby a maximal free subset $\{f_i\}_{i\in I}$ of $G=SS^{-1}$. Then $|I|=\alpha$ and G/H is a torsion abelian group. Let $\beta=r_1g_1+r_2g_2+\cdots+r_ng_n$ be an element of $R[G]$ with $r_i\in R$ and $g_i\in G$, $i=1,2,\cdots,n$. Consider the subgroup H_0 generated by H and $g_1,g_2,\cdots,g_n$. Then H_0/H is finite and b is in $R[H_0]$. Observing that $R[H_0]$ is a finite centralizing extension of $R[H]$, $R[H_0]$ is an intergral extension of $R[H]$ by Schelter[6]. So b is integral over $R[H]$ and hence $R[G]$ is integral over $R[H]$. In fact, since $R[H]$ is free abelian group of rank α, $H\cong Z_\alpha$. By Theorem, $R[H]$ is a Hilbert ring. By application of previous mentioned Schelter's result, $R[G]$ is a Hilbert ring.

(2) implies(3): Suppore that $R[S]$ is not a Hilbert ring. Since α is infinite, $|S|=\alpha$. Observing that $R[S]$ is a epimorphic image of $R[x]$ by sending $x_i\to s_i$. Since $R[G]$ is integral over $R[H]$, $R[G]$ is Hilbert ring if and only if

$R[H]$ is Hilbert. By Theorem 5, $R[X]$ is a Hilbert ring because $R[G]$ is a Hilbert ring. But this contradicts the fact that $R[S]$ is not a Hilbert ring.

(3) implies (1): Let $F=\{f_i\}_{i\in I}$ bea maximal free subset of G. But since $G=SS^{-1}$ we may assume that F is a subset of S. Let H be the subgroup of G generated by $\{f_i\}_{i\in I}$. Then ofcause, $H\cong Z^{\alpha}$ and $T=H\cap S$ is a free submonoid of S. First our claim is that $R[S]$ is integral over $R[T]$. For this argument, let $\beta=a_1s_1+a_2s_2+\cdots+a_ks_k$ be an element of $R[S]$ with $a_i\in R$ and $s_i\in S$, $i=1,2,\cdots,k$. Then for each s_i, there is a positive integer n_i such that $n_is_i\in H$ and so $n_is_i\in T$. So if we denote T_0 as the submonoid generated by T and $s_1,s_2,\cdots,s_k$, then $R[T_0]$ is a finite centralizing extension of $R[T]$ generated by finite centralizing elements $\{m_1s_1+\cdots+m_ks_k|1\leq m_i\leq n_i,\ i=1,2,\cdots k\}$ over $R[T]$. But since β is in $R[T_0]$, β is integral over $R[T]$. In other words, every element of $R[S]$ is integral over $R[T]$, that is $R[S]$is an integral extension of $R[T]$.

Now for our proof that (3) implies (1), assume to the contrary that contrary that $R[x]$ is not a Hilbert ring. Then by Theorem 5, $R[Z^{\alpha}]$ is not a Hilbert ring. By the condition (2) in Theorem 5, there is a prime ideal P of R such that the ring R/P admits an α-generated extension that is a *G-ring* but not simple. In this situation any ring sitting between $(R/P)[Z_0{}^{\alpha}]$ and $(R/P)[Z^{\alpha}]$ can not be Hilbert, where $Z_0{}^{\alpha}$ denotes the monoid of nonnegative intergers. Now since

$$(R/P)[Z_0{}^{\alpha}]\subseteq(R/P)[T]\subseteq(R/P)[Z^{\alpha}],$$

$(R/P)[T]$ is not a Hilbert ring and hence $R[T]$ is not a Hilbert ring. But as we already proved since $R[S]$ integral over $R[T]$, $R[S]$ is not a Hilbert ring, which is a contradiction So So $R[x]$ is Hilbert.

Case 2. α is finite

As in the proof of (3) impleis (1), still we can verify $R[S]$ is integral over $R[T]$ when α is finite. Now since T is a free submonoid and $|F|=\alpha$, we have that $R[T]=R[X]$. So we have $R[X]$ is Hilbert if and only if $R[T]$ is Hilbert if and only if $R[S]$ is Hilbert. On the other hand by Theorem 5, $R[X]$ is Hilbert if and only if $R[G]$ is Hilbert.

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Pusan National University
Pusan 609-735
Korea

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